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ON SOME PROBLEM FOR ENTIRE FUNCTIONS OF UNBOUNDED INDEX IN ANY DIRECTION

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In this paper, we select a class of entire functions $F(z_1,\ldots,z_n)$ such that for any direction $(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{0\}$ and for every point $(z_1^0,\ldots,z_n^0)\in\mathbb{C}$ the function $F(z_1^0+tb_1,\ldots,z_n^0+tb_n)$ is of bounded index as a function in variable $t\in\mathbb{C}$, but the function F is of unbounded index in every direction $(b_1,\ldots,b_n)$. This result partially solves Problem 18 from the paper A. I. Bandura, O. B. Skaskiv, *Open problems for entire functions of bounded index in direction*, Mat. Stud., **43** (2015), no.1, 103–109. This problem concerns the existence of entire functions of unbounded L-index in any direction, where $L:\mathbb{C}^n\to\mathbb{R}_+$ is a continuous function and $n\geq 3$. Our result solves the problem in the case $L\equiv 1$.

Let us introduce some notations. Let $\mathbf{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be a fixed direction, $L\colon\mathbb{C}^n\to\mathbb{R}_+$ be a positive continuous function.

An entire function $F\colon\mathbb{C}^n\to\mathbb{C}$ is called a function of *bounded L-index [5, 7] in the direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, if there exists $m_0\in\mathbb{Z}_+$ such that for every $m\in\mathbb{Z}_+$ and for each $z\in\mathbb{B}^n$

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m!L^m(z)}\leq\max_{0\leq k\leq m_0}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)},\tag{1}$$

where $\partial_{\mathbf{b}}^0F(z)=F(z),\partial_{\mathbf{b}}F(z)=\sum\limits_{j=1}^n\frac{\partial F(z)}{\partial z_j}b_j$, $\partial_{\mathbf{b}}^kF(z)=\partial_{\mathbf{b}}\Big(\partial_{\mathbf{b}}^{k-1}F(z)\Big)$, $k\geq 2$.

The least such integer $m_0=m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b}$ of the analytic function F and is denoted by $N_{\mathbf{b}}(F,L)=m_0$. If $n=1$ and $L=l$ we obtain the definition of bounded l-index for entire functions of one variable [10], and if, in addition, $l\equiv 1$ we have the definition of entire function of bounded index [11].

The theory of entire functions of bounded L-index in direction was initialized in paper [5]. Mostly, these functions were investigated by A. Bandura and O. Skaskiv (see the bibliography in their monograph [7]). These functions have many applications in analytic theory of differential equations [6, 7, 13, 14].

Despite there are many papers on this topic the multidimensional theory of bounded index is not completely accomplished. It has many interesting problems. Particularly, the most impressive problems were listed in [1]. Recently, O. Skaskiv published overview [9] of solved problems in theory of functions of bounded index and posed new problems.

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The present paper is devoted to Problems 17 and 18 from [1]. Below we give their full formulations.

Problem 1 ([1, Problem 17]). *What are conditions on zero set and growth of entire functions providing the index boundedness of $F(z_1^0+b_1t, z_2^0+b_2t)$ for every $(z_1^0, z_2^0)\in\mathbb{C}^2$ and the index unboundedness of $F(z_1,z_2)$ in the direction $\mathbf{b}=(b_1,b_2)$?*

A. Bandura and O. Skaskiv gave some answer to the question in [2,3]. The most general their result is the following.

Theorem 1 ([3]). *Let $f(t)$, $t\in\mathbb{C}$, be an even entire transcendental function of bounded index. Then: 1) for each direction $\mathbf{b}=(b_1,b_2)\in\mathbb{C}^2\setminus\{0\}$ and for every fixed z_1^0, $z_2^0\in\mathbb{C}$ the function $g(t)=f(\sqrt{(z_1^0+b_1t)(z_2^0+b_2t)})$ is an entire function of bounded index ($t\in\mathbb{C}$); 2) the function $f(\sqrt{z_1z_2})$ is of unbounded index in each direction $\mathbf{b}$.*

Problem 2 ([1, Problem 18]). *Construct an entire function F of n variables such that $F(z^0+t\mathbf{b})$ is of bounded l_{z^0}-index for any $z^0\in\mathbb{C}^n$, but $F(z)$ is of unbounded L-index in the direction $\mathbf{b}=(b_1,\dots,b_n)$, where $n\geq 3$, $l_{z^0}(t)=L(z^0+t\mathbf{b})$.*

Our present goal is *to construct an entire function with the properties described in the problem.*

To solve Problem 2 we need some notations and propositions. For $\eta>0$, $z\in\mathbb{C}^n$, $\mathbf{b}=(b_1,\dots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ and a positive continuous function $L\colon\ \mathbb{C}^n\to\mathbb{R}_+$ we define

$$\lambda_{\mathbf{b}}(\eta)=\sup_{z\in\mathbb{C}^n}\sup_{t_1,t_2\in\mathbb{C}}\left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})}\colon |t_1-t_2|\leq\frac{\eta}{\min\{L(z+t_1\mathbf{b}),L(z+t_2\mathbf{b})\}}\right\}.$$

By $Q_{\mathbf{b}}^n$ we denote the class of functions L such that $\lambda(\eta)$ is finite for any $\eta>0$.

