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ON ε-FRIEDRICHS INEQUALITIES AND ITS APPLICATION

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Let $n \in \mathbb{N}$ be a fixed number, Ω be a bounded domain in $\mathbb{R}^n$, $L^2(\Omega)$, $L^\infty(\Omega)$ be the Lebesgue spaces, $H^1(\Omega)$ and $H^1_0(\Omega)$ be the Sobolev spaces,

$$\Pi_\ell(\alpha) := \overset{n}{\underset{k=1}{\otimes}} (\alpha_k; \alpha_k + \ell), \ \ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n, \ \ \ell > 0.$$

There are proved the following assertions about ε-Friedrichs inequality (Theorem 1) in the space $H^1(\Omega)$ (Theorem 1) and $H^1_0(\Omega)$ (Theorem 2) : for every $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and $\omega_1, \dots, \omega_{N_\varepsilon} \in L^\infty(\Omega)$ such that the inequality

$$\int_\Omega |v(x)|^2\, dx \le \varepsilon \int_\Omega |\nabla v(x)|^2\, dx + \sum_{j=1}^{N_\varepsilon} \Big(\int_\Omega v(x)\omega_j(x)\, dx \Big)^2$$

holds for every $v \in H^1(\Omega)$ (Theorem 1) and $v \in H^1_0(\Omega)$ (Theorem 2), where Ω is a bounded domain in $\mathbb{R}^n$ for which there exist numbers $\ell > 0$, $m \in \mathbb{N}$, and $\alpha^1, \dots, \alpha^m \in \mathbb{R}^n$ satisfying the following conditions

1) $\overline{\Omega} = \overline{\Pi_\ell(\alpha^1)} \cup \ldots \cup \overline{\Pi_\ell(\alpha^m)}$;
2) for every $i, j \in \{1, \dots, m\}$ with $i \ne j$ we obtain: $\Pi_\ell(\alpha^i) \cap \Pi_\ell(\alpha^j) = \varnothing$.

Let $n \in \mathbb{N}$ be a fixed number, Ω be a bounded domain in $\mathbb{R}^n$, $L^2(\Omega)$, $L^\infty(\Omega)$ be the Lebesgue spaces (see [1, p. 618]), $H^1(\Omega)$ and $H^1_0(\Omega)$ be the Sobolev spaces (see [1, p. 245-246]), H be a Hilbert space, $C([0,T];H)$ be the space of the H-valued continuous functions defined on $[0,T]$ (see [2, p. 147]), $L^2(0,T;H)$ be the Lebesgue-Bochner space (see [2, p. 155]). We define

$$C_{\text{weak}}([0,T];H) := \{u\colon [0,T] \to H \ \ | \ \ \forall\, z \in H \quad (u(\cdot), z)_H \in C([0,T])\}.$$

Note that $C([0,T];H) \subsetneq C_{\text{weak}}([0,T];H)$ (see [2, p. 147] for more details).

We recall (see Lemma 1.26 [2, Chapter 2, §1]) *the standard Friedrichs inequality*:

$$\int\limits_\Omega |v(x)|^2\, dx \le M_\Omega \int\limits_\Omega |\nabla v(x)|^2\, dx, \quad v \in H^1_0(\Omega), \tag{1}$$

where $\nabla v := (v_{x_1}, \dots, v_{x_n})$. Notice that the constant $M_\Omega > 0$ depends on Ω and does not depend on v. The various generalizations of formula (1) are considered in [3], [4], [5], etc.

In the paper, we consider some modification of (1). We denote

$$\Pi_\ell(\alpha) := (\alpha_1; \alpha_1 + \ell) \times (\alpha_2; \alpha_2 + \ell) \times \ldots \times (\alpha_n; \alpha_n + \ell), \ \ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n, \ \ \ell > 0. \tag{2}$$

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Clearly, $\Pi_\ell(\alpha)$ is a cube in $\mathbb{R}^n$ with the volume ℓ^n. Let us define the set $\Pi(\mathbb{R}^n)$ as follows. For every $\Omega \in \Pi(\mathbb{R}^n)$ we have that Ω is a bounded domain in $\mathbb{R}^n$ and there exist numbers $\ell > 0$, $m \in \mathbb{N}$, and $\alpha^1, \ldots, \alpha^m \in \mathbb{R}^n$ such that the following conditions hold:

1) $\overline{\Omega} = \overline{\Pi_\ell(\alpha^1)} \cup \ldots \cup \overline{\Pi_\ell(\alpha^m)}$;

2) for every $i, j \in \{1, \ldots, m\}$ such that $i \neq j$ we obtain: $\Pi_\ell(\alpha^i) \cap \Pi_\ell(\alpha^j) = \varnothing$.

