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SEQUENTIAL COARSE STRUCTURES OF TOPOLOGICAL GROUPS

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We endow a topological group (G, τ) with a coarse structure defined by the smallest group ideal S_τ on G containing all converging sequences and denote the obtained coarse group by (G, S_τ). If G is discrete, then (G, S_τ) is a finitary coarse group studding in *Geometric Group Theory*. The main result: if a topological abelian group (G, τ) contains a non-trivial converging sequence then $asdim\,(G, S_\tau) = \infty$. We study metrizability, normality and functional boundedness of sequential coarse groups and put some open questions.

To Anatoly Plichko on 70-th birthday

1. Introduction. Let X be a set. A family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x, x) \colon x \in X\}$;

- if $E, E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) \colon \exists z((x, z) \in E, \ (z, y) \in E')\}$, $E^{-1} = \{(y, x) \colon (x, y) \in E\}$;

- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;

- for any $x, y \in X$, there exists $E \in \mathcal{E}$ such that $(x, y) \in E$.

A subset $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ and $E \in \mathcal{E}$, we denote $E[x] = \{y \in X \colon (x, y) \in E\}$, $E[A] = \bigcup_{a \in A} E[a]$ and say that $E[x]$ and $E[A]$ are *balls of radius E around x and A*.

The pair $(X, \mathcal{E})$ is called a *coarse space* [14] or a *ballean* [9], [12].

Each subset $Y \subseteq X$ defines the *subballean* $(Y, \mathcal{E}_Y)$, where $\mathcal{E}_Y$ is the restriction of $\mathcal{E}$ to $Y \times Y$. A subset Y is called *bounded* if $Y \subseteq E[x]$ for some $x \in X$ and $E \in \mathcal{E}$.

A family $\mathcal{F}$ of subsets of X is called *E-bounded (E-disjoint)* if, for each $A \in \mathcal{F}$, there exists $x \in X$ such that $A \subseteq E[x]$ ($E[A] \cap E[B] = \varnothing$ for all distinct $A, B \in \mathcal{F}$).

By the definition [14, Chapter 9], $asdim\,(X, \mathcal{E}) \geq n$ if, for each $E \in \mathcal{E}$, there exist $F \in \mathcal{E}$ and F-bounded covering $\mathcal{M}$ of X which can be partitioned $\mathcal{M} = \mathcal{M}_0 \cup \ldots \cup \mathcal{M}_n$ so that each family M_i is E-disjoint. If there exists the minimal n with this property then $asdim\,(X, \mathcal{E}) = n$, otherwise $asdim\,(X, \mathcal{E}) = \infty$.

Given two coarse spaces $(X, \mathcal{E})$, $(X', \mathcal{E}')$, a mapping $f \colon X \to X'$ is called *macro-uniform* if, for each $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $f(E[x]) \subseteq E'[f(x)]$ for each $x \in X$. If f is a bijection such that f and f^{-1} are macro-uniform then f is called an *asymorphism*.

Now let G be a group. A family $\mathcal{I}$ of subsets of G is called a *group ideal* [10], [12] if $\bigcup \mathcal{I} = G$ and for any $A, B \in \mathcal{I}$ any subset of the set $AB^{-1} := \{ab^{-1} \colon a \in A,\ b \in B\}$ belongs to $\mathcal{I}$. Every group ideal $\mathcal{I}$ defines a coarse structure on G with the base $\{\{(x, y) \colon x \in Ay\} \colon A \in \mathcal{I}\}$. We denote G endowed with this coarse structure by $(G, \mathcal{I})$.

If G is discrete, then the coarse space $(G, [G]^{<\omega})$ is the main subject of *Geometric Group Theory*, see [5]. For coarse structures on G defined by the ideal $[G]^{<\kappa}$, where κ is a cardinal, see [11].

Every topological group G can be endowed with a coarse structure defined by the ideal of all totally bounded subsets of G. These coarse structures were introduced and studied in [6]. For asymptotic dimensions of locally compact abelian groups endowed with coarse structures defined by ideals of precompact subsets see [7]. We recall that a subset Y of a topological space is *precompact* if the closure of Y is compact.

