ON \( \varepsilon \)-FRIEDRICHS INEQUALITIES AND ITS APPLICATION


Let \( n \in \mathbb{N} \) be a fixed number, \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( L^2(\Omega) \), \( L^\infty(\Omega) \) be the Lebesgue spaces, \( H^1(\Omega) \) and \( H^1_0(\Omega) \) be the Sobolev spaces,

\[
\Pi_\varepsilon(\alpha) := \bigoplus_{k=1}^n (\alpha_k; \alpha_k + \ell), \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n, \quad \ell > 0.
\]

There are proved the following assertions about \( \varepsilon \)-Friedrichs inequality (Theorem 1) in the space \( H^1(\Omega) \) (Theorem 1) and \( H^1_0(\Omega) \) (Theorem 2): for every \( \varepsilon > 0 \) there exist \( N_\varepsilon \in \mathbb{N} \) and \( \omega_1, \ldots, \omega_N \in L^\infty(\Omega) \) such that the inequality

\[
\int_\Omega |v(x)|^2 \, dx \leq \varepsilon \int_\Omega |\nabla v(x)|^2 \, dx + \sum_{j=1}^{N_\varepsilon} \left( \int_\Omega v(x)\omega_j(x) \, dx \right)^2
\]

holds for every \( v \in H^1(\Omega) \) (Theorem 1) and \( v \in H^1_0(\Omega) \) (Theorem 2), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) for which there exist numbers \( \ell > 0, m \in \mathbb{N} \), and \( \alpha^1, \ldots, \alpha^m \in \mathbb{R}^n \) satisfying the following conditions

1) \( \Omega = \Pi_\varepsilon(\alpha^1) \cup \ldots \cup \Pi_\varepsilon(\alpha^m) \);
2) for every \( i, j \in \{1, \ldots, m\} \) with \( i \neq j \) we obtain: \( \Pi_\varepsilon(\alpha^i) \cap \Pi_\varepsilon(\alpha^j) = \emptyset \).

Let \( n \in \mathbb{N} \) be a fixed number, \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( L^2(\Omega) \), \( L^\infty(\Omega) \) be the Lebesgue spaces (see [1, p. 618]), \( H^1(\Omega) \) and \( H^1_0(\Omega) \) be the Sobolev spaces (see [1, p. 245-246]), \( H \) be a Hilbert space, \( C([0, T]; H) \) be the space of the \( H \)-valued continuous functions defined on \([0, T]\) (see [2, p. 147]), \( L^2(0, T; H) \) be the Lebesgue-Bochner space (see [2, p. 155]). We define

\[
C_{\text{weak}}([0, T]; H) := \left\{ u : [0, T] \to H \mid \forall z \in H \quad (u(\cdot), z)_H \in C([0, T]) \right\}.
\]

Note that \( C([0, T]; H) \subset C_{\text{weak}}([0, T]; H) \) (see [2, p. 147] for more details).

We recall (see Lemma 1.26 [2, Chapter 2, §1]) the standard Friedrichs inequality:

\[
\int_\Omega |v(x)|^2 \, dx \leq M_\Omega \int_\Omega |\nabla v(x)|^2 \, dx, \quad v \in H^1_0(\Omega),
\]

where \( \nabla v := (v_1, \ldots, v_n) \). Notice that the constant \( M_\Omega > 0 \) depends on \( \Omega \) and does not depend on \( v \). The various generalizations of formula (1) are considered in [3], [4], [5], etc.

In the paper, we consider some modification of (1). We denote

\[
\Pi_\varepsilon(\alpha) := (\alpha_1; \alpha_1 + \ell) \times (\alpha_2; \alpha_2 + \ell) \times \ldots \times (\alpha_n; \alpha_n + \ell), \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n, \quad \ell > 0.
\]

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Clearly, \( \Pi_\ell(\alpha) \) is a cube in \( \mathbb{R}^n \) with the volume \( \ell^n \). Let us define the set \( \Pi(\mathbb{R}^n) \) as follows.

