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AN ALTERNATIVE LOOK AT THE STRUCTURE OF GRAPH INVERSE SEMIGROUPS[1]

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For any graph inverse semigroup $G(E)$ we describe subsemigroups $D^0 = D \cup \{0\}$ and $J^0 = J \cup \{0\}$ of $G(E)$ where D and J are arbitrary $\mathcal{D}$-class and $\mathcal{J}$-class of $G(E)$, respectively. In particular, we prove that for each $\mathcal{D}$-class D of a graph inverse semigroup over an acyclic graph the semigroup D^0 is isomorphic to a semigroup of matrix units. Also we show that for any elements a, b of a graph inverse semigroup $G(E)$, $J_a \cdot J_b \cup J_b \cdot J_a \subset J_b^0$ if there exists a path w such that $s(w) \in J_a$ and $r(w) \in J_b$.

1. Preliminaries. We shall follow the terminology of [11] and [19]. By $\mathbb{N}$ we denote the set of positive integers. The cardinality of a set X is denoted by $|X|$. A semigroup S is called an *inverse semigroup* if for each element $a \in S$ there exists a unique element $a^{-1} \in S$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$.

By $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{H}$ we denote Green's relations on a semigroup S which are defined as follows: for each $a, b \in S$

$$\begin{array}{lll} a\mathcal{R}b & \text{if and only if} & aS \cup \{a\} = bS \cup \{b\}; \\ a\mathcal{L}b & \text{if and only if} & Sa \cup \{a\} = Sb \cup \{b\}; \\ a\mathcal{J}b & \text{if and only if} & SaS \cup aS \cup Sa \cup \{a\} = SbS \cup bS \cup Sb \cup \{b\}; \\ & \mathcal{D} = \mathcal{L}\circ\mathcal{R} = \mathcal{R}\circ\mathcal{L}; & \mathcal{H} = \mathcal{L} \cap \mathcal{R}. \end{array}$$

Let S be a semigroup with zero 0_S and X be a non-empty set. By $\mathcal{B}_X(S)$ we denote the set $X \times S \times X \sqcup \{0\}$ endowed with the following semigroup operation:

$$(a, s, b) \cdot (c, t, d) = \begin{cases} (a, s \cdot t, d), & \text{if } b = c; \\ 0, & \text{if } b \neq c, \end{cases}$$

$$\text{and } (a, s, b) \cdot 0 = 0 \cdot (a, s, b) = 0 \cdot 0 = 0, \text{ for each } a, b, c, d \in X \text{ and } s, t \in S.$$

The semigroup $\mathcal{B}_X(S)$ is called the *Brandt X-extension of the semigroup S*. Obviously, the set $J = \{(a, 0_S, b) \mid a, b \in X\} \cup \{0\}$ is a two-sided ideal of the semigroup $\mathcal{B}_X(S)$. The Rees factor semigroup $\mathcal{B}_X(S)/J$ is called the *Brandt X^0-extension of the semigroup S* and is denoted by $\mathcal{B}^0_X(S)$. If S is the semilattice $(\{0, 1\}, \min)$ then we denote the semigroup

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$\mathcal{B}^0_X(S)$ by $\mathcal{B}^0_X$. The semigroup $\mathcal{B}^0_X$ is well-known (see page 86 from [19]) and is called the *semigroup of* $X\times X$*-matrix units*. Observe that semigroups $\mathcal{B}^0_X(S)$ and $\mathcal{B}^0_Y(S)$ are isomorphic iff $|X| = |Y|$. A Brandt X^0-extension of a group play an important role in the structure of primitive inverse semigroups (see [19, Chapter 3.3]). Algebraic and topological properties of a Brandt X^0-extension of a semigroup were investigated in [14] and [15].

For a cardinal λ *polycyclic monoid* $\mathcal{P}_\lambda$ is the semigroup with identity 1 and zero 0 given by the presentation

$$\mathcal{P}_\lambda = \left\langle 0, 1, \{p_i\}_{i\in\lambda}, \left\{p_i^{-1}\right\}_{i\in\lambda} \mid p_i^{-1}p_i = 1, p_j^{-1}p_i = 0 \text{ for } i \neq j \right\rangle.$$

Observe that polycyclic monoid $\mathcal{P}_0$ is isomorphic to the semilattice $(\{0,1\}, \min)$. Polycyclic monoid is a generalization of the well-known bicyclic monoid (see [19, Chapter 3.4]). More precisely, the bicyclic monoid with adjoined zero is isomorphic to the polycyclic monoid $\mathcal{P}_1$. Polycyclic monoid $\mathcal{P}_k$ over a finite non-zero cardinal k was introduced in [24]. Algebraic and topological properties of polycyclic monoids were investigated in [4, 9, 10, 13, 20, 21].