For a topological group (G, τ), we denote by $\mathcal{C}_\tau$ the group ideal of precompact subsets of G, and by $\mathcal{S}_\tau$ the minimal group ideal containing all converging sequences in G. Clearly, $\mathcal{S}_\tau \subseteq \mathcal{C}_\tau$.

2. Asympotic dimension. We recall [13] that a sequence $(a_n)_{n \in \omega}$ in an abelian group G is a *T-sequence* if there exists a Hausdorff group topology on G in which $(a_n)_{n \in \omega}$ converges to 0. For a T-sequences $(a_n)_{n \in \omega}$ on G, we denote by $\tau_{(a_n)}$ the strongest group topology on G in which $(a_n)_{n \in \omega}$ converges to 0. We put $A = \{0, a_n, -a_n \colon n \in \omega\}$ and denote by $A_n = \{a_1 + \cdots + a_n \colon a_1, \ldots, a_n \in A\}$ the sum on n copies of A.

By [13, Theorem 2.3.11], $(G, \tau_{(a_n)})$ is complete. Hence, a subset S of G is totally bounded in $(G, \tau_{(a_n)})$ if and only if S is precompact.

We use the following three theorems proved by the author in [3].

Theorem 1. *For any T-sequences $(a_n)_{n \in \omega}$ on G, the family $\{F + A_n \colon F \in [G]^{<\omega}\}$, $n \in \omega\}$ is a base for the ideal $\mathcal{C}_{\tau(a_n)}$ and $\mathcal{C}_{\tau(a_n)} = \mathcal{S}_{\tau(a_n)}$. If G is generated by the set $\{a_n \colon n \in \omega\}$ then $\{A_n \colon n \in \omega\}$ is a base for $\mathcal{C}_{\tau(a_n)}$.*

Proof. Apply Lemma 2.3.2 from [13]. ☐

Given an arbitrary subset S of G, the Cayley graph Cay (G, S) is a graph with the set of vertices G and the set of edges $\{(x, y) \colon x - y \in S \cup (-S)\}$.

Theorem 2. *If a T-sequences $(a_n)_{n \in \omega}$ generates G then the coarse group $(G, \mathcal{C}_{\tau(a_n)})$ is asymorphic to Cay $(G, \{a_n \colon n \in \omega\})$.*

Proof. Apply and Theorem 1 and Theorem 5.1.1 from [12]. ☐

We recall that the Hamming space $\mathbb{H}$ is the set $[\omega]^{<\omega}$ endowed with the metric $h(F, H) = |F \triangle H|$. To see that *asdim* $\mathbb{H} = \infty$, it suffices to find an asymorphic copy of $\mathbb{N}$ in $\mathbb{H}$ and observe that $\mathbb{H}$ is asymorphic to $\mathbb{H} \times \mathbb{H}$.

Example 1. Let G be the direct sum of groups $\{\langle a_n \rangle \colon n \in \omega\}$ of order 2. Clearly, $(a_n)_{n \in \omega}$ is a T-sequence on G. By Theorem 1, the canonical bijection between $(G, \mathcal{C}_{\tau(a_n)})$ and the Hamming space $\mathbb{H}$ of all finite subsets of ω is an asymorphism.

A T-sequence $(a_n)_{n \in \omega}$ is called *trivial* if $a_n = 0$ for all but finitely many $n \in \omega$.

Theorem 3. *For any non-trivial T-sequences $(a_n)_{n \in \omega}$ on G, the coarse group $(G, \mathcal{C}_{\tau(a_n)})$ contains a subspace asymorphic to the Hamming space $\mathbb{H}$ so asdim $(G, \mathcal{C}_{\tau(a_n)}) = \infty$.*

Proof. Without loss of generality, we suppose that $\{a_n \colon n \in \omega\}$ generates G and $a_n \neq 0$ for each $n \in \omega$.

