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COMPOSITIONS OF DIRICHLET SERIES SIMILAR TO THE HADAMARD COMPOSITIONS, AND CONVERGENCE CLASSES

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Let (λ_n) be a positive sequence increasing to $+\infty$, $m \geq 2$ and Dirichlet series $F_j(s) = \sum_{n=0}^{\infty} a_{n,j} \exp\{s\lambda_n\}$ $(j = 1, 2, \ldots, m)$ have the abscissa $A \in (-\infty, +\infty]$ of absolute convergence. We say that Dirichlet series $F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}$ is similar to Hadamard compositions of of Dirichlet series F_j if $a_n = w(a_{n,1}, a_{n,2})$ for all n, where $w\colon \mathbb{C}^2 \to \mathbb{C}$ is some function. Clearly, if $w(a_{n,1}, a_{n,2}) = a_{n,1}a_{n,2}$ then F is the Hadamard composition of the functions F_1 and F_2.

In the case $|a_n| \asymp \prod_{j=1}^{m} |a_{n,j}|^{\omega_j}$ as $n \to +\infty$, where $\omega_j > 0$ and $\sum_{j=1}^{m} \omega_j = 1$, it is investigated the belonging of F to some convergence class with respect of the belonging to this class of functions F_j.

1. Introduction.

Let

$$f_1(z) = \sum_{n=0}^{\infty} a_{n,1} z^n, \quad f_2(z) = \sum_{n=0}^{\infty} a_{n,2} z^n \tag{1}$$

be entire transcendental functions. We say that the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2}$$

is *similar to the Hadamard composition* of the functions f_1 and f_2 if $a_n = w(a_{n,1}, a_{n,2})$ for all n, where $w\colon \mathbb{C}^2 \to \mathbb{C}$ is some function. Clearly, if $w(u, v) = u \cdot v$ then f is the Hadamard composition of the functions f_1 and f_2.

E. G. Calys [1] investigated the functions similar to the Hadamard compositions of the special kind and proved the following theorem.

Theorem A ([1]). *Let entire functions f_1, f_2 of form* (1) *have the same order $\varrho[f_1] = \varrho[f_2] = \varrho \in (0, +\infty)$ and types $\sigma[f_1] = \sigma_1$, $\sigma[f_2] = \sigma_2$, respectively. Suppose that $a_{n,1} \neq 0$ and $|a_{n,2}| \geq |a_{n,1}|/l(1/|a_{n,1}|)$ for all $n \geq n_0$, where l is a slowly varying function ([9]). If $|a_n| = (1 + o(1))\sqrt{|a_{n,1}||a_{n,2}|}$, $n \to +\infty$, then the function f of form* (2) *has the order $\varrho[f] = \varrho$ and type $\sigma[f] \leq \sqrt{\sigma_1\sigma_2}$.*

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In paper [2], the results of E.G. Calys are generalized on the case of entire Dirichlet series of finite generalized orders. Moreover, there are considered $m \geq 2$ entire Dirichlet series instead of two entire functions.

Let $\Lambda = (\lambda_n)$ be an increasing to $+\infty$ sequence of nonnegative numbers, $S(\Lambda, A)$ be the class of Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{3}$$

with a given sequence (λ_n) of exponents and an abscissa $\sigma_a = A \in (-\infty, +\infty)$ of absolutely convergence. We write $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ for $\sigma \in (-\infty, A)$.

As in [3], by L we denote the class of positive continuous functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$ and $0 < \alpha(x) \uparrow +\infty$ as $x_0 \leq x \uparrow +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

If $\alpha \in L$, $\beta \in L$ and $F \in S(\Lambda, +\infty)$, that is series (3) is entire, then the value

$$\varrho_{\alpha,\beta}[F] = \varlimsup_{\sigma \to +\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)}$$

is called generalized order of F. If $\varrho_{\alpha,\beta}[F] \in (0, +\infty)$ then we define the generalized type

$$T_{\alpha,\beta}[F] = \varlimsup_{\sigma \to +\infty} \frac{\ln M(\sigma, F)}{\alpha^{-1}(\varrho_{\alpha,\beta}[F]\beta(\sigma))}.$$

The following theorem is proved in [2].