The main results of the paper are contained in the next propositions.

Theorem 1 (ε-Friedrichs inequality for the functions from $H^1(\Omega)$)**.** *If $\Omega \in \Pi(\mathbb{R}^n)$, then for every $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and $\omega_1, \ldots, \omega_{N_\varepsilon} \in L^\infty(\Omega)$ such that the inequality*

$$\int\limits_\Omega |v(x)|^2 \, dx \le \varepsilon \int\limits_\Omega |\nabla v(x)|^2 \, dx + \sum_{j=1}^{N_\varepsilon} \Big(\int\limits_\Omega v(x) \omega_j(x) \, dx \Big)^2 \tag{3}$$

holds for every $v \in H^1(\Omega)$.

Theorem 2 (ε-Friedrichs inequality for functions from $H_0^1(\Omega)$)**.** *If Ω is a bounded domain in $\mathbb{R}^n$, then for every $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and $\omega_1, \ldots, \omega_{N_\varepsilon} \in L^\infty(\Omega)$ such that inequality (3) holds for every $v \in H_0^1(\Omega)$.*

Corollary 1. *Suppose that Ω is taken from either Theorem 1 or Theorem 2, $T > 0$ is given number, $H := L^2(\Omega)$, $V := H^1(\Omega)$ if $\Omega \in \Pi(\mathbb{R}^n)$, $V := H_0^1(\Omega)$ if Ω satisfies the conditions of Theorem 2, and $u^m \underset{m\to\infty}{\longrightarrow} u$ in $C_{\mathrm{weak}}([0,T]; H)$. Then the following statements are fulfilled:*

(i) if $u^m \underset{m\to\infty}{\longrightarrow} u$ slowly in $L^2(0,T;V)$, then $u^m \underset{m\to\infty}{\longrightarrow} u$ strongly in $L^2(0,T;H)$;

(ii) if $u^m \underset{m\to\infty}{\longrightarrow} u$ $*$-slowly in $L^\infty(0,T;V)$, then $u^m \underset{m\to\infty}{\longrightarrow} u$ in $C([0,T];H)$.

Note that the case $n = 2$ of Theorem 2 is considered in [6, p. 553] and [7, p. 172]. We give the proof of Theorem 2 in case $n \neq 2$ for convenience. As we know the ε-Friedrichs inequalities for functions from $H^1(\Omega)$ is not studied yet.

Let us prove the main results. Clearly,

$$(a_1 + \ldots + a_n)^2 = (1 \cdot a_1 + \ldots + 1 \cdot a_n)^2 \le \Big(\sum_{i=1}^n 1^2 \Big) \Big(\sum_{i=1}^n a_i^2 \Big) \le n(a_1^2 + \ldots + a_n^2). \tag{4}$$

The following lemma is needed for the sequel.

Lemma 1 (Poincaré's inequality for cube in $\mathbb{R}^n$)**.** *If $n \in \mathbb{N}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $\ell > 0$, and $G = \Pi_\ell(\alpha)$, where $\Pi_\ell(\alpha)$ is chosen from (2), then for every $v \in H^1(G)$ we have*

$$\int\limits_G |v(\xi)|^2 \, d\xi \le \frac{n}{2} \ell^2 \int\limits_G |\nabla v(\xi)|^2 \, d\xi + \frac{1}{\ell^n} \Big(\int\limits_G v(\xi) \, d\xi \Big)^2. \tag{5}$$

Proof. Note that the case $n = 2$ is considered in [6, p. 552]. Suppose that $n \in \mathbb{N} \setminus \{2\}$. For simplicity, we prove Lemma 1 only if $n = 3$ and $\alpha_1 = \ldots = \alpha_n = 0$.