Given an arbitrary T-sequence $(b_n)_{n \in \omega}$ in G, we denote

$$FS(b_n)_{n \in \omega} = \left\{ \sum_{i \in F} b_i \colon F \in [\omega]^{<\omega} \right\}$$

and say that $(b_n)_{n \in \omega}$ is FS-*strict* if, for any $H, F \in [\omega]^{<\omega}$,

$$\sum_{i \in H} b_i = \sum_{i \in F} b_i \quad \Longrightarrow \quad F = H.$$

We note that $(b_n)_{n \in \omega}$ is FS-strict if, for each $n \in \omega$,

$$b_{n+1} \notin \left\{ \sum_{i \in F} b_i - \sum_{i \in H} b_i \colon H, F \subseteq \{0, \ldots, n\} \right\}. \tag{1}$$

We assume that $(b_n)_{n \in \omega}$ is FS-strict and

$$\text{if } \; b = \sum_{i \in F} b_i, \; a \in A_n \text{ and } b + a = \sum_{i \in H} b_i \text{ then } |F \triangle H| \leq n. \tag{2}$$

Then the canonical bijection $f \colon \mathbb{H} \longrightarrow FS(b_n)$, $f(H) = \sum_{i \in H} b_i$ is an asymorphism.
To construct the desired sequence $(b_n)_{n \in \omega}$ we rewrite (2) in the following equivalent form

$$\text{if } \; i_0 < i_1 < \ldots < i_n < \omega \; \text{ and } t_{i_0}, \ldots, t_{i_n} \in \{1, -1\} \text{ then } t_{i_0} b_{i_0} + \ldots + t_{i_n} b_{i_n} \notin A_n. \tag{3}$$

We put $b_0 = a_0$ and assume that $b_0, \ldots, b_n$ have been chosen. We show how to choose b_{n+1} to satisfy (1) and

$$\sum_{s=0}^{k} t_{i_s} b_{i_s} + t_{n+1} b_{n+1} \notin A_{k+1} \text{ for } i_0 < \ldots < i_k \leq n \text{ and}$$

$$t_i \in \{1, -1\} \text{ for } i \in \{i_0, \ldots, i_k, n+1\}. \tag{4}$$

We assume that there exists a subsequence $(c_m)_{m \in \omega}$ of $(a_n)_{n \in \omega}$ such that $t_{i_0} b_{i_0} + \ldots + t_{i_k} b_{i_k} + t c_m \in A_{k+1}$ for $t \in \{1, -1\}$ and for each $m \in \omega$. Every infinite subset of A_{k+1} has a limit point in A_k. Hence, $t_{i_0} b_{i_0} + \ldots + t_{i_k} b_{i_k} \in A_{k+1}$ contradicting the choice of $b_0, \ldots, b_n$. Thus, b_{n+1} can be taken from $\{a_{m_0}, a_{m_0+1}, \ldots\}$ for some $m_0 \in \omega$. $\qquad\square$

Theorem 4. *Let G be a group and let $\mathcal{I}$ be a group ideal on G. Assume that there exists a family $\mathfrak{F}$ of group ideals on G such that $\bigcup \mathfrak{F} = \mathcal{I}$ and asdim$(G, \mathcal{J}) \geq n$ for each $\mathcal{J} \in \mathfrak{F}$. Then asdim$(G, \mathcal{I}) \geq n$.*

Proof. Given any $A \in \mathcal{I}$ and $n \in \omega$, we choose $\mathcal{J} \in \mathfrak{F}$ such that $A \in \mathcal{J}$ and a uniformly bounded covering $\mathcal{M}$ of $(G, \mathcal{J})$ and a partition $\mathcal{M}_0, \ldots, \mathcal{M}_n$ of $\mathcal{M}$ witnessing asdim$(G, \mathcal{J}) \geq n$. Then these $\mathcal{M}, \mathcal{M}_0, \ldots, \mathcal{M}_n$ witness that asdim$(G, \mathcal{I}) \geq n$. $\qquad\square$

Theorem 5. *If a topological abelian group (G, τ) contains a non-trivial converging sequence then asdim $(G, \mathcal{S}_\tau) = \infty$.*

Proof. We denote

$$\mathcal{F} = \{\mathcal{S}_{\tau(a_n)}\colon (a_n) \text{ is a non-trivial sequence in } (G, \tau) \text{ converging to } 0 \text{ in } (G, \tau)\}.$$