For every \( \Omega \in \Pi(\mathbb{R}^n) \) we have that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and there exist numbers \( \ell > 0, m \in \mathbb{N}, \) and \( \alpha^1, \ldots, \alpha^m \in \mathbb{R}^n \) such that the following conditions hold:

1) \( \Omega = \overline{\Pi}(\alpha^1) \cup \ldots \cup \overline{\Pi}(\alpha^m); \)

2) for every \( i, j \in \{1, \ldots, m\} \) such that \( i \neq j \) we obtain: \( \Pi_\ell(\alpha^i) \cap \Pi_\ell(\alpha^j) = \emptyset . \)

The main results of the paper are contained in the next propositions.

**Theorem 1** (\( \varepsilon \)-Friedrichs inequality for the functions from \( H^1(\Omega) \)). If \( \Omega \in \Pi(\mathbb{R}^n) \), then for every \( \varepsilon > 0 \) there exist \( N_\varepsilon \in \mathbb{N} \) and \( \omega_1, \ldots, \omega_{N_\varepsilon} \in L^\infty(\Omega) \) such that the inequality

\[
\int_{\Omega} |v(x)|^2 \, dx \leq \varepsilon \int_{\Omega} |\nabla v(x)|^2 \, dx + \sum_{j=1}^{N_\varepsilon} \left( \int_{\Omega} v(x) \omega_j(x) \, dx \right)^2
\]

holds for every \( v \in H^1(\Omega) \).

**Theorem 2** (\( \varepsilon \)-Friedrichs inequality for functions from \( H^1_0(\Omega) \)). If \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), then for every \( \varepsilon > 0 \) there exist \( N_\varepsilon \in \mathbb{N} \) and \( \omega_1, \ldots, \omega_{N_\varepsilon} \in L^\infty(\Omega) \) such that inequality (3) holds for every \( v \in H^1_0(\Omega) \).

**Corollary 1.** Suppose that \( \Omega \) is taken from either Theorem 1 or Theorem 2, \( T > 0 \) is given number, \( H := L^2(\Omega), V := H^1(\Omega) \) if \( \Omega \in \Pi(\mathbb{R}^n), V := H^1_0(\Omega) \) if \( \Omega \) satisfies the conditions of Theorem 2, and \( u^m \to u \) in \( C_{\text{weak}}([0, T]; H) \). Then the following statements are fulfilled:

(i) if \( u^m \to u \) slowly in \( L^2(0, T; V) \), then \( u^m \to u \) strongly in \( L^2(0, T; V) \);

(ii) if \( u^m \varrightharpoonup u \) \#-slowly in \( L^\infty(0, T; V) \), then \( u^m \to u \) in \( C([0, T]; H) \).

Note that the case \( n = 2 \) of Theorem 2 is considered in [6, p. 553] and [7, p. 172]. We give the proof of Theorem 2 in case \( n \neq 2 \) for convenience. As we know the \( \varepsilon \)-Friedrichs inequalities for functions from \( H^1(\Omega) \) is not studied yet.

Let us prove the main results. Clearly,

\[
(a_1 + \ldots + a_n)^2 = (1 \cdot a_1 + \ldots + 1 \cdot a_n)^2 \leq \left( \sum_{i=1}^{n} 1^2 \right) \left( \sum_{i=1}^{n} a_i^2 \right) \leq n(a_1^2 + \ldots + a_n^2).
\]

The following lemma is needed for the sequel.

**Lemma 1** (Poincaré’s inequality for cube in \( \mathbb{R}^n \)). If \( n \in \mathbb{N}, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n, \ell > 0, \) and \( G = \Pi_\ell(\alpha), \) where \( \Pi_\ell(\alpha) \) is chosen from (2), then for every \( v \in H^1(G) \) we have

\[
\int_{G} |v(\xi)|^2 \, d\xi \leq \frac{n}{2} \ell^2 \int_{G} |\nabla v(\xi)|^2 \, d\xi + \frac{1}{\ell^n} \left( \int_{G} v(\xi) \, d\xi \right)^2.
\]

**Proof.** Note that the case \( n = 2 \) is considered in [6, p. 552]. Suppose that \( n \in \mathbb{N} \setminus \{2\} \). For simplicity, we prove Lemma 1 only if \( n = 3 \) and \( \alpha_1 = \ldots = \alpha_n = 0 \).