Theorem B ([2]). *Let the functions $\alpha \in L_{si}$ and $\beta \in L^0$ be continuously differentiable,*

$$\frac{d \ln \alpha^{-1}(\varrho\beta(x))}{d \ln x} \to \varrho, \quad \alpha(x) = (1 + o(1)) \ln x \quad x \to +\infty$$

and Dirichlet series $F_j \in S(\Lambda, +\infty)$ of form

$$F_j(s) = \sum_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}, \quad 2 \leq j \leq m, \tag{4}$$

have the same generalized order $\varrho_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$ and types $T_{\alpha,\beta}[F_j] \in (0, +\infty)$. If $\ln n = o(\lambda_n)$ $(n \to +\infty)$, $a_{n,1} \neq 0$ for all $n \geq n_0$, and for some $\omega_j > 0$ and $\sum_{j=1}^{m} \omega_j = 1$

$$\alpha^{-1}\left(\varrho\beta\left(\frac{1}{\varrho} + \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)\right) = (1 + o(1)) \prod_{j=1}^{m} \alpha^{-1}\left(\varrho\beta\left(\frac{1}{\varrho} + \frac{1}{\lambda_n} \ln \frac{1}{|a_{n,j}|}\right)\right)^{\omega_j} \quad (n \to +\infty),$$

and

$$\beta\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_{n,j}|}\right) \leq (1 + o(1))\beta\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_{n,1}|}\right) \quad (n \to +\infty)$$

for all $2 \leq j \leq m$, then Dirichlet series F of form (3) has the generalized order $\varrho_{\alpha,\beta}[F] = \varrho$ and the type $T_{\alpha,\beta}[F] \leq \prod_{j=1}^{m} T_{\alpha,\beta}[F_j]^{\omega_j}$.

Here we study the belonging of Dirichlet series to classes of convergence. Thus, we distinguish cases $A=+\infty$ and $-\infty<A<+\infty$.

2. Case $A=+\infty$. If $T_{\alpha,\beta}[F]=0$ then to characterize of the growth of entire Dirichlet series (3) we define the generalized convergence class $\mathcal{C}_\rho$ of functions $F\in S(\Lambda,+\infty)$ by the condition

$$\int\limits_{\sigma_0}^{\infty}\frac{\ln M(\sigma,F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))}d\sigma<+\infty,\quad \varrho=\varrho_{\alpha,\beta}[F]. \tag{5}$$

Theorem 1. *Let $\alpha\in L$ and $\beta\in L$ be positive continuously differentiable functions such that for each $\varrho\in(0,+\infty)$*

$$\frac{d\ln\alpha^{-1}(\varrho\beta(\sigma))}{d\sigma}=O(1),\quad \sigma\to\infty. \tag{6}$$

Suppose that $\ln n=O(\lambda_n)$ as $n\to+\infty$ and for some $\omega_j>0$ such that $\sum\limits_{j=1}^{m}\omega_j=1$

$$|a_n|\asymp\prod_{j=1}^{m}|a_{n,j}|^{\omega_j},\quad n\to+\infty. \tag{7}$$

If for all j, $1\le j\le m$, functions of form (4) $F_j\in\mathcal{C}_\rho$, i.e. they belong to the same generalized convergence class, then the function of form (3) $F\in\mathcal{C}_\rho$, i.e. it also belongs to this class.

If, in addition, $a_{n,1}\neq 0$ for all $n\ge 0$ and $|a_{n,j}|\asymp|a_{n,1}|$ as $n\to+\infty$ for all $j=2,\dots,m$, then the condition $F\in\mathcal{C}_\rho$ implies that $F_j\in\mathcal{C}_\rho$ for every j, $1\le j\le m$.