Let $A \in \mathcal{S}_\tau$. By the definition of $\mathcal{S}_\tau$, there exists a subset $B \in \mathcal{S}_\tau$, $A \subseteq B$ and a finite number $(a_{1n})_{n \in \omega}, \ldots, (a_{mn})_{n \in \omega}$ of sequences converging to 0 in (G, τ) such that B can be obtained from the sets $\{a_{1n}, -a_{1n}\colon n \in \omega\}, \ldots, \{a_{mn}, -a_{mn}\colon n \in \omega\}$ by the finite number of additions of these sets and join of finite subsets of G. We choose a sequence $(b_n)_{n \in \omega}$ converging to 0 in (G, τ) and containing each $(a_{1n})_{n \in \omega}, \ldots, (a_{mn})_{n \in \omega}$ as a subsequence. Then $A \in \mathcal{S}_{\tau(b_n)}$, $\mathcal{S}_{\tau(b_n)} \subseteq \mathcal{S}_\tau$. Apply Theorems 3 and 4. $\square$

We note that Theorem 5 answers Question 2 from [3].

Question 1. *Let (G, τ) be a countable non-discrete metrizable abelian group. Does $(G, \mathcal{S}_\tau)$ contain an asymorphic copy of $\mathbb{H}$?*

In [4], the authors ask about asymptotic dimension of $(G, \mathcal{C}_\tau)$, where G is an infinite cyclic subgroup of the circle. We put this question in more genera form.

Question 2. *Let (G, τ) be a countable non-discrete metrizable group. Is asdim $(G, \mathcal{C}_\tau) = \infty$?*

Example 2. Let G be the direct sum of ω copies of $\mathbb{Z}_2$ endowed with the topology induced by the Tikhonov topology of $\mathbb{Z}_2^\omega$. Since *asdim* $(G, \mathcal{C}_\tau) > 0$ and $(G, \mathcal{C}_\tau)$ is asymorphic to $(G, \mathcal{C}_\tau) \times (G, \mathcal{C}_\tau)$, we see that *asdim* $(G, \mathcal{C}_\tau) = \infty$.

Let $(X, \mathcal{E})$ be a coarse space. A function $f\colon X \longrightarrow \{0, 1\}$ is called *slowly oscillating* if, for every $E \in \mathcal{E}$, there exists a bounded subset B of X such that $f|_{E[x]} = const$ for each $x \in X \setminus B$. We endow X with the discrete topology, identify the Stone-Čech compactification βX of X with the set of ultrafilters on X and denote $X^\sharp = \{p \in \beta X\colon \text{ each } P \in p \text{ is unbounded }\}$. We define an equivalence $\sim$ on $X^\sharp$ by the rule: $p \sim q$ if and only if $f^\beta(p) = f^\beta(q)$ for every slowly oscillating function $f\colon X \longrightarrow \{0, 1\}$. The quotient $X^\sharp/_\sim$ is called a space of *ends* or *binary corona* of $(X, \mathcal{E})$, see [12, Chapter 8].

Theorem 6. *If a non-trivial T-sequences $A = \{a_n\}_{n \in \omega}$ generates G, then the space of ends of $(G, \mathcal{C}_{\tau(a_n)})$ is a singleton.*

Proof. First we show that for every slowly oscillating function $f\colon G \longrightarrow \{0, 1\}$ there exists an m such that

$$f|_{G \setminus A_m} = \text{const}. \tag{5}$$

Indeed, by the definition of slow oscillation and Theorem 1, there exists $m \in \omega$ such that $f|_{x+A} = \text{const}$ for each $x \in G \setminus A_m$. We show now that (5) holds true for this m.

We take arbitrary $y, z \in G \setminus A_m$. Since A generates G and contains 0, there exists an index k such that $y, z \in A_k$, i.e.,

$$y = b_1 + \ldots + b_k \quad \text{and} \quad z = c_1 + \ldots + c_k,$$

for appropriate $b_1, \ldots, b_k, \ c_1, \ldots, c_k \in A$. By a property of T-sequences established at the end of the proof of Theorem 3, there exists a member a_{m_1} of (a_n) such that

$$a_{m_1} + y \notin A_{m+1} \quad \text{and} \quad a_{m_1} + z \notin A_{m+1},$$

since (a_n) is a T-sequence. Then $a_{m_1} + b_2 + \ldots + b_k \notin A_m$, $a_{m_1} + c_2 + \ldots + c_k \notin A_m$. Therefore,

$$f(a_{m_1} + b_2 + \ldots + b_k) = f(y) \quad \text{and} \quad f(a_{m_1} + c_2 + \ldots + c_k) = f(z). \tag{6}$$

Repeating this trick k times, we can replace $b_2, \ldots, b_k$ and $c_2, \ldots, c_k$, by appropriate members $a_{m_2}, \ldots, a_{m_k}$ of (a_n), as before. Hence, we can replace $b_2 + \ldots + b_k$ and $c_2 + \ldots + c_k$, by $a_{m_2} + \ldots + a_{m_k}$ in (6). This obviously gives $f(y) = f(z)$ and proves (5).