A *directed* graph $E = (E^0, E^1, r, s)$ consists of disjoint sets E^0, E^1 of *vertices* and *edges*, respectively, together with functions $s, r\colon E^1 \to E^0$ which are called *source* and *range*, respectively. In this paper we refer to a directed graph simply as "graph". We consider each vertex as a path of length zero. A path of non-zero length $x = e_1 \dots e_n$ in a graph E is a finite sequence of edges $e_1, \dots, e_n$ such that $r(e_i) = s(e_{i+1})$ for each positive integer $i < n$. By $\operatorname{Path}^+(E)$ we denote the set of all paths of a graph E which have a non-zero length. We extend functions s and r on the set $\operatorname{Path}(E) = E^0 \cup \operatorname{Path}^+(E)$ of all paths in the graph E as follows: for each vertex $e \in E^0$ put $s(e) = r(e) = e$ and for each path of non-zero length $x = e_1 \dots e_n \in \operatorname{Path}^+(E)$ put $s(x) = s(e_1)$ and $r(x) = r(e_n)$. By $|x|$ we denote the length of a path x. Let $a = e_1 \dots e_n$ and $b = f_1 \dots f_m$ be two paths such that $|a| > 0$, $|b| > 0$ and $r(a) = s(b)$. Then by ab we denote the path $e_1 \dots e_n f_1 \dots f_m$. If a is a vertex and b is a path such that $s(b) = a$ ($r(b) = a$, resp.) then put $ab = b$ ($ba = b$, resp.). A path x is called a *prefix* of a path y if there exists a path z such that $y = xz$. An edge e is called a *loop* if $s(e) = r(e)$. A path x is called a *cycle* if $s(x) = r(x)$ and $|x| > 0$. Vertices a and b of a graph E are called *strongly connected* if there exist paths $u, v \in \operatorname{Path}(E)$ such that $a = s(u) = r(v)$ and $b = s(v) = r(u)$. Define a relation R on the set E^0 as follows: $(a, b) \in R$ if and only if vertices a and b are strongly connected. Simple verifications show that R is an equivalence relation. Equivalence classes of the relation R are called *strongly connected components* of a graph E. A graph E is called *acyclic* if it contains no cycles.

For a given directed graph $E = (E^0, E^1, r, s)$ a graph inverse semigroup (or simply GIS) $G(E)$ over the graph E is the semigroup with zero generated by the sets E^0, E^1 together with the set $E^{-1} = \{e^{-1} \mid e \in E^1\}$ which is disjoint with $E^0 \cup E^1$ satisfying the following relations for all $a, b \in E^0$ and $e, f \in E^1$:

(1) $a \cdot b = a$ if $a = b$ and $a \cdot b = 0$ if $a \neq b$;
(2) $s(e) \cdot e = e \cdot r(e) = e$;
(3) $e^{-1} \cdot s(e) = r(e) \cdot e^{-1} = e^{-1}$;
(4) $e^{-1} \cdot f = r(e)$ if $e = f$ and $e^{-1} \cdot f = 0$ if $e \neq f$.

Graph inverse semigroups are generalizations of the polycyclic monoids. In particular, for each cardinal λ a polycyclic monoid $\mathcal{P}_\lambda$ is isomorphic to the graph inverse semigroup over the graph E which consists of one vertex and λ distinct loops. However, by [5, Theorem 1], each graph inverse semigroup $G(E)$ is isomorphic to a subsemigroup of the polycyclic monoid $\mathcal{P}_{|G(E)|}$.

According to [16, Chapter 3.1], each non-zero element of a graph inverse semigroup $G(E)$ can be uniquely represented as uv^{-1} where $u, v \in \operatorname{Path}(E)$ and $r(u) = r(v)$. A semigroup operation in $G(E)$ is defined by the following way:

$$u_1v_1^{-1} \cdot u_2v_2^{-1} = \begin{cases} u_1wv_2^{-1}, & \text{if } u_2 = v_1w \text{ for some } w \in \operatorname{Path}(E); \\ u_1(v_2w)^{-1}, & \text{if } v_1 = u_2w \text{ for some } w \in \operatorname{Path}(E); \\ 0, & \text{otherwise,} \end{cases}$$

and $uv^{-1} \cdot 0 = 0 \cdot uv^{-1} = 0 \cdot 0 = 0$.