Proof. It follows from (6) that if C is a positive number then for some $\xi\in(\sigma,\sigma+C)$

$$\ln\alpha^{-1}(\varrho\beta(\sigma+C))-\ln\alpha^{-1}(\varrho\beta(\sigma))=C\cdot\frac{d\ln\alpha^{-1}(\varrho\beta(\sigma))}{d\sigma}\Big|_{\sigma=\xi}\le C_1<+\infty,$$

i. e. the function $\beta_1(\sigma)=\sigma\alpha^{-1}(\varrho\beta(\sigma))$ satisfies the condition

$$\beta_1(\sigma+O(1))=O(\beta_1(\sigma)),\quad \sigma\to+\infty.$$

On the other hand, since $\ln n=O(\lambda_n)$ as $n\to+\infty$ and $F\in S(\Lambda,+\infty)$, one has [6, c. 184] $\ln M(\sigma,F)\le\ln\mu(\sigma+O(1),F)$ as $\sigma\to+\infty$, where $\mu(\sigma,F)=\max\{|a_n|\exp\{\sigma\lambda_n\}:\ n\ge 0\}$ is the maximal term of series (3). Therefore, in view of Cauchy's inequality $\mu(\sigma,F)\le M(\sigma,F)$ condition (5) can be replaced by the condition

$$\int\limits_{a}^{\infty}\frac{\ln\mu(\sigma,F)}{\beta_1(\sigma)}d\sigma<+\infty. \tag{8}$$

Thus, if all functions (4) belong to generalized convergence class defined by condition (5) then

$$\int\limits_{\sigma_0}^{\infty}\frac{\ln\mu(\sigma,F_j)}{\beta_1(\sigma)}d\sigma<+\infty. \tag{9}$$

From (8) it follows that

$$\sum_{j=1}^{m} \omega_j \ln |a_{n,j}| + h \leq \ln |a_n| \leq \sum_{j=1}^{m} \omega_j \ln |a_{n,j}| + H \tag{10}$$

for some numbers h and H and all n. Since $\sum_{j=1}^{m} \omega_j = 1$, we have

$$\ln |a_n| + \sigma\lambda_n \leq \sum_{j=1}^{m} \omega_j \ln |a_{n,j}| + \sum_{j=1}^{m} \omega_j \sigma\lambda_n + H = \sum_{j=1}^{m} \omega_j (\ln |a_{n,j}| + \sigma\lambda_n) + H. \tag{11}$$

Hence, it follows that

$$\ln \mu(\sigma, F) \leq \sum_{j=1}^{m} \omega_j \ln \mu(\sigma, F_j) + H. \tag{12}$$

Since $H = o(\ln \mu(\sigma, F)$ as $\sigma \to +\infty$, condition (9) yields (8). The first part of Theorem 1 is proved.

If $a_{n,1} \neq 0$ for all $n \geq 0$ and $|a_{n,j}| \asymp |a_{n,1}|$ as $n \to +\infty$ for all $j = 2, \ldots, m$, that is $\ln |a_{n,j}| + h_1 \leq \ln |a_{n,1}| \leq \ln |a_{n,j}| + H_1$ for some numbers h_1 and H_1 and all $n \geq 0$ and $j = 2, \ldots, m$. Therefore, from (10) we obtain

$$\ln |a_n| \geq \omega_1 \ln |a_{n,1}| + \sum_{j=2}^{m} \omega_j (\ln |a_{n,1}| - H_1) + h = \ln |a_{n,1}| - H_1 \sum_{j=2}^{m} \omega_j + h,$$

whence

$$\ln |a_{n,1}| + \sigma\lambda_n \leq \ln |a_n| + \sigma\lambda_n + H_1 \sum_{j=2}^{m} \omega_j + h.$$

On the other hand, for $2 \leq j \leq m$ we have $\ln |a_{n,j}| + \sigma\lambda_n \leq \ln |a_{n,1}| + \sigma\lambda_n + h_1$. Hence, it follows that $\ln \mu(\sigma, F_j) \leq \ln \mu(\sigma, F) + \text{const}$ for all $j = 1, \ldots, m$. □

For $\alpha \in L$, $\beta \in L$ and entire Dirichlet series in [7] a modified generalized convergence $\alpha\beta$-class is defined by condition

$$\int_{\sigma_0}^{\infty} \frac{1}{\beta(\sigma)} \alpha \left(\frac{\ln M(\sigma, F)}{\sigma} \right) d\sigma < +\infty. \tag{13}$$

Denote this convergence class by $\mathcal{C}_{\alpha\beta}$.