Finally, to prove the assertion of the theorem, pick $p, q \in X^\sharp$. In order to check that $p \sim q$ fix an arbitrary slowly oscillating function $f \colon X \longrightarrow \{0, 1\}$. We have to prove that $f^\beta(p) = f^\beta(q)$. Pick an m with (5). Since $G \setminus A_m \in p \cap q$, for every $P \in p$ and for every $Q \in q$ we have $P_1 := P \setminus A_m \in p$ and $Q_1 := Q \setminus A_m \in q$ and $f|_{P_1} = f|_{Q_1}$ is constant in view of (5). This proves that $f^\beta(p) = f^\beta(q)$. $\square$

Question 3. *Let (G, τ) be a countable non-discrete metrizable abelian group. Is the space of ends of $(G, \mathcal{S}_\tau)$ a singleton? The same question for $(G, \mathcal{C}_\tau)$.*

3. Metrizability and normality. By [12, Theorem 2.1.1], a coarse space $(X, \mathcal{E})$ is metrizable if and only if $\mathcal{E}$ has a countable base.

Theorem 7. *For an infinite abelian group G and a T-sequence $(a_n)_{n \in \omega}$ in G, $(G, \mathcal{S}_{\tau(a_n)})$ is metrizable if and only if G is countable.*

Proof. Apply Theorem 1. $\square$

Theorem 8. *Let (G, τ) be a non-discrete metrizable group. Then the coarse structure $\mathcal{E}_\tau$ of (G, S_τ) does not have a linearly ordered base. In particular, (G, S_τ) is not metrizable.*

Proof. We assume that $\mathcal{E}_\tau$ has a linear base $\mathcal{E}$ and choose a sequence $(A_n)_{n \in \omega}$ in $\mathcal{E}$ such that $A_n \subset A_{n+1}$ and the closure of $\bigcup_{n \in \omega} A_n$ is not compact. If $B \in \mathcal{E}$ then $B \subseteq A_m$ for some m so $\{A_n \colon n \in \omega\}$ is a base for $\mathcal{E}_\tau$.

Now let $\{U_n \colon n \in \omega\}$ is a base of neighbourhoods of the identity of (G, τ). We choose an injective sequence $(b_n)_{n \in \omega}$ in G such that $b_n \in U_n$, $b_n \notin A_n$. Then $\{b_n \colon n \in \omega\} \in \mathcal{S}_\tau$ but $\{b_n \colon n \in \omega\} \setminus A_n \neq \varnothing$. Hence, $\{A_n \colon n \in \omega\}$ is not a base for $\mathcal{E}_\tau$ and we get a contradiction. $\square$

Let $(X, \mathcal{E})$ be a ballean. A subset U of X is called an *asymptotic neighbourhood* of a subset $Y \subseteq X$ if, for every $E \in \mathcal{E}$, $E[Y] \setminus U$ is bounded.

Two subsets Y, Z of X are called

- *asymptotically disjoint if, for every $E \in \mathcal{E}$, $E[Y] \cap E[Z]$ is bounded;*
- *asymptotically separated if Y, Z have disjoint asymptotic neighbourhoods.*

A ballean $(X, \mathcal{E})$ is called *normal* [8] if any two asymptotically disjoint subsets of X are asymptotically separated. Every ballean with linearly ordered base is normal [8, Proposition 1.1].

We suppose that a non-discrete metrizable group (G, τ) is topologically isomorphic to the product $G_1 \times G_2$ of infinite groups. Applying Theorem 8 and Theorem 1.4 from [1], we conclude that $(G, \mathcal{S}_\tau)$ is not normal.