Further, when we write an element of $G(E)$ in a form uv^{-1} we always mean that $u, v \in \operatorname{Path}(E)$ and $r(u) = r(v)$. Simple verifications show that $G(E)$ is an inverse semigroup and $(uv^{-1})^{-1} = vu^{-1}$, for each element $uv^{-1} \in G(E)$.

Graph inverse semigroups play an important role in the study of rings and C^*-algebras (see [1, 3, 12, 18, 25]). Algebraic theory of graph inverse semigroups is well developed (see [2, 5, 16, 17, 20, 22]). Topological properties of graph inverse semigroups were investigated in [6–8, 23].

This paper is inspired by the paper of Mesyan and Mitchell [22] and can be regarded as an alternative look at the structure of graph inverse semigroups.

2. A local structure of graph inverse semigroups. By [22, Corollary 2], two non-zero elements ab^{-1} and cd^{-1} of a GIS $G(E)$ are $\mathcal{D}$-equivalent iff $r(a) = r(b) = r(c) = r(d)$. Observe that each non-zero $\mathcal{D}$-class contains exactly one vertex of E. By D_e we denote the $\mathcal{D}$-class which contains vertex $e \in E^0$. Put $D_e^0 = D_e \cup \{0\}$.

Lemma 1. *Let $G(E)$ be a GIS, $ab^{-1} \in D_e^0$ and $cd^{-1} \in D_f^0$. Then $ab^{-1} \cdot cd^{-1} \in D_e^0 \cup D_f^0$.*

Proof. Fix any elements $ab^{-1} \in D_e^0$ and $cd^{-1} \in D_f^0$. If $ab^{-1} \cdot cd^{-1} = 0$ then $ab^{-1} \cdot cd^{-1} \in D_e^0 \cup D_f^0$. Assume that $ab^{-1} \cdot cd^{-1} \neq 0$. Then there exists a path $w \in \operatorname{Path}(E)$ such that either $ab^{-1} \cdot cd^{-1} = awd^{-1}$ or $ab^{-1} \cdot cd^{-1} = a(dw)^{-1}$. In the first case $c = bw$ which yields that $r(aw) = r(w) = r(c) = r(d) = f$. Hence $ab^{-1} \cdot cd^{-1} \in D_f$. In the second case $b = cw$ which implies that $r(dw) = r(w) = r(b) = r(a) = e$. Hence $ab^{-1} \cdot cd^{-1} \in D_e$. □

Observe that if $uv^{-1} \in D_e$ then $(uv^{-1})^{-1} = vu^{-1} \in D_e$ which provides the following:

Corollary 1. *For each vertex e of a graph E, D_e^0 is an inverse subsemigroup of $G(E)$.*

Further we need the following denotations. For any vertex e of a graph E put:

$$I_e = \{u \in \operatorname{Path}(E) \mid r(u) = e\};$$

$$Q_e = \{u \in I_e \mid r(v) \neq e, \text{ for each non-trivial prefix } v \text{ of } u\};$$

$$C_e = \{u \in I_e \mid s(u) = r(u) = e\};$$

$$C_e^1 = C_e \cap Q_e = \{u \in C_e \mid r(v) \neq e, \text{ for each non-trivial prefix } v \text{ of } u\}.$$

By $\langle C_e \rangle$ ($\langle C_e^1 \rangle$, resp.) we denote the inverse subsemigroup of $G(E)$ which is generated by the set $C_e \cup \{0\}$ ($C_e^1 \cup \{0\}$, resp.). Observe that $e \in C_e^1$ and e is the identity of the semigroup $\langle C_e \rangle$. The following theorem describes the structure of the semigroup $\langle C_e \rangle$.

Theorem 1. *For each vertex e of any graph E the semigroup $\langle C_e\rangle$ is isomorphic to the polycyclic monoid $\mathcal{P}_{|C_e^1\setminus\{e\}|}$.*

Proof. Fix any vertex $e\in E^0$. Put $\lambda=|C_e^1\setminus\{e\}|$. Let $C_e^1\setminus\{e\}=\{u_\alpha\}_{\alpha\in\lambda}$. For convenience we denote e by u_{-1}. It is easy to check that $\langle C_e^1\rangle=\{uv^{-1}\mid u,v\in C_e\}\cup\{0\}=\langle C_e\rangle$.