For such convergence class the following theorem is true.

Theorem 2. *Let $\alpha \in L^0$, $\beta(x) = x\gamma(x)$, $\gamma \in L^0$,*

$$\ln n = O\left(\lambda_n \gamma^{-1}(\alpha(\lambda_n))\right), \quad n \to +\infty, \tag{14}$$

and (7) holds. If all functions of form (4) $F_j \in \mathcal{C}_{\alpha\beta}$, then also $F \in \mathcal{C}_{\alpha\beta}$. If, in addition, $a_{n,1} \neq 0$ for all $n \geq 0$ and $\ln |a_{n,j}| \asymp \ln |a_{n,1}|$ as $n \to +\infty$ for all $j = 2, \ldots, m$, then the condition $F \in \mathcal{C}_{\alpha\beta}$ implies $F_j \in \mathcal{C}_{\alpha\beta}$ for all j, $1 \leq j \leq m$.

Proof. It is proved in [7], if $\alpha\in L^0$, $\beta(x)=x\gamma(x)$, $\gamma\in L^0$ and (14) holds then condition (13) is equivalent to the condition

$$\int\limits_{\sigma_0}^{\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\ln\mu(\sigma,F)}{\sigma}\right)d\sigma<+\infty. \tag{15}$$

Thus, if for all functions of form (4) $F_j\in\mathcal{C}_{\alpha\beta}$ then

$$\int\limits_{\sigma_0}^{\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\ln\mu(\sigma,F_j)}{\sigma}\right)d\sigma<+\infty,\quad 1\le j\le m. \tag{16}$$

Since $\ln\mu(\sigma,F_j)\to+\infty$ as $\sigma\to+\infty$ and $\alpha\in L^0$, we obtain from (12)

$$\alpha\left(\frac{\ln\mu(\sigma,F)}{\sigma}\right)\le\alpha\Big(\frac{1}{\sigma}\Big(\sum_{j=1}^{m}\omega_j\ln\,\mu(\sigma,F_j)+H\Big)\Big)\le\alpha\Big(\frac{1}{\sigma}\Big(\max_{1\le j\le m}\ln\,\mu(\sigma,F_j)+H\Big)\Big)\le$$
$$\le(1+o(1))\max\Big\{\alpha\left(\frac{\ln\mu(\sigma,F_j)}{\sigma}\right):1\le j\le m\Big\}\le(1+o(1))\sum_{j=1}^{m}\alpha\left(\frac{\ln\mu(\sigma,F_j)}{\sigma}\right)$$

as $\sigma\to+\infty$. Therefore, in view of (16) we get (15), and the first part of Theorem 2 is proved.

If $a_{n,1}\ne0$ for all $n\ge0$ and $\ln|a_{n,j}|\asymp\ln|a_{n,1}|$ for all $j=2,\dots,m$ as $n\to+\infty$ then $h_1\ln|a_{n,j}|\le\ln|a_{n,1}|\le H_1\ln|a_{n,j}|$ for some numbers h_1 and H_1 and all $n\ge0$ and $j=2,\dots,m$. So from (10) we obtain

$$\ln|a_n|\ge\omega_1\ln|a_{n,1}|+\sum_{j=2}^{m}\frac{\omega_j}{H_1}\ln|a_{n,1}|+h=\left(\omega_1+\sum_{j=2}^{m}\frac{\omega_j}{H_1}\right)\ln|a_{n,1}|.$$