Let \( n = 3, \ell > 0, \) and \( G = (0; \ell)^n \). Take arbitrary points \( \xi_1 = (x_1, y_1, z_1) \in G \) and \( \xi_2 = (x_2, y_2, z_2) \in G \). Then, for every \( v \in H^1(G) \), we have

\[
v(x_2, y_2, z_2) - v(x_1, y_1, z_1) = v(x_2, y_2, z_2) - v(x_1, y_2, z_2) + v(x_1, y_2, z_2) - v(x_1, y_1, z_2) +
\]
\[ + v(x_1, y_1, z) - v(x_1, y_1, z_1) = \int_{x_1}^{x_2} v_x(x, y_2, z_2) \, dx + \int_{y_1}^{y_2} v_y(x_1, y, z_2) \, dy + \int_{z_1}^{z_2} v_z(x_1, y_1, z) \, dz. \]

Squaring this equality and using inequality (4) with \( n = 3 \), the Cauchy-Bunyakowski-Schwarz inequality, and the conditions

\[ |x_2 - x_1| \leq \ell, \quad |y_2 - y_1| \leq \ell, \quad |z_2 - z_1| \leq \ell, \]

we obtain

\[ |v(x_2, y_2, z) - v(x_1, y_1, z_1)|^2 \leq \]

\[ \leq \int_G \left( \int_{x_1}^{x_2} |v_x(x, y_2, z_2)|^2 \, dx + \int_{y_1}^{y_2} |v_y(x_1, y, z_2)|^2 \, dy + \int_{z_1}^{z_2} |v_z(x_1, y_1, z)|^2 \, dz \right) \leq \]

\[ n \ell \left\{ \int_0^\ell |v_x(x_2, y_2, z_2)|^2 \, dx + \int_0^\ell |v_y(x_1, y, z_2)|^2 \, dy + \int_0^\ell |v_z(x_1, y_1, z)|^2 \, dz \right\}. \]

Integrating this inequality in \((x_1, y_1, z_1), (x_2, y_2, z_2) \in G\), we deduce

\[ J \leq n \ell \int_G dxdydz \int_0^\ell \left\{ \int_0^\ell |v_x(x, y_2, z_2)|^2 \, dx + \int_0^\ell |v_y(x_1, y, z_2)|^2 \, dy + \right. \]

\[ + \left. \int_0^\ell |v_z(x_1, y_1, z)|^2 \, dz \right\} dx_2 dy_2 dz_2 = n \ell \cdot \ell^{n+1} \int_G \left[ |v_x(x, y, z)|^2 + |v_y(x, y, z)|^2 + 
\]

\[ + |v_z(x, y, z)|^2 \right] dx dy dz = n \ell^{n+2} \int_G |\nabla v(x, y, z)|^2 \, dx dy dz, \]

where

\[ J := \int_G dxdydz \int_0^\ell |v(x_2, y_2, z_2) - v(x_1, y_1, z_1)|^2 \, dx_2 dy_2 dz_2. \]

Clearly, (8) implies

\[ J = \int_G dxdydz \int_0^\ell \left[ |v(x_2, y_2, z_2)|^2 + |v(x_1, y_1, z_1)|^2 - 2v(x_2, y_2, z_2) v(x_1, y_1, z_1) \right] \, dx_2 dy_2 dz_2 = \]

\[ = 2 \ell^n \int_G |v(x, y, z)|^2 \, dx dy dz - 2 \left( \int_G v(x_2, y_2, z_2) \, dx_2 dy_2 dz_2 \right) \cdot \left( \int_G v(x_1, y_1, z_1) \, dx_1 dy_1 dz_1 \right) = \]

\[ = 2 \ell^n \int_G |v(x, y, z)|^2 \, dx dy dz - 2 \left( \int_G v(x, y, z) \, dx dy dz \right)^2. \]

Then from (7) we obtain

\[ 2 \ell^n \int_G |v|^2 \, dx dy dz - 2 \left( \int_G v \, dx dy dz \right)^2 \leq n \ell^{n+2} \int_G |\nabla v|^2 \, dx dy dz, \]

Therefore, (5) holds and Lemma 1 is proved. \( \square \)
Proof of Theorem 1. Since \( \Omega \in \Pi(\mathbb{R}^n) \), we have \( L > 0 \), \( m \in \mathbb{N} \), and \( \alpha_1, \ldots, \alpha_m \in \mathbb{R}^n \) such that \( \Omega = \Pi_k(\alpha_1) \cup \cdots \cup \Pi_k(\alpha_m) \) and \( \Pi_k(\alpha_i) \cap \Pi_k(\alpha_j) = \emptyset \) if \( i \neq j \).

