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THE INTERPOLATION FUNCTIONAL POLYNOMIAL: THE ANALOGUE OF THE TAYLOR FORMULA

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The paper deals with a functional Newton type polynomial, which has two properties: the first one is that interpolation nodes are continual, that is, they depend on continuous parameters, and the second one is the invariance of the interpolation formulas with respect to polynomials of the corresponding degree. The first property is provided by the substitution rule, the fulfillment of which for a given functional is a sufficient condition for the possibility of the interpolation in the space of piecewise continuous functions on $[0,1]$ with a finite number of points of discontinuity of the first kind. On the basis of Newton's interpolation formulas, using the multiplicity of nodes by means of a passage to the limit, an interpolation functional Taylor type polynomial, which has the above-mentioned properties, is constructed.

1. Introduction. The generalization of the classical theory of single variable function interpolation to the case of nonlinear functionals and operators has been studied in many papers (see, e.g., [1, 3–7, 10–13]). The operator interpolation Newton-type formulas occupy here rather a significant place. Since the majority of known formulas include continual information about interpolated operators, and the set of nodes is discrete, V. L. Makarov and V. V. Khlobystov (see, e.g. [6, 7, 9]) propose to construct Newton-type polynomials on the continual set of nodes. These interpolation polynomials differ from those described in the previous papers by obtaining simultaneously two properties: the first one is that interpolation nodes are continual, that is, they depend on continuous parameters, and the second one is the invariance of interpolation formulas with respect to polynomials of the respective degree, i.e. there is a property of preservation of polynomial. The first property is provided by the substitution rule, the fulfillment of which for a given functional is a sufficient condition for the possibility of the interpolation in the space $Q[0,1]$ of piecewise continuous functions on $[0,1]$ with a finite number of points of discontinuity of the first kind.

In the current paper, it is proposed to construct functional Taylor type polynomials on the basis of Newton interpolation formulas using the multiplicity of nodes and applying passage to a limit.

2. Statement of the problem. In paper [8], it is constructed and studied an interpolation Newton type polynomial for the functional $F\colon Q[0,1]\to\mathbb{R}^1$ on the continual set of nodes

$$x^n(\cdot,\boldsymbol{\xi}^n)=x_0(\cdot)+\sum_{i=1}^{n}H(\cdot-\xi_i)[x_i(\cdot)-x_{i-1}(\cdot)], \tag{1}$$

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$$\boldsymbol{\xi}^n = (\xi_1, \xi_2, \ldots, \xi_n) \in \Omega_{z^n} = \{\boldsymbol{z}^n \colon 0 \le z_1 \le \ldots\ldots \le z_n \le 1\},$$

where $x_i(t) \in Q[0,1]$, $i = \overline{1,n}$, and $H(t)$ is the Heaviside function.

According to [8], the following theorem is valid.

Theorem 1. *The functional $F\colon Q[0,1] \to \mathbb{R}^1$ has the following representation*

$$F(x(\cdot)) = P_n^I(x(\cdot)) + R_n(x(\cdot)), \tag{2}$$

where the interpolation polynomial can be written as

$$P_n^I(x(\cdot)) = F(x_0(\cdot)) + \int_0^1 K_1(z_1)[x(z_1) - x_0(z_1)]\,dz_1 + \ldots + \\ + \int_0^1 \int_{z_1}^1 \ldots \int_{z_{n-1}}^1 K_n(\boldsymbol{z}^n) \prod_{i=1}^n [x(z_i) - x_{i-1}(z_i)]\,dz_n \ldots dz_1 \tag{3}$$

with kernels

$$K_p(\boldsymbol{z}^p) = (-1)^p \prod_{i=1}^p [x_i(z_i) - x_{i-1}(z_i)]^{-1} \frac{\partial^p}{\partial z_1 \ldots \partial z_p} F\big(x^p(\cdot, \boldsymbol{z}^p)\big), \quad p = \overline{1,n},$$

and the remainder

$$R_n(x(\cdot)) = \\ = (-1)^{n+1} \int_0^1 \int_{z_1}^1 \ldots \int_{z_n}^1 \frac{\partial^{n+1} F\big(x^{n+1}(\cdot, \boldsymbol{z}^{n+1})\big)}{\partial z_1 \partial z_2 \ldots \partial z_{n+1}} \prod_{i=1}^n \left[\frac{x(z_i) - x_{i-1}(z_i)}{x_i(z_i) - x_{i-1}(z_i)}\right] dz_{n+1} dz_n \ldots dz_1, \tag{4} \\ x_{n+1}(z) = x(z)$$

if and only if the following substitution rule is true

$$\frac{\partial^p}{\partial z_1 \partial z_2 \ldots \partial z_p} \left[F\big(x^{p+1}(\cdot, \boldsymbol{z}^{p+1})\big)\Big|_{z_{p+1} = z_p} \right] = \\ = \left[\frac{\partial^p}{\partial z_1 \partial z_2 \ldots \partial z_p} F\big(x^{p+1}(\cdot, \boldsymbol{z}^{p+1})\big)\right]\Bigg|_{z_{p+1} = z_p} \frac{x_{p+1}(z_p) - x_{p-1}(z_p)}{x_p(z_p) - x_{p-1}(z_p)}, \quad p = \overline{1,n}.$$