Hence, in view of the inequality $h_1\ln|a_{n,j}|\le\ln|a_{n,1}|$ we obtain the inequality $\ln|a_{n,j}|\le q\ln|a_n|$ for some number $q>0$ and all n and j. Therefore,

$$\ln\mu(\sigma,F_j)=\max\{\ln|a_{n,j}|+\sigma\lambda_n:\,n\ge0\}\le\max\{q\ln|a_n|+\sigma\lambda_n:\,n\ge0\}=$$
$$=q\max\{\ln|a_n|+(\sigma/q)\lambda_n:\,n\ge0\}=q\ln\mu(\sigma/q,F),$$
$$\int\limits_{\sigma_0}^{\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\ln\mu(\sigma,F_j)}{\sigma}\right)d\sigma\le q\int\limits_{\sigma_0}^{\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\ln\mu(\sigma/q,F)}{\sigma}\right)d\sigma=$$
$$=q^2\int\limits_{\sigma_0/q}^{\infty}\frac{1}{\beta(q\sigma)}\alpha\left(\frac{\ln\mu(\sigma,F)}{q\sigma}\right)d\sigma. \tag{17}$$

It is proved in [8] that if $\alpha\in L^0$ then α is *RO*-increasing [9], i. e. for every $h\in[1,a]$, $1<a<+\infty$, and all $x\ge x_0$ the inequality $1\le\frac{\alpha(hx)}{\alpha(x)}\le M(a)<+\infty$ is true. Therefore, from (15) and (17) we get (16), and the proof of Theorem 2 is completed. □

3. Case $-\infty<A<\infty$. In mathematical literature there are usually studied Dirichlet series with a zero abscissa of absolute convergence. Choosing $A=0$, for such series we get a corresponding result. On the other hand, if in Dirichlet series (3) with the absolute convergence abscissa $\sigma_a = A \in (-\infty, +\infty)$ the variable s is replaced by $s - A$ then we obtain Dirichlet series with zero abscissa of absolute convergence. Thus, without loss of generality we consider the class $S_0(\Lambda) := S(\Lambda, 0)$ instead of class $S(\Lambda, A)$ with $-\infty < A < +\infty$.

The analog of Theorem B for the class $S(\Lambda, A)$ with $-\infty < A < +\infty$ was obtained in [4]. For such Dirichlet series the quantity

$$\varrho^0_{\alpha,\beta}[F] = \overline{\lim_{\sigma\uparrow 0}} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)},$$

is called a generalized order. If $\varrho^0_{\alpha,\beta}[F] \in (0, +\infty)$ then we define the generalized type

$$T^0_{\alpha,\beta}[F] = \overline{\lim_{\sigma\uparrow 0}} \frac{\ln M(\sigma, F)}{\alpha^{-1}(\varrho\beta(1/|\sigma|))}, \quad \varrho = \varrho^0_{\alpha,\beta}[F].$$

The following theorem is an analog of Theorem B.

Theorem C ([4]). *Let $\beta \in L_{si}$, $\alpha(e^x) \in L^0$, $\alpha^{-1}(c\beta(x)) \in L_{si}$ such that*

$$\frac{x}{\alpha^{-1}(c\beta(x))} \uparrow +\infty, \quad \alpha^{-1}\left(c\beta\left(\frac{x}{\alpha^{-1}(c\beta(x))}\right)\right) = (1 + o(1))\alpha^{-1}(c\beta(x))$$

as $x_0 \le x \to +\infty$ for each $c \in (0, +\infty)$, and Dirichlet series $F_j \in S_0(\Lambda)$ of form (4) have the same generalized order $\varrho^0_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$ and the types $T^0_{\alpha,\beta}[F_j] \in (0, +\infty)$. If $\ln n = o(\alpha^{-1}(c\beta(\lambda_n)))$ as $n \to +\infty$ for each $c \in (0, +\infty)$, $a_{n,1} \ne 0$ for all $n \ge n_0$, for some $\omega_j > 0$ $(1 \le j \le m)$ such that $\sum_{j=1}^m \omega_j = 1$

$$\ln (|a_n|) = (1 + o(1)) \prod_{j=1}^m (\ln (|a_{n,j}|))^{\omega_j}, \quad n \to +\infty,$$

and for all $2 \le j \le m$

$$\ln \ln (|a_{n,j}|) \ge (1 + o(1)) \ln \ln (|a_{n,1}|), \quad n \to +\infty.$$

then Dirichlet series F of form (3) has the generalized order $\varrho^0_{\alpha,\beta}[F] = \varrho$ and the type