Question 4. *Let (G, τ) be a non-discrete metrizable group. Is the ballean $(G, \mathcal{S}_\tau)$ non-normal?*

4. Functional boundedness. Let $(X, \mathcal{E})$ be a coarse space. Following [2], we say that a function $f \colon (X, \mathcal{E}) \longrightarrow \mathbb{R}$ is

- *bornologous* if $f(B)$ is bounded in $\mathbb{R}$ for each bounded subset B of X;
- *macro-uniform* if, for every $E \in \mathcal{E}$, the supremum $\sup_{x \in X} \ diam \ f(E[x])$ is finite;
- *eventually macro-uniform* if, for every $E \in \mathcal{E}$, there exists a bounded subset B of X such that $\sup_{x \in X \setminus B} \ diam f(E[x])$ is finite;
- *slowly oscillating* if, for every $E \in \mathcal{E}$ and $\varepsilon > 0$, there exists a bounded subset B of X such that $diam f(E[x]) < \varepsilon$ for each $x \in X \setminus B$.

We say that a coarse space $(X, \mathcal{E})$ is

- *b-bounded* if each bornologous function on X is bounded;
- *mu-bounded* if each macro-uniform function on X is bounded;
- *emu-bounded* if, for every macro-uniform function f on X, there exists a bounded subset B of X such that f is bounded on $X \setminus B$;
- *so-bounded* if, for every slowly oscillating function f on X, there exists a bounded subset B of X such that f is bounded on $X \setminus B$.

If (G, τ) is discrete then every function $f \colon (G, \mathcal{S}_\tau) \longrightarrow \mathbb{R}$ is bornologous. If (G, τ) has a non-trivial converging sequence then there is a non-bornologous function $f \colon (G, \mathcal{S}_\tau) \longrightarrow \mathbb{R}$.

Theorem 9. *If (G, τ) is metrizable and compact, then $(G, \mathcal{S}_\tau)$ is b-bounded. If (G, τ) is countable and metrizable and non-discrete, then $(G, \mathcal{S}_\tau)$ is not b-bounded.*

Proof. In the first case, we assume that there exists an unbounded bornologous function $f \colon (G, \mathcal{S}_\tau) \longrightarrow \mathbb{R}$.

We choose a sequence $(a_n)_{n \in \omega}$ in G such that $|f(a_n)| > n$. Passing to a subsequence, we may suppose that $(a_n)_{n \in \omega}$ converges to some point a. Then f is not bounded on the set $\{a, a_n \colon n \in \omega\} \in \mathcal{S}_\tau$ so f is not bornologous.

In the second case, we denote by H the completion of G and choose *an* injective sequence $(b_n)_{n \in \omega}$ in G converging to some point $h \in H \setminus G$. Then we define a function $f \colon (G, \tau) \longrightarrow \mathbb{R}$ by $f(b_n) = n$ and $f(x) = 0$ for each $x \in G \setminus \{b_n \colon n \in \omega\}$. Since $K \cap \{b_n \colon n \in \omega\}$ is finite for each compact subset K of G, we see that f is bornologous. Hence, $(G, \mathcal{S}_\tau)$ is not b-bounded. $\square$

If (G, τ) is discrete and $(G, \mathcal{S}_\tau)$ is mu-bounded, then G is a Bergman group [2].

Theorem 10. *If (G, τ) is metrizable and totally bounded then $(G, \mathcal{S}_\tau)$ is mu-bounded.*

Proof. We assume the contrary. Then there exists a macro-uniform function $f \colon (G, \mathcal{S}_\tau) \to \mathbb{R}^+$ such that $f(a_{n+1}) - f(a_n) > n$ for some sequence $(a_n)_{n \in \omega}$ in (G, τ). Passing to a subsequence, we may suppose that $(a_n)_{n \in \omega}$ converges to some point $h \in H$, where H is the completion of G. We denote $b_n = a_n a_{n+1}^{-1}$ and observe that the sequence $(b_n)_{n \in \omega}$ converges to the identity e of G. We put $B = \{e, b_n \colon n \in \omega\}$ and note that $a_n \in B a_{n+1}$. Since $B \in \mathcal{S}_\tau$, we see that f is not macro-uniform. $\square$

Question 5. *Let be a compact metrizable abelian group. Is $(G, \mathcal{S}_\tau)$ so-bounded? mu-bounded?*

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