Let $G=\{p_\alpha\}_{\alpha\in\lambda}\cup\{p_\alpha^{-1}\}_{\alpha\in\lambda}$ be the set of generators of the polycyclic monoid $\mathcal{P}_\lambda$. Define a map $f\colon\langle C_e\rangle\to\mathcal{P}_\lambda$ consecutively extending it as follows. At first we define f on C_e^1 by putting $f(u_{-1})=1$ and $f(u_\alpha)=p_\alpha$ for each $\alpha\in\lambda$. Let $u\in C_e\setminus\{e\}$ be any element. It is easy to check that u has a unique representation $u=u_{\alpha_1}u_{\alpha_2}\dots u_{\alpha_n}$ as a product of elements of $C_e^1\setminus\{e\}$. We put $f(u)=p_{\alpha_1}p_{\alpha_2}\dots p_{\alpha_n}$. Observe that any non-zero element of $\langle C_e\rangle$ has a unique representation in a form uv^{-1} for some paths $u,v\in C_e$. Finally, put $f(uv^{-1})=f(u)f(v)^{-1}$ and $f(0)=0$.

Obviously, f is a bijection. Let us show that f is a homomorphism. Fix any elements $ab^{-1},cd^{-1}\in\langle C_e\rangle$. Let

$$a=u_{\alpha_1}\dots u_{\alpha_n},\quad b=u_{\beta_1}\dots u_{\beta_m},\quad c=u_{\gamma_1},\dots u_{\gamma_k},\quad d=u_{\delta_1}\dots u_{\delta_t}$$

be (unique) representations of elements a,b,c,d as a product of elements of C_e^1 (here we agree that if some of the elements a,b,c or d are equal to e, then their representations are equal to u_{-1}). There are three cases to consider:

(1) b is a prefix of c;

(2) c is a prefix of b;

(3) $ab^{-1}\cdot cd^{-1}=0$.

Suppose that case 1 holds, i.e., $u_{\gamma_1}\dots u_{\gamma_k}=u_{\beta_1}\dots u_{\beta_m}u_{\gamma_{m+1}}\dots u_{\gamma_k}$. Observe that

$$ab^{-1}\cdot cd^{-1}=u_{\alpha_1}\dots u_{\alpha_n}u_{\gamma_{m+1}}\dots u_{\gamma_k}(u_{\delta_1}\dots u_{\delta_t})^{-1}.$$

Then

$$f(ab^{-1}\cdot cd^{-1})=f(u_{\alpha_1}\dots u_{\alpha_n}u_{\gamma_{m+1}}\dots u_{\gamma_k}(u_{\delta_1}\dots u_{\delta_t})^{-1})= \\ =p_{\alpha_1}\dots p_{\alpha_n}p_{\gamma_{m+1}}\dots p_{\gamma_k}(p_{\delta_1}\dots p_{\delta_t})^{-1}.$$

On the other hand,

$$f(ab^{-1})\cdot f(cd^{-1})=p_{\alpha_1}\dots p_{\alpha_n}(p_{\beta_m}^{-1}\dots p_{\beta_1}^{-1}\cdot p_{\beta_1}\dots p_{\beta_m})p_{\gamma_{m+1}}\dots p_{\gamma_k}(p_{\delta_1}\dots p_{\delta_t})^{-1}= \\ =p_{\alpha_1}\dots p_{\alpha_n}p_{\gamma_{m+1}}\dots p_{\gamma_k}(p_{\delta_1}\dots p_{\delta_t})^{-1}=f(ab^{-1}\cdot cd^{-1}).$$

Case 2 is similar to case 1. Consider case 3. In this case there exists a positive integer i such that $u_{\beta_j}=u_{\gamma_j}$ for every $j<i$ and $u_{\beta_i}\neq u_{\gamma_i}$. Observe that $f(ab^{-1}\cdot cd^{-1})=f(0)=0$ and

$$f(ab^{-1})\cdot f(cd^{-1})=p_{\alpha_1}\dots p_{\alpha_n}p_{\beta_m}^{-1}\dots p_{\beta_i}^{-1}(p_{\beta_{i-1}}^{-1}\dots p_{\beta_1}^{-1}\cdot p_{\beta_1}\dots p_{\beta_{i-1}})p_{\gamma_i}\dots p_{\gamma_k}(p_{\delta_1}\dots p_{\delta_t})^{-1}= \\ =p_{\alpha_1}\dots p_{\alpha_n}p_{\beta_m}^{-1}\dots(p_{\beta_i}^{-1}\cdot p_{\gamma_i})\dots p_{\gamma_k}(p_{\delta_1}\dots p_{\delta_t})^{-1}=0=f(ab^{-1}\cdot cd^{-1}).$$

Hence f is an isomorphism between semigroups $\langle C_e\rangle$ and $\mathcal{P}_{|C_e^1\setminus\{e\}|}$. □

The following theorem describes the structure of a subsemigroup D_e^0 of an arbitrary GIS.