This interpolant is unique, it is invariant in its class and satisfies the following interpolation conditions

$$P_n^I\big(x^n(\cdot, \boldsymbol{\xi}^n)\big), \quad \forall \boldsymbol{\xi}^n \in \boldsymbol{\Omega}_n.$$

Let $F\big(x(\cdot)\big)$ be a nonlinear functional, which acts from the class of piecewise continuous functions $F\colon Q[0,1] \to \mathbb{R}^1$, for which there are known its values and the values of its Gateaux differentials in the following directions

$$F^{(j)}(x_0(\cdot)) V_1(\cdot) H(\cdot - \xi_1) V_2(\cdot) H(\cdot - \xi_2) \ldots V_j(\cdot) H(\cdot - \xi_j) = w(x_0(\cdot), \xi_1, \xi_2, \ldots, \xi_j) \equiv \\ \equiv w(x_0(\cdot), \boldsymbol{\xi}^j), \quad j = \overline{1,n},$$

where $V_j(z)$, $\overline{1,n}$ are known functions from $Q[0,1]$.

Set the following interpolation problem. In the class of polynomial of type (2), (3), (4) find such the polynomial $P_n^T(x(\cdot))$ that fulfills conditions

$$P_n^T(x(\cdot))V_1(\cdot)H(\cdot-\xi_1)V_2(\cdot)H(\cdot-\xi_2)\ldots V_j(\cdot)H(\cdot-\xi_j)=w(x_0(\cdot),\boldsymbol{\xi}^j),$$
$$j=\overline{0,n},\ \forall\xi_i\in[0,1],\ i=\overline{1,n}. \tag{5}$$

Here we use the following definition of Gateaux differential of the functional $F(x(\cdot))$ at the point $x_0(\cdot)$ in the directions $V_1(z)H(z-\xi_1)\ldots V_j(z)H(z-\xi_j)$

$$F^{(j)}(x_0(\cdot))V_1(\cdot)H(\cdot-\xi_1)\ldots V_j(\cdot)H(\cdot-\xi_j)=$$
$$=\lim_{\alpha_1,\ldots,\alpha_j\to0}\frac{\partial^j}{\partial\alpha_1\partial\alpha_2\ldots\partial\alpha_j}F\big(x_0(\cdot)+\alpha_1V_1(\cdot)H(\cdot-\xi_1)+\ldots+\alpha_jV_j(\cdot)H(\cdot-\xi_j)\big).$$

If the substitution rule is true, then $V_1(\cdot)$ can be substituted by (1), thus the condition takes the form

$$P_n^{T(j)}(x_0(\cdot))H(\cdot-\xi_1)H(\cdot-\xi_2)\ldots H(\cdot-\xi_j)=w\big(x_0(\cdot),\boldsymbol{\xi}^j\big),\quad j=\overline{0,n},\forall\xi_i\in[0,1],i=\overline{1,n}. \tag{6}$$

3. Solution of the problem. The problem formulated above will be solved in the following way. Consider interpolant (3), (4). The sequence of functions $x_0(\cdot),x_1(\cdot),\ldots,x_n(\cdot)$ is represented as follows

$$x_0(\cdot),x_1(\cdot)=x_0(\cdot)+\alpha_1V_1(\cdot),x_2(\cdot)=x_0(\cdot)+\alpha_1V_1(\cdot)+\alpha_2V_2(\cdot),\ldots,x_n(\cdot)=x_0(\cdot)+\sum_{i=1}^{n}\alpha_iV_i(\cdot), \tag{7}$$

where $\alpha_i\in R$.