Let $n = 3$, $\ell > 0$, and $G = (0;\ell)^n$. Take arbitrary points $\xi_1 = (x_1, y_1, z_1) \in G$ and $\xi_2 = (x_2, y_2, z_2) \in G$. Then, for every $v \in H^1(G)$, we have

$$v(x_2, y_2, z_2) - v(x_1, y_1, z_1) = v(x_2, y_2, z_2) - v(x_1, y_2, z_2) + v(x_1, y_2, z_2) - v(x_1, y_1, z_2) +$$

$$+ v(x_1, y_1, z_2) - v(x_1, y_1, z_1) = \int_{x_1}^{x_2} v_x(x, y_2, z_2)\, dx + \int_{y_1}^{y_2} v_y(x_1, y, z_2)\, dy + \int_{z_1}^{z_2} v_z(x_1, y_1, z)\, dz.$$

Squaring this equality and using inequality (4) with $n = 3$, the Cauchy-Bunyakowski-Schwarz inequality, and the conditions

$$|x_2 - x_1| \le \ell, \quad |y_2 - y_1| \le \ell, \quad |z_2 - z_1| \le \ell, \tag{6}$$

we obtain

$$|v(x_2, y_2, z_2) - v(x_1, y_1, z_1)|^2 \le$$

$$\le n\Big\{\Big(\int_{x_1}^{x_2} v_x(x, y_2, z_2)\, dx\Big)^2 + \Big(\int_{y_1}^{y_2} v_y(x_1, y, z_2)\, dy\Big)^2 + \Big(\int_{z_1}^{z_2} v_z(x_1, y_1, z)\, dz\Big)^2\Big\} \le$$

$$\le n\ell\Big\{\int_0^\ell |v_x(x, y_2, z_2)|^2\, dx + \int_0^\ell |v_y(x_1, y, z_2)|^2\, dy + \int_0^\ell |v_z(x_1, y_1, z)|^2\, dz\Big\}.$$

Integrating this inequality in $(x_1, y_1, z_1), (x_2, y_2, z_2) \in G$, we deduce

$$J \le n\ell \int_G dx_1dy_1dz_1 \int_G \Big\{\int_0^\ell |v_x(x, y_2, z_2)|^2\, dx + \int_0^\ell |v_y(x_1, y, z_2)|^2\, dy +$$

$$+ \int_0^\ell |v_z(x_1, y_1, z)|^2\, dz\Big\}\, dx_2dy_2dz_2 = n\ell \cdot \ell^{n+1} \int_G \Big[|v_x(x, y, z)|^2 + |v_y(x, y, z)|^2 +$$

$$+ |v_z(x, y, z)|^2\Big]\, dxdydz = n\ell^{n+2} \int_G |\nabla v(x, y, z)|^2\, dxdydz, \tag{7}$$

where

$$J := \int_G dx_1dy_1dz_1 \int_G |v(x_2, y_2, z_2) - v(x_1, y_1, z_1)|^2\, dx_2dy_2dz_2. \tag{8}$$

Clearly, (8) implies

$$J = \int_G dx_1dy_1dz_1 \int_G \Big[|v(x_2, y_2, z_2)|^2 + |v(x_1, y_1, z_1)|^2 - 2v(x_2, y_2, z_2)v(x_1, y_1, z_1)\Big]\, dx_2dy_2dz_2 =$$

$$= 2\ell^n \int_G |v(x, y, z)|^2\, dxdydz - 2\Big(\int_G v(x_2, y_2, z_2)\, dx_2dy_2dz_2\Big) \cdot \Big(\int_G v(x_1, y_1, z_1)\, dx_1dy_1dz_1\Big) =$$

$$= 2\ell^n \int_G |v(x, y, z)|^2\, dxdydz - 2\Big(\int_G v(x, y, z)\, dxdydz\Big)^2.$$

Then from (7) we obtain

$$2\ell^n \int_G |v|^2\, dxdydz - 2\Big(\int_G v\, dxdydz\Big)^2 \le n\ell^{n+2} \int_G |\nabla v|^2\, dxdydz, \qquad (9).$$

Therefore, (5) holds and Lemma 1 is proved. $\square$

Proof of Theorem 1. Since $\Omega \in \Pi(\mathbb{R}^n)$, we have $L > 0$, $m \in \mathbb{N}$, and $\alpha^1, \dots, \alpha^m \in \mathbb{R}^n$ such that $\overline{\Omega} = \overline{\Pi_L(\alpha^1)} \cup \ldots \cup \overline{\Pi_L(\alpha^m)}$ and $\Pi_L(\alpha^i) \cap \Pi_L(\alpha^j) = \varnothing$ if $i \neq j$.