Theorem 2. *Let E be any graph and $e \in E^0$. Then the semigroup D_e^0 is isomorphic to the Brandt Q_e^0-extension of the polycyclic monoid $\mathcal{P}_{|C_e^1 \setminus \{e\}|}$.*

Proof. Recall that $D_e = \{uv^{-1} \mid r(u) = r(v) = e\}$. The proof is based on the following obvious fact: each element $u \in I_e$ can be uniquely represented as follows: $u = u_1 u_2$ where $u_1 \in Q_e$ and $u_2 \in C_e$ (here both u_1 and u_2 can be equal to e). By Theorem 1, the semigroup $\langle C_e \rangle$ is isomorphic to the polycyclic monoid $\mathcal{P}_{|C_e^1 \setminus \{e\}|}$. Define the map $h\colon D_e^0 \to B_{Q_e}^0(\mathcal{P}_{|C_e^1 \setminus \{e\}|})$ as follows: $h(0) = 0$ and $h(uv^{-1}) = (u_1, f(u_2 v_2^{-1}), v_1)$ for each non-zero element $uv^{-1} = u_1 u_2 (v_1 v_2)^{-1} \in D_e$ where $u_1, v_1 \in Q_e$, $u_2, v_2 \in C_e$ and f is an isomorphism between semigroups $\langle C_e \rangle$ and $\mathcal{P}_{|C_e^1 \setminus \{e\}|}$ defined in Theorem 1. We remark that $f(e) = (e, 1, e)$. Suppose that $u \neq v$ for some paths $u, v \in I_e$. Then $u_1 \neq v_1$ or $u_2 \neq v_2$ which implies that the map h is injective. Since for each non-zero element $(a, uv^{-1}, b) \in B_{Q_e}^0(\mathcal{P}_{|C_e^1 \setminus \{e\}|})$ we have

$$h(af^{-1}(u)(bf^{-1}(v))^{-1}) = (a, f(f^{-1}(u)f^{-1}(v)^{-1}), b) = (a, ff^{-1}(uv^{-1}), b) = (a, uv^{-1}, b)$$

the map h is bijective. Now it remains to show that h is a homomorphism. Fix any elements $ab^{-1}, cd^{-1} \in D_e$. Following the main idea of the proof we can uniquely represent elements $a, b, c, d \in I_e$ as follows: $a = a_1 a_2, b = b_1 b_2, c = c_1 c_2$ and $d = d_1 d_2$ where $a_1, b_1, c_1, d_1 \in Q_e$ and $a_2, b_2, c_2, d_2 \in C_e$. There are three cases to consider:

(1) There exists $w \in \operatorname{Path}(E)$ such that $ab^{-1} \cdot cd^{-1} = awd^{-1}$, i.e., $c = bw$;

(2) there exists $w \in \operatorname{Path}(E)$ such that $ab^{-1} \cdot cd^{-1} = a(dw)^{-1}$, i.e., $b = cw$;

(3) $ab^{-1} \cdot cd^{-1} = 0$.

Consider case 1. Observe that $s(w) = r(b) = r(c) = r(w)$ which implies that $c_1 = b_1$ and $c_2 = b_2 w$. Hence

$$\begin{aligned} h(ab^{-1}) \cdot h(cd^{-1}) &= (a_1, f(a_2 b_2^{-1}), b_1) \cdot (b_1, f(b_2 w d_2^{-1}), d_1) = (a_1, f(a_2 b_2^{-1}) \cdot f(b_2 w d_2^{-1}), d_1) = \\ &= (a_1, f(a_2 b_2^{-1} \cdot b_2 w d_2^{-1}), d_1) = (a_1, f(a_2 w d_2^{-1}), d_1) = h(awd^{-1}). \end{aligned}$$

Consider case 2. Observe that $s(w) = r(c) = r(b) = r(w)$ which implies that $c_1 = b_1$ and $b_2 = c_2 w$. Similar calculations as in case 1 show that $h(ab^{-1}) \cdot h(cd^{-1}) = h(a(dw)^{-1})$.

Consider case 3. Observe that for each path $w \in \operatorname{Path}(E)$ neither $b = cw$ nor $c = bw$. Then one of the following two subcases holds:

(3.1) $b_1 \neq c_1$;

(3.2) $b_1 = c_1$, but for any $w \in \operatorname{Path}(E)$ neither $b_2 = c_2 w$ nor $c_2 = b_2 w$.