$$T^0_{\alpha,\beta}[F] \le \prod_{j=1}^m T^0_{\alpha,\beta}[F_j]^{\omega_j}.$$

If $T^0_{\alpha,\beta}[F] = 0$ then to characterize of the growth of Dirichlet series (3) from $S_0(\Lambda)$ we define the generalized convergence class $\mathcal{C}^0_\rho$ by the condition

$$\int_{\sigma_0}^{0} \frac{\ln M(\sigma, F)}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))} d\sigma < +\infty, \quad \varrho = \varrho^0_{\alpha,\beta}[F]. \tag{18}$$

Theorem 3. *Let the function $\alpha \in L^0$, $\beta \in L^0$ and $\beta_1(x) = \alpha^{-1}(\varrho\beta(x)) \in L^0$ for each $\varrho \in (0, +\infty)$. If all functions of form (4) $F_j \in \mathcal{C}^0_\rho$,*

$$\overline{\lim_{n\to+\infty}} \frac{\ln n}{\ln |a_{n,j}|} \le h_0 < +\infty \tag{19}$$

for all $j = 1, 2, \ldots m$, and condition (7) holds, then the function of form (3) $F \in \mathcal{C}^0_\rho$. If, in addition, $a_{n,1} \ne 0$ for all $n \ge 0$ and $|a_{n,j}| \asymp |a_{n,1}|$ as $n \to +\infty$ for all $j = 2, \ldots, m$, then the condition $F \in \mathcal{C}^0_\rho$ implies that all functions of form (4) $F_j \in \mathcal{C}^0_\rho$.

Proof. In [10], it is proved that by condition (19) for each $\varepsilon \in (0,\,1)$ and all $\sigma \in [\sigma_0(\varepsilon),\,0)$

$$\ln\, M(\sigma, F_j) \le (1+h_0+\varepsilon)\ln\,\mu\left(\sigma + \frac{h+\varepsilon}{h_0+1}|\sigma|, F_j\right) + \ln\, K(\varepsilon),$$

where $K(\varepsilon)$ is a positive constant depending only on ε. Therefore, in view of Cauchy's inequality

$$\int\limits_{\sigma_0}^{0} \frac{\ln\,\mu(\sigma,F_j)}{|\sigma|^2\beta_1(1/|\sigma|)}d\sigma \le \int\limits_{\sigma_0}^{0} \frac{\ln\,M(\sigma,F_j)}{|\sigma|^2\beta_1(1/|\sigma|)}d\sigma \le$$

$$\le (1+h_0+\varepsilon)\int\limits_{\sigma_0}^{0} \frac{\ln\,\mu\left(\sigma+\frac{h_0+\varepsilon}{h_0+1}|\sigma|, F_j\right)}{|\sigma|^2\beta_1(1/|\sigma|)}d\sigma + \text{const} =$$

$$= \frac{(1+h_0+\varepsilon)(1-\varepsilon)}{1+h_0}\int\limits_{\sigma_1}^{0} \frac{\ln\,\mu(\sigma,F_j)}{|\sigma|^2\beta_1\left(\frac{1-\varepsilon}{(1+h_0)|\sigma|}\right)}d\sigma + \text{const},$$

Since the function $\beta_1 \in L^0$, we repeat considerations from the proof of Theorem 2 and obtain that the integrals

$$\int\limits_{\sigma_0}^{0} \frac{\ln\,M(\sigma,F_j)}{|\sigma|^2\beta_1(1/|\sigma|)}d\sigma, \quad \int\limits_{\sigma_0}^{0} \frac{\ln\,\mu(\sigma,F_j)}{|\sigma|^2\beta_1(1/|\sigma|)}d\sigma$$

converge or diverge simultaneously.