Take \( k \in \mathbb{N} \) and put \( N = k^n \). For every \( j \in \{1, \ldots, m\} \) we partition every cube \( \Pi_k(\alpha_j) \) into cubes \( Q_{k,j}^{k_1}, Q_{k,j}^{k_2}, \ldots, Q_{k,j}^{k_n} \) such that every cube \( Q_{k,j}^{k} \) has its side with length \( \ell = \frac{L}{k} \).

Let us consider an arbitrary \( v \in H^1(\Omega) \). For every \( j \in \{1, \ldots, m\} \) and \( \mu \in \{1, \ldots, N\} \), we use Poincaré’s inequality (5) with \( G = Q_{k,j}^{k} \). Summing these inequalities, we get

\[
J_1 \leq J_2 + J_3,
\]

where (for the sake of convenience we replace \( \xi \) by \( x \))

\[
J_1 := \sum_{j=1}^m \sum_{\mu=1}^N \int_{Q_{k,j}^{k}} |v(x)|^2 \, dx, \quad J_2 := \sum_{j=1}^m \sum_{\mu=1}^N \frac{n}{2} \ell^2 \int_{Q_{k,j}^{k}} |\nabla v(x)|^2 \, dx,
\]

\[
J_3 := \sum_{j=1}^m \sum_{\mu=1}^N \frac{1}{k^n} \left( \int_\Omega v(x) \, dx \right)^2.
\]

Clearly, \( J_1 = \int_\Omega |v|^2 \, dx \), \( J_2 = \frac{n}{2} \ell^2 \int_\Omega |\nabla v|^2 \, dx \). If we take

\[
\chi_{k,j}^k(x) := \begin{cases} 1, & x \in Q_{k,j}^{k}, \\ 0, & x \in \mathbb{R}^n \setminus Q_{k,j}^{k}, \end{cases} \quad j = 1, m, \quad \mu = 1, N,
\]

then

\[
J_3 = \sum_{j=1}^m \sum_{\mu=1}^N \frac{1}{k^n} \left( \int_\Omega v(x) \chi_{k,j}^k(x) \, dx \right)^2.
\]

Thus, (10) yields that

\[
\int_\Omega |v|^2 \, dx \leq \frac{nL^2}{2} \cdot \frac{1}{k^2} \int_\Omega |\nabla v|^2 \, dx + \frac{1}{L^2} \cdot k^n \sum_{j=1}^m \sum_{\mu=1}^N \left( \int_\Omega v(x) \chi_{k,j}^k(x) \, dx \right)^2, \quad k \in \mathbb{N}.
\]

Finally, for every \( \varepsilon > 0 \) we choose \( k \in \mathbb{N} \) such that \( \frac{nL^2}{2} \cdot \frac{1}{k^2} \leq \varepsilon \). Then, from (11) we obtain (3) and Theorem 1 is proved.

Proof of Theorem 2. Note that the case \( n = 2 \) is considered in [6, p. 542, 543, 553]. Suppose that \( n \in \mathbb{N} \setminus \{2\} \) and modify the proof of Theorem 1.

Let \( Q \) be a cube in \( \mathbb{R}^n \) such that \( \Omega \subset Q \) and the length of the side of \( Q \) equals \( L > 0 \). Take \( k \in \mathbb{N} \) and put \( N = k^n \). We partition the cube \( Q \) into the cubes \( Q_{1,1}^{k}, Q_{1,2}^{k}, \ldots, Q_{N}^{k} \) such that every cube \( Q_{\mu}^{k} \) has its side with length \( \ell = \frac{L}{k} \).