Take $k \in \mathbb{N}$ and put $N = k^n$. For every $j \in \{1, \dots, m\}$ we partition every cube $\Pi_L(\alpha^j)$ into cubes $Q_1^{k,j}, Q_2^{k,j}, \dots, Q_N^{k,j}$ such that every cube $Q_\mu^{k,j}$ has its side with length $\ell = \frac{L}{k}$.

Let us consider an arbitrary $v \in H^1(\Omega)$. For every $j \in \{1, \dots, m\}$ and $\mu \in \{1, \dots, N\}$, we use Poincaré's inequality (5) with $G = Q_\mu^{k,j}$. Summing these inequalities, we get

$$J_1 \leq J_2 + J_3, \tag{10}$$

where (for the sake of convenience we replace ξ by x)

$$J_1 := \sum_{j=1}^{m} \sum_{\mu=1}^{N} \int\limits_{Q_\mu^{k,j}} |v(x)|^2 \, dx, \qquad J_2 := \sum_{j=1}^{m} \sum_{\mu=1}^{N} \frac{n}{2} \, \ell^2 \int\limits_{Q_\mu^{k,j}} |\nabla v(x)|^2 \, dx,$$

$$J_3 := \sum_{j=1}^{m} \sum_{\mu=1}^{N} \frac{1}{\ell^n} \Big(\int\limits_{Q_\mu^{k,j}} v(x) \, dx \Big)^2.$$

Clearly, $J_1 = \int_\Omega |v|^2 \, dx$, $\quad J_2 = \frac{n}{2} \, \ell^2 \int_\Omega |\nabla v|^2 \, dx$. If we take

$$\chi_\mu^{k,j}(x) := \begin{cases} 1, & x \in Q_\mu^{k,j}, \\ 0, & x \in \mathbb{R}^n \setminus Q_\mu^{k,j}, \end{cases} \qquad j = \overline{1, m}, \quad \mu = \overline{1, N},$$

then

$$J_3 = \sum_{j=1}^{m} \sum_{\mu=1}^{N} \frac{1}{\ell^n} \Big(\int\limits_{\Omega} v(x) \chi_\mu^{k,j}(x) \, dx \Big)^2.$$

Thus, (10) yields that

$$\int\limits_\Omega |v|^2 \, dx \leq \frac{nL^2}{2} \cdot \frac{1}{k^2} \int\limits_\Omega |\nabla v|^2 \, dx + \frac{1}{L^n} \cdot k^n \sum_{j=1}^{m} \sum_{\mu=1}^{k^n} \Big(\int\limits_{\Omega} v(x) \chi_\mu^{k,j}(x) \, dx \Big)^2, \quad k \in \mathbb{N}. \tag{11}$$

Finally, for every $\varepsilon > 0$ we choose $k \in \mathbb{N}$ such that $\frac{nL^2}{2} \cdot \frac{1}{k^2} \leq \varepsilon$. Then, from (11) we obtain (3) and Theorem 1 is proved. □

Proof of Theorem 2. Note that the case $n = 2$ is considered in [6, p. 542, 543, 553]. Suppose that $n \in \mathbb{N} \setminus \{2\}$ and modify the proof of Theorem 1.

Let Q be a cube in $\mathbb{R}^n$ such that $\Omega \subset Q$ and the length of the side of Q equals $L > 0$. Take $k \in \mathbb{N}$ and put $N = k^n$. We partition the cube Q into the cubes $Q_1^k, Q_2^k, \dots, Q_N^k$ such that every cube Q_μ^k has its side with length $\ell = \frac{L}{k}$.