Further, from (10) and (19) we have

$$\varliminf_{n\to+\infty} \frac{\ln\,|a_n|}{\ln\,n} \ge \varliminf_{n\to+\infty} \sum_{j=1}^{m} \frac{\omega_j \ln\,|a_{n,j}| + h}{\ln\,n} \ge \sum_{j=1}^{m} \omega_j \varliminf_{n\to+\infty} \frac{\ln\,|a_{n,j}|}{\ln\,n} \ge \frac{1}{h_0}\sum_{j=1}^{m}\omega_j = \frac{1}{h_0},$$

i. e. (19) also holds with $|a_n|$ instead of $|a_{n,j}|$. Thus, in (18) $M(\sigma,F)$ can be replaced by $\mu(\sigma,F)$.

From (19) it follows that $|a_n| \to +\infty$ as $n \to +\infty$, whence $\mu(\sigma,F) \to +\infty$ as $\sigma \uparrow 0$. Therefore, since (19) implies (12), from (12) it follows that the condition

$$\int\limits_{\sigma_0}^{0} \frac{\ln\,\mu(\sigma,F_j)}{|\sigma|^2\beta_1(1/|\sigma|)}d\sigma, \quad 1 \le j \le m, \tag{20}$$

implies the condition

$$\int\limits_{\sigma_0}^{0} \frac{\ln\,\mu(\sigma,F)}{|\sigma|^2\beta_1(1/|\sigma|)}d\sigma. \tag{21}$$

The first part of Theorem 3 is proved.

If $a_{n,1} \ne 0$ for all $n \ge 0$ and $|a_{n,j}| \asymp |a_{n,1}|$ as $n \to +\infty$ for all $j = 2,\dots,m$ then as in the proof of Theorem 1 we obtain the inequality $\ln\,\mu(\sigma,F_j) \le \ln\,\mu(\sigma,F) + \text{const}$ for all $j = 1,\dots,m$. Thus, (21) implies (20). □

For Dirichlet series (3) with zero abscissa of absolute convergence in [11] the generalized convergence $\alpha\beta$-class $\mathcal{C}^0_{\alpha\beta}$ is defined by the condition

$$\int\limits_{\sigma_0}^{0} \frac{\alpha(\ln M(\sigma,F))}{|\sigma|^2\beta(1/|\sigma|)} d\sigma < +\infty. \tag{22}$$

where $\alpha \in L$ and $\beta \in L$.

Theorem 4. *Let $\alpha \in L^0$, $\beta \in L^0$,*

$$\int\limits_{x_0}^{\infty} \frac{\alpha(\gamma^{-1}(1/x)))}{\beta(x)} dx < +\infty, \tag{23}$$

and a sequence (λ_n) satisfy the condition

$$\varlimsup_{n\to+\infty} \frac{\ln n}{\lambda_n \gamma(\lambda_n)} \le h_0 < +\infty, \tag{24}$$

where γ is some positive continuous decreasing to 0 function on $[0, +\infty)$ such that $t\gamma(t) \uparrow +\infty$ as $t \to +\infty$. If all functions of form (4) $F_j \in \mathcal{C}^0_{\alpha\beta}$ and condition (7) holds then the function of form (3) $F \in \mathcal{C}^0_{\alpha\beta}$.

If, in addition, $a_{n,1} \neq 0$ for all $n \ge 0$ and $\ln |a_{n,j}| \asymp \ln |a_{n,1}|$ as $n \to +\infty$ for all $j = 2, \ldots, m$, then the condition $F \in \mathcal{C}^0_{\alpha\beta}$ for a function of form (3) implies $F_j \in \mathcal{C}^0_{\alpha\beta}$ for all functions F_j of form (4).

Proof. At first, we remark that in view of Cauchy's inequality condition (22) implies

$$\int\limits_{\sigma_0}^{0} \frac{\alpha(\ln \mu(\sigma,F))}{|\sigma|^2\beta(1/|\sigma|)} d\sigma < +\infty. \tag{25}$$

It is proved in [5] that if condition (24) holds then for every $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that for all $\sigma < 0$

$$M(\sigma,F) \le \mu\left(\frac{\sigma}{1+\varepsilon}\right)\left(\exp\left\{\frac{\varepsilon|\sigma|}{1+\varepsilon}\gamma^{-1}\left(\frac{\varepsilon|\sigma|}{(1+\varepsilon)^2(h_0+\varepsilon^2}\right)\right\} + K(\varepsilon)\right).$$