Consider subcase 3.1. Then

$$h(ab^{-1}) \cdot h(cd^{-1}) = (a_1, f(a_2 b_2^{-1}), b_1) \cdot (c_1, f(c_2 d_2^{-1}), d_1) = 0 = f(0).$$

Consider subcase 3.2. Then

$$\begin{aligned} h(ab^{-1}) \cdot h(cd^{-1}) &= (a_1, f(a_2 b_2^{-1}), b_1) \cdot (b_1, f(c_2 d_2^{-1}), d_1) = (a_1, f(a_2 b_2^{-1}) \cdot f(c_2 d_2^{-1}), d_1) = \\ &= (a_1, f(a_2 b_2^{-1} \cdot c_2 d_2^{-1}), d_1) = (a_1, 0, d_1) = 0 = h(0). \end{aligned}$$

Hence the map h is an isomorphism between semigroups D_e^0 and $B_{Q_e}^0(\mathcal{P}_{|C_e^1 \setminus \{e\}|})$. $\square$

Graph E is called *acyclic at a vertex* $e \in E^0$ if $C_e = \{e\}$.

Corollary 2. *Let E be a graph which is acyclic at a vertex e. Then the subsemigroup D_e^0 of $G(E)$ is isomorphic to the semigroup of $I_e \times I_e$-matrix units $\mathcal{B}_{I_e}^0$.*

Proof. Recall that by B_X^0 we denote the semigroup $B_X^0(\mathcal{P}_0)$. Since graph E is acyclic at a vertex e we obtain that $I_e = Q_e$ and $C_e = \{e\}$. By Theorem 2, the semigroup D_e^0 is isomorphic to the semigroup $B_{I_e}^0(\mathcal{P}_0)$. □

By [22, Corollary 2], two non-zero elements ab^{-1} and cd^{-1} of a GIS $G(E)$ are $\mathcal{J}$-equivalent iff there exist elements $u, v \in \operatorname{Path}(E)$ such that $s(u) = r(a) = r(v)$ and $r(u) = r(c) = s(v)$. There exists a one to one correspondence between the set of strongly connected components of a graph E and non-zero $\mathcal{J}$-classes of a GIS $G(E)$. More precisely, $J \cap E^0$ is a strongly connected component of a graph E for each non-zero $\mathcal{J}$-class J of $G(E)$. Therefore, by J_A we denote a $\mathcal{J}$-class which contains a strongly connected component $A \subset E^0$.

Observe that for each strongly connected component A of a graph E, $\mathcal{J}_A = \bigcup_{e \in A} D_e$. Hence Lemma 1 provides the following:

Corollary 3. *For each strongly connected component A of a graph E the set $J_A^0 = J_A \cup \{0\}$ is an inverse subsemigroup of $G(E)$.*

Let E be a graph and X be any non-empty subset of E^0. By E_X we denote the induced (by the set X) subgraph of the graph E, i.e., $E_X^0 = X$, $E_X^1 = \{x \in E^1 \mid s(x) \in X \text{ and } r(x) \in X\}$ and source (resp., range) function s_X (resp., r_X) of the graph E_X is the restriction of the source function s (resp., range function r) of the graph E on the set E_X^1. Let A be a strongly connected component of a graph E. Put

$$I_A = \{u \in \operatorname{Path}(E) \mid r(u) \in A\};$$
$$Q_A = \{u \in I_A \mid r(v) \notin A, \text{ for each non-trivial prefix } v \text{ of } u\}.$$

Lemma 2. *Let A be a strongly connected component of a graph E and w be a path such that $s(w) \in A$ and $r(w) \in A$. Then $w \in \operatorname{Path}(E_A)$ where E_A is an induced subgraph of E.*

Proof. The proof is obvious if $w = e \in A$. Let $w = a_1 \dots a_n$ be a path of non-zero length such that $s(w) = s(a_1) = e_1 \in A$ and $r(w) = r(a_n) = f \in A$. Put $s(a_i) = e_i$, for each $i \leq n$. Since vertices e_1 and f belong to A there exists a path u such that $s(u) = f$ and $r(u) = e_1$. We claim that for each $i \leq n$ vertices e_i belong to A. Indeed, put $x = a_1 \dots a_{i-1}$ and $y = a_i \dots a_n u$. Then $s(x) = e_1, r(x) = e_i$ and $s(y) = e_i, r(y) = e_1$ which provides that $\{e_i\}_{i \leq n} \subseteq A$. Hence $a_i \in E_A^1$ for every $i \leq n$ and, as a consequence, $w \in \operatorname{Path}(E_A)$. □

The following theorem describes the structure of a subsemigroup J_A^0 of $G(E)$ where A is any strongly connected component of a graph E.