We take an arbitrary $v \in H_0^1(\Omega)$. Assume that the function v equals zero outside Ω. For every $\mu \in \{1, \dots, N\}$, we use Poincaré's inequality (5) with $G = Q_\mu^k$. Summing these inequalities from 1 to N, we get

$$I_1 \leq I_2 + I_3, \tag{12}$$

where (for the sake of convenience we replace ξ by x again)

$$I_1 := \sum_{\mu=1}^{N} \int\limits_{Q_\mu^k} |v(x)|^2 \, dx, \quad I_2 := \sum_{\mu=1}^{N} \frac{n}{2} \, \ell^2 \int\limits_{Q_\mu^k} |\nabla v(x)|^2 \, dx, \quad I_3 := \sum_{\mu=1}^{N} \frac{1}{\ell^n} \Big(\int\limits_{Q_\mu^k} v(x) \, dx \Big)^2.$$

Clearly,

$$I_1 = \int\limits_Q |v|^2\,dx = \int\limits_\Omega |v|^2\,dx, \qquad I_2 = \frac{n}{2}\,\ell^2 \int\limits_Q |\nabla v|^2\,dx = \frac{n}{2}\,\ell^2 \int\limits_\Omega |\nabla v|^2\,dx.$$

If we choose

$$\chi^k_\mu(x) = \begin{cases} 1, & x \in Q^k_\mu, \\ 0, & x \in \mathbb{R}^n \setminus Q^k_\mu, \end{cases} \qquad \mu = \overline{1, N},$$

then

$$I_3 = \sum_{\mu=1}^{N} \frac{1}{\ell^n} \Big(\int\limits_Q v(x)\chi^k_\mu(x)\,dx \Big)^2 = \sum_{\mu=1}^{N} \frac{1}{\ell^n} \Big(\int\limits_\Omega v(x)\chi^k_\mu(x)\,dx \Big)^2.$$

Thus, (12) yields that

$$\int\limits_\Omega |v|^2\,dx \le \frac{nL^2}{2} \cdot \frac{1}{k^2} \int\limits_\Omega |\nabla v|^2\,dx + \frac{1}{L^n} \cdot k^n \sum_{\mu=1}^{k^n} \Big(\int\limits_\Omega v(x)\chi^k_\mu(x)\,dx \Big)^2, \quad k \in \mathbb{N}. \tag{13}$$

Finally, for every $\varepsilon > 0$ we choose $k \in \mathbb{N}$ such that $\frac{nL^2}{2} \cdot \frac{1}{k^2} \le \varepsilon$. Then, from (13) we obtain (3). □

Proof of Corollary 1. Suppose that $\{u^m\}_{m\in\mathbb{N}}$ satisfies condition (i) of Corollary 1. For a.e. $t \in (0, T)$, we use (3) with $v(x) := u^m(x,t) - u^k(x,t)$ for $x \in \Omega$. Integrating in t, we have

$$\int\limits_{Q_{0,T}} |u^m - u^k|^2\,dxdt \le \varepsilon \int\limits_{Q_{0,T}} |\nabla u^m - \nabla u^k|^2\,dxdt + \int\limits_0^T \sum_{j=1}^{N_\varepsilon} \Big(\int\limits_\Omega \big[u^m(x,t) - u^k(x,t)\big]\omega_j(x)\,dx \Big)^2 dt,$$

where $Q_{0,T} = \Omega \times (0,T)$ and $m, k \in \mathbb{N}$. The first integral on the right-hand side of this inequality does not exceed a fixed constant for any m and k. Moreover, the last integral can be made arbitrary small for sufficiently large m and k because $(u^m(\cdot,t), \omega_j(\cdot))_{L^2(\Omega)}$ uniform converges in t as $m \to \infty$ (see [7, p. 173] for comparison). Therefore, the right-hand side of the inequality can be made arbitrary small for sufficiently large m and k. Hence, the sequence $\{u^m\}_{m\in\mathbb{N}}$ converges strongly to u in $L^2(0,T;H) = L^2(Q_{0,T})$.

If $\{u^m\}_{m\in\mathbb{N}}$ satisfies condition (ii), then the proof is similar. □

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