We take an arbitrary \( v \in H_0^1(\Omega) \). Assume that the function \( v \) equals zero outside \( \Omega \). For every \( \mu \in \{1, \ldots, N\} \), we use Poincaré’s inequality (5) with \( G = Q_{\mu}^{k} \). Summing these inequalities from 1 to \( N \), we get

\[
I_1 \leq I_2 + I_3,
\]

where (for the sake of convenience we replace \( \xi \) by \( x \) again)

\[
I_1 := \sum_{\mu=1}^N \int_{Q_{\mu}^{k}} |v(x)|^2 \, dx, \quad I_2 := \sum_{\mu=1}^N \frac{n}{2} \ell^2 \int_{Q_{\mu}^{k}} |\nabla v(x)|^2 \, dx, \quad I_3 := \sum_{\mu=1}^N \frac{1}{k^n} \left( \int_\Omega v(x) \, dx \right)^2.
\]
Clearly,
\[ I_1 = \int_Q |v|^2 \, dx = \int_\Omega |v|^2 \, dx, \quad I_2 = \frac{n}{2} \ell^2 \int_Q |\nabla v|^2 \, dx = \frac{n}{2} \ell^2 \int_\Omega |\nabla v|^2 \, dx. \]

If we choose
\[ \chi_\mu^k(x) = \begin{cases} 1, & x \in Q_\mu^k, \\ 0, & x \in \mathbb{R}^n \setminus Q_\mu^k, \end{cases} \quad \mu = 1, N, \]
then
\[ I_3 = \sum_{\mu=1}^N \frac{1}{\ell^n} \left( \int_Q v(x) \chi_\mu^k(x) \, dx \right)^2 = \sum_{\mu=1}^N \frac{1}{\ell^n} \left( \int_\Omega v(x) \chi_\mu^k(x) \, dx \right)^2. \]

Thus, (12) yields that
\[ \int_\Omega |v|^2 \, dx \leq \frac{nL^2}{2} \cdot \frac{1}{k^2} \int_\Omega |\nabla v|^2 \, dx + \frac{1}{L^n} \cdot k^2 \sum_{\mu=1}^k \left( \int_\Omega v(x) \chi_\mu^k(x) \, dx \right)^2, \quad k \in \mathbb{N}. \quad (13) \]

Finally, for every \( \varepsilon > 0 \) we choose \( k \in \mathbb{N} \) such that \( \frac{nL^2}{2} \cdot \frac{1}{k^2} \leq \varepsilon \). Then, from (13) we obtain (3).

**Proof of Corollary 1.** Suppose that \( \{u^m\}_{m \in \mathbb{N}} \) satisfies condition (i) of Corollary 1. For a.e. \( t \in (0, T) \), we use (3) with \( v(x) := u^m(x, t) - u^k(x, t) \) for \( x \in \Omega \). Integrating in \( t \), we have
\[ \int_{Q_0,T} |u^m - u^k|^2 \, dx dt \leq \varepsilon \int_{Q_0,T} |\nabla u^m - \nabla u^k|^2 \, dx dt + \int_0^T \sum_{j=1}^{N_k} \left( \int_{\Omega} [u^m(x, t) - u^k(x, t)] \omega_j(x) \, dx \right)^2 \, dt, \]
where \( Q_{0,T} = \Omega \times (0, T) \) and \( m, k \in \mathbb{N} \). The first integral on the right-hand side of this inequality does not exceed a fixed constant for any \( m \) and \( k \). Moreover, the last integral can be made arbitrary small for sufficiently large \( m \) and \( k \) because \( (u^m(\cdot, t), \omega_j(\cdot))_{L^2(\Omega)} \) uniform converges in \( t \) as \( m \to \infty \) (see [7, p. 173] for comparison). Therefore, the right-hand side of the inequality can be made arbitrary small for sufficiently large \( m \) and \( k \). Hence, the sequence \( \{u^m\}_{m \in \mathbb{N}} \) converges strongly to \( u \) in \( L^2(0, T; H) = L^2(Q_{0,T}) \).

If \( \{u^m\}_{m \in \mathbb{N}} \) satisfies condition (ii), then the proof is similar.

**REFERENCES**


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