Theorem 3. *Let E be any graph and $A \subseteq E^0$ be a strongly connected component. Then the semigroup J_A^0 is isomorphic to a subsemigroup of the Brandt Q_A^0-extension of the graph inverse semigroup $G(E_A)$ over the induced subgraph E_A.*

Proof. The proof of this theorem is based on the following fact which follows from Lemma 2. Each element $u \in I_A$ can be uniquely represented as follows: $u = u_1 u_2$ where $u_1 \in Q_A$ and $u_2 \in \operatorname{Path}(E_A) \subset \operatorname{Path}(E)$. Observe that u_1 and u_2 could be equal to some vertex $e \in A$. Define the map $f \colon J_A^0 \to B_{Q_A}^0(G(E_A))$ by the following way: $f(0) = 0$ and for each non-zero element $uv^{-1} = u_1 u_2 (v_1 v_2)^{-1} \in G(E)$ where $u_1, v_1 \in Q_A$ and $u_2, v_2 \in \operatorname{Path}(E_A) \subset \operatorname{Path}(E)$

put $f(uv^{-1}) = (u_1, u_2v_2^{-1}, v_1)$. The injectivity of the map f is straightforward. Next we show that the map f is a homomorphism. Fix any elements $ab^{-1}, cd^{-1} \in J_A$. Following the main idea of the proof we can uniquely represent elements $a, b, c, d \in I_A$ as follows: $a = a_1a_2, b = b_1b_2, c = c_1c_2$ and $d = d_1d_2$ where $a_1, b_1, c_1, d_1 \in Q_A$ and $a_2, b_2, c_2, d_2 \in G(E_A)$. There are three cases to consider:

(1) there exists $w \in \text{Path}(E)$ such that $ab^{-1} \cdot cd^{-1} = awd^{-1}$, i.e., $c = bw$;

(2) there exists $w \in \text{Path}(E)$ such that $ab^{-1} \cdot cd^{-1} = a(dw)^{-1}$, i.e., $b = cw$;

(3) $ab^{-1} \cdot cd^{-1} = 0$.

Consider case 1. Observe that $s(w) = r(b) \in A$ and $r(w) = r(c) \in A$. By Lemma 2, $w \in \text{Path}(E_A)$ which implies that $c_1 = b_1$ and $c_2 = b_2w$. Hence

$$f(ab^{-1}) \cdot f(cd^{-1}) = (a_1, a_2b_2^{-1}, b_1) \cdot (b_1, b_2wd_2^{-1}, d_1) = (a_1, a_2b_2^{-1} \cdot b_2wd_2^{-1}, d_1) = \\ = (a_1, a_2wd_2^{-1}, d_1) = f(awd^{-1}).$$

Consider case 2. Observe that $s(w) = r(c) \in A$ and $r(w) = r(b) \in A$. By Lemma 2, $w \in \text{Path}(E_A)$ which implies that $c_1 = b_1$ and $b_2 = c_2w$. Similar calculations as in case 1 show that $f(ab^{-1}) \cdot f(cd^{-1}) = f(a(dw)^{-1})$.

Consider case 3. Observe that neither $b = cw$ nor $c = bw$. Then one of the following two subcases holds:

(3.1) $b_1 \neq c_1$;

(3.2) $b_1 = c_1$, but for any $w \in \text{Path}(E)$ neither $b_2 = c_2w$ nor $c_2 = b_2w$.

Consider subcase 3.1. Then $f(ab^{-1}) \cdot f(cd^{-1}) = (a_1, a_2b_2^{-1}, b_1) \cdot (c_1, c_2d_2^{-1}, d_1) = 0 = f(0)$. Consider subcase 3.2. Then

$$f(ab^{-1}) \cdot f(cd^{-1}) = (a_1, a_2b_2^{-1}, b_1) \cdot (b_1, c_2d_2^{-1}, d_1) = \\ = (a_1, a_2b_2^{-1} \cdot c_2d_2^{-1}, d_1) = (a_1, 0, d_1) = 0 = f(0).$$

Hence the map f is an isomorphic embedding of the semigroup J_A^0 into $B_{Q_A}^0(G(E_A))$. □

3. A global structure of graph inverse semigroups. By $\mathcal{A}$ we denote the set of all strongly connected components of a graph E. The set $\mathcal{A}$ admits a natural partial order $\leq$: for each $X, Y \in \mathcal{A}$, $X \leq Y$ iff there exists a path $u \in \text{Path}(E)$ such that $s(u) \in Y$ and $r(u) \in X$.