If we choose $\varepsilon = 1$ then we obtain

$$\ln M(\sigma,F) \le \ln \mu\left(\frac{\sigma}{2}\right) + \gamma^{-1}\left(\frac{|\sigma|}{4(h_0+1)}\right) + \text{const},$$

whence in view of condition $\alpha \in L^0$ we deduce

$$\alpha(\ln M(\sigma,F)) \le \alpha\left(3\max\left\{\ln \mu\left(\frac{\sigma}{2}\right), \gamma^{-1}\left(\frac{|\sigma|}{4(h_0+1)}\right)\right\}\right) \le$$
$$\le M(3)\alpha\left(\max\left\{\ln \mu\left(\frac{\sigma}{2}\right), \gamma^{-1}\left(\frac{|\sigma|}{4(h_0+1)}\right)\right\}\right) =$$
$$= M(3)\max\left\{\alpha\left(\ln \mu\left(\frac{\sigma}{2}\right)\right), \alpha\left(\gamma^{-1}\left(\frac{|\sigma|}{4(h_0+1)}\right)\right)\right\} \le$$

$$\leq M(3)\left(\alpha\left(\ln\mu\left(\frac{\sigma}{2}\right)\right)+\alpha\left(\gamma^{-1}\left(\frac{|\sigma|}{4(h_0+1)}\right)\right)\right). \tag{26}$$

Since $\beta \in L^0$, from condition (23) it follows that

$$\int\limits_{\sigma_0}^{0} \frac{\alpha\left(\gamma^{-1}\left(\frac{|\sigma|}{4(h_0+1)}\right)\right)}{|\sigma|^2\beta(1/|\sigma|)} d\sigma < +\infty$$

and, therefore, (25) and (26) imply (22).

If all functions (4) belong to the generalized convergence $\alpha\beta$-class then (25) holds with F_j instead of F. Since (7) implies (12) and $\alpha \in L^0$, we have

$$\alpha(\ln\mu(\sigma,F)) \leq \alpha\left(\sum_{j=1}^{m}\omega_j \ln\,\mu(\sigma,F_j)+H\right) \leq \alpha(\max\{\ln\mu(\sigma,F_j): 1\leq j\leq m\}+H) \leq$$
$$\leq \alpha(2\max\{\max\{\ln\mu(\sigma,F_j): 1\leq j\leq m\}, H\}) \leq$$
$$\leq M(2)\max\{\max\{\alpha(\ln\mu(\sigma,F_j)): 1\leq j\leq m\}, \alpha(H)\} \leq \sum_{j=1}^{m}\alpha(\ln\mu(\sigma,F_j))+\alpha(H).$$

We remark that from (23) it follows that

$$\int_{x_0}^{\infty} \frac{dx}{\beta(x)} < +\infty.$$

Therefore, the first part of Theorem 4 is proved.

If $a_{n,1} \neq 0$ for all $n \geq 1$ and $\ln|a_{n,j}| \asymp \ln|a_{n,1}|$ as $n \to +\infty$ for all $j = 2,\dots,m$ then as in the proof of Theorem 2 we obtain the inequality $\ln\mu(\sigma,F_j) \leq q\ln\mu(\sigma/q,F)$ for some numbers $q > 0$ and all $j = 2,\dots,m$ and $\sigma < 0$. Hence

$$\int\limits_{\sigma_0}^{0}\frac{\alpha(\ln\mu(\sigma,F_j))}{|\sigma|^2\beta(1/|\sigma|)}d\sigma \leq \int\limits_{\sigma_0}^{0}\frac{\alpha(q\ln\mu(\sigma/q,F))}{|\sigma|^2\beta(1/|\sigma|)}d\sigma = \int\limits_{\sigma_0/q}^{0}\frac{\alpha(q\ln\mu(\sigma,F))}{q|\sigma|^2\beta(1/(q|\sigma|))}d\sigma.$$

Since $\alpha \in L^0$ and $\beta \in L^0$, the proof of Theorem 4 is completed. □

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