Let us write $G_r^{\mathbf{b}}(F)=\bigcup_{z\colon F(z)=0}\{z+t\mathbf{b}\colon |t|<r/L(z)\}$ and let a_k^0 be zeros of the function $F(z^0+t\mathbf{b})$ for a given $z^0\in\mathbb{C}^n$. By $n_{\mathbf{b}}\big(r,z^0,1/F\big):=\sum_{|a_k^0|\leq r}1$ we denote the counting function of the zeros a_k^0 of the slice function $F(z^0+t\mathbf{b})$ in the disc $\{t\in\mathbb{C}\colon |t|\leq r\}$. If for a given $z^0\in\mathbb{C}^n$ and for all $t\in\mathbb{C}$ $F(z^0+t\mathbf{b})\equiv 0$, then we put $n_{\mathbf{b}}\big(r,z^0,1/F\big)=-1$. Denote $n_{\mathbf{b}}(r,F)=\sup_{z\in\mathbb{C}^n}n_{\mathbf{b}}(r/L(z),z,F)$.

The following theorem and its analogues for various classes of analytic functions are called the logarithmic criterion (one-dimensional proposition was obtained in [8,12]).

Theorem 2 ([5,7]). *Let $L\in Q_{\mathbf{b}}^n$, $F(z)$ be an entire function in $\mathbb{C}^n$. The function F is of bounded L-index in the direction $\mathbf{b}$ if and only if the following conditions hold*

1) *for any $r>0$ there exists $P=P(r)>0$ such that for each $z\in\mathbb{C}^n\setminus G_r(F)$ inequality $\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right|\leq PL(z)$ holds;*
2) *for any $r>0$ $n_{\mathbf{b}}(r,F)$ is finite.*

Note that the class of entire functions of bounded L-index in direction is closed under multiplication. It follows from the next theorem.

Theorem 3 ([4]). *Let $L\in Q_{\mathbf{b}}^n$, $F(z)$ be an entire function of bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, $\Phi(z)$ be an entire function in $\mathbb{C}^n$ and $\Psi(z)=F(z)\Phi(z)$. The function $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$ if and only if the function $\Phi(z)$ is of bounded L-index in the direction $\mathbf{b}$.*

One-dimensional analog of the theorem was obtained for bounded index in [8] and for bounded l-index in [12].

Remark. Reading the proof of Theorem 1 in [3], one should observe that authors proves $n_{\mathbf{b}}(r^*) = +\infty$ for some $r^* > 0$. By Theorem 2 it yields unboundedness of index in the direction $\mathbf{b}$.

Our main result is following.

Theorem 4. *Let $f(t)$, $t \in \mathbb{C}$, be an even entire transcendental function of bounded index and $F(z) = \prod_{1\le i<j\le n} f(\sqrt{z_i z_j})$, $z \in \mathbb{C}^n$. Then: 1) for each direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and for every fixed $z^0 \in \mathbb{C}^n$ the function $g(t) = F(z^0 + t\mathbf{b})$ is an entire function of bounded index ($t \in \mathbb{C}$); 2) the function $F(z)$ is of unbounded index in each direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.*

Proof. Let $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be given. Then there exists at least one $j' \in \{1, \ldots, n\}$ such that $b_{j'} \neq 0$.

Consider the function $f(\sqrt{z_i z_j})$.

If $b_i = 0$ and $b_j = 0$ then for every fixed $z_i^0 \in \mathbb{C}$ and $z_j^0 \in \mathbb{C}$ the function

$$f(\sqrt{(z_i^0 + tb_1)(z_j^0 + tb_2)}) \equiv f(\sqrt{z_i^0 z_j^0}),$$

that is a constant function independent of t. Thus, it is of bounded index.

If $b_i \neq 0$ or $b_j \neq 0$ then by item 1) of Theorem 1 the function $f(\sqrt{(z_i^0 + tb_i)(z_j^0 + tb_j)})$ has bounded index as a function of variable $t \in \mathbb{C}$. Applying Theorem 3 in the case $n = 1$, $\mathbf{b} = 1$ and $L(z) \equiv 1$ we conclude that for every fixed $z^0 = (z_1^0, \ldots, z_n^0) \in \mathbb{C}^n$ the function

$$F(z^0 + t\mathbf{b}) = \prod_{1\le i<j\le n} f\Big(\sqrt{(z_i^0 + tb_i)(z_j^0 + tb_j)}\Big)$$

is of bounded index. Thus, item 1) of Theorem 4 is completely proved.

Denote $\mathbf{b}_{i,j} = (b_i, b_j)$ and $f_{i,j} = f(\sqrt{z_i z_j})$. Remind that $F(z) = \prod_{1\le i<j\le n} f(\sqrt{z_i z_j})$. Therefore, $n_{\mathbf{b}}(r, F) = \sum_{1\le i<j\le n} n_{\mathbf{b}_{i,j}}(r, f_{i,j})$. By Theorem 1 and the remark there exists $r^* > 0$ such that $n_{\mathbf{b}_{i,j'}}(r, f_{i,j'}) = +\infty$ or $n_{\mathbf{b}_{j',i}}(r, f_{j',i}) = +\infty$, where $b_{j'} \neq 0$. Hence, $n_{\mathbf{b}}(r^*, F) = +\infty$, i.e. the condition 2) of Theorem 2 is not satisfied. So, by Theorem 2 the function F is of unbounded index in the direction $\mathbf{b}$. □

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