Theorem 4. *For any graph E the following statements hold:*

(1) $G(E) = \bigcup_{X \in \mathcal{A}} J_X^0$;

(2) J_X^0 *is isomorphic to a subsemigroup of* $B_{Q_X}^0(G(E_X))$, *for each* $X \in \mathcal{A}$.

(3) $J_X^0 \cap J_Y^0 = \{0\}$ *for each distinct elements* $X, Y \in \mathcal{A}$;

(4) *if* $X \leq Y$ *then* $J_X^0 \cdot J_Y^0 \cup J_Y^0 \cdot J_X^0 \subseteq J_X^0$;

(5) *if* $X \not\leq Y$ *and* $Y \not\leq X$ *then* $J_X^0 \cdot J_Y^0 \cup J_Y^0 \cdot J_X^0 \subseteq \{0\}$.

Proof. Statements 1 and 3 follows from the fact that $\mathcal{J}$ is an equivalence relation. Statement 2 follows from Theorem 3.

Consider statement 4. Assume that $X, Y \in \mathcal{A}$ and $X \leq Y$. Fix any elements $ab^{-1} \in J_X^0$ and $cd^{-1} \in J_Y^0$. Observe that the case $ab^{-1} \cdot cd^{-1} = 0$ is trivial, because $0 \in J_X^0$. Suppose that

$ab^{-1}\cdot cd^{-1}\neq 0$. In this case there exists a path w such that either $c=bw$ or $b=cw$. If $b=cw$ then $ab^{-1}\cdot cd^{-1}=a(dw)^{-1}$ and $r(dw)=r(w)=r(b)=r(a)\in X$. Hence $a(dw)^{-1}\in J_X$. If $c=bw$ then $s(w)=r(b)\in X$ and $r(w)=r(c)\in Y$ which implies that $Y\leq X$. Since the order $\leq$ is antisymmetric we obtain that $X=Y$. Hence Corollary 3 provides that $ab^{-1}\cdot cd^{-1}\in J_X$.

Consider statement 5. Assume that $X\nleq Y$ and $Y\nleq X$. Fix any elements $ab^{-1}\in J^0_X$ and $cd^{-1}\in J^0_Y$. We claim that neither b is a prefix of c nor c is a prefix of b. Indeed, if b is a prefix of c, i.e., $c=bw$ for some path w. Then $s(w)=r(b)\in X$ and $r(w)=r(c)\in Y$ witnessing that $Y\leq X$ which contradicts to the assumption. If c is a prefix of b, i.e., $b=cw$ for some path w. Then $s(w)=r(c)\in Y$ and $r(w)=r(b)\in X$ witnessing that $X\leq Y$ which contradicts to the assumption. Hence $ab^{-1}\cdot cd^{-1}=0$. □

The proof of the following lemma follows from the definition of Green's relations $\mathcal{D}$ and $\mathcal{J}$.

Lemma 3. *For a graph inverse semigroup $G(E)$ the following conditions are equivalent:*

1. *relations $\mathcal{J}$ and $\mathcal{D}$ coincide on $G(E)$;*
2. *graph E is acyclic.*

Now we apply our results to graph inverse semigroups over acyclic graphs. Observe that each strongly connected component of an acyclic graph E coincides with some vertex $e\in E^0$. Hence each acyclic graph E admits a natural partial order $\leq$ on the set E^0. For each $e,f\in E^0$, $e\leq f$ iff there exists a path u such that $s(u)=f$ and $r(u)=e$. The following theorem describes the structure of graph inverse semigroups over acyclic graphs.

Theorem 5. *Let E be an acyclic graph. Then the following statements hold:*

1. $G(E)=\bigcup_{e\in E^0}D^0_e$;
2. *D^0_e is isomorphic to the semigroup of $I_e\times I_e$-matrix units $\mathcal{B}^0_{I_e}$, for each vertex $e\in E^0$;*
3. *$D^0_e\cap D^0_f=\{0\}$, for each distinct vertices $e,f\in E^0$;*
4. *If $e\leq f$ then $D^0_e\cdot D^0_f\cup D^0_f\cdot D^0_e\subseteq D^0_e$;*
5. *If $e\nleq f$ and $f\nleq e$ then $D^0_e\cdot D^0_f\cup D^0_f\cdot D^0_e=\{0\}$.*

Proof. Statements 1 and 3 are obvious. Statement 2 follows from Corollary 2. Statement 4 (resp., 5) follows from Lemma 3 and statement 4 (resp., 5) of Theorem 4. □

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