Let us substitute expressions (7) in formula (3) and pass to the limit as $\alpha_i\to0$. If we assume that corresponding Gateaux derivatives and integrals exist and the substitution rule is true, making some calculations and transformations, we get a limit polynomial $P_n^T(x(\cdot))$ of the following form

$$P_n^T(x(\cdot))=F(x_0(\cdot))+\sum_{k=1}^{n}(-1)^kq_k^T(x(\cdot)), \tag{8}$$

where

$$q_1^T(x(\cdot))=-\int_0^1(x(z_1)-x_0(z_1))\frac{\partial}{\partial z_1}\big[F'(x_0(\cdot))H(\cdot-z_1)\big]\,dz_1,$$

$$q_2^T(x(\cdot))=\int_0^1\int_{z_1}^1(x(z_1)-x_0(z_1))(x(z_2)-x_0(z_2))\frac{\partial^2}{\partial z_1\partial z_2}\big[F''(x_0(\cdot))H(\cdot-z_1)H(\cdot-z_2)\big]\,dz_2dz_1.$$

By analogy, we get

$$q_n^T(x(\cdot))=\int_0^1\int_{z_1}^1\cdots\int_{z_{n-1}}^1\frac{\partial^n}{\partial z_1\partial z_2\ldots\partial z_n}F^{(n)}(x_0(\cdot))\sum_{i=1}^{n}H(\cdot-z_i)\times$$
$$\times\prod_{j=1}^{n}(x(z_j)-x_0(z_j))\,dz_{2k}\ldots dz_1. \tag{9}$$

Theorem 2. *Let the substitution rule is true and there exist integrals* (9) *on the corresponding subset of* $Q[0,1]$. *Then the Taylor type interpolant* (8), (9) *satisfies continual interpolational conditions* (6) *and does not depend on the directions of differentiation.*

Proof. By Theorem 1 we get

$$P_n^T(x_0(\cdot)) = F(x_0(\cdot)).$$

Let us check the possibility of the interpolation of derivatives. If $n = 1$ then we obtain

$$P_n^{T'}(x_0(\cdot)) = \lim_{\alpha_1 \to 0} \frac{1}{\alpha_1}\left[P_n^T\big(x_0(\cdot) + \alpha_1 H(\cdot - \xi_1)\big) - P_n^T\big(x_0(\cdot)\big)\right] =$$
$$= \lim_{\alpha_1 \to 0} \frac{1}{\alpha_1}\left[P_1^T\big(x_0(\cdot) + \alpha_1 H(\cdot - \xi_1)\big) - P_1^T\big(x_0(\cdot)\big)\right] =$$
$$= \lim_{\alpha_1 \to 0} \frac{1}{\alpha_1}\left[F\big(x_0(\cdot) + \alpha_1 H(\cdot - \xi_1)\big) - F\big(x_0(\cdot)\big)\right] = F'\big(x_0(\cdot)\big)H(\cdot - \xi_1).$$

Fix an arbitrary node, e.g. $x_l(\cdot)$, $l = \overline{1, n}$. In view of the definition of the Gateaux derivative we get

$$P_n^{T(l)}\left(x_0(\cdot) + \sum_{i=1}^{l} \alpha_i H(\cdot - \xi_i)\right) = \lim_{\alpha_1, \ldots, \alpha_l \to 0} \frac{\partial^l}{\partial \alpha_1 \ldots \partial \alpha_l} P_n^T\left(x_0(\cdot) + \sum_{i=1}^{l} \alpha_i H(\cdot - \xi_i)\right) =$$
$$= \lim_{\alpha_1, \ldots, \alpha_l \to 0} \frac{\partial^l}{\partial \alpha_1 \ldots \partial \alpha_l} P_l^T\left(x_0(\cdot) + \sum_{i=1}^{l} \alpha_i H(\cdot - \xi_i)\right) =$$
$$= \lim_{\alpha_1, \ldots, \alpha_l \to 0} \frac{\partial^l}{\partial \alpha_1 \ldots \partial \alpha_l} F\left(x_0(\cdot) + \sum_{i=1}^{l} \alpha_i H(\cdot - \xi_i)\right) = F^{(l)}(x_0(\cdot)) \prod_{i=1}^{l} H(\cdot - \xi_i).$$

Thus, condition (6) is fulfilled.

To find the remainder of formula (8) let us substitute expressions in the form (7) and $x_{n+1}(\cdot) = x(\cdot)$ into (4) instead of $x_0(\cdot), x_1(\cdot), \ldots, x_n(\cdot)$, then pass to the limit as $\alpha_i \to 0$. We get the remainder of the formula (8) in the form

$$R_n^T(x(\cdot)) =$$
$$= (-1)^{n+1} \frac{\partial^n}{\partial \alpha_1 \ldots \partial \alpha_n} \int_0^1 \int_{z_1}^1 \cdots \int_{z_n}^1 \frac{\partial^{n+1} F^{(n)}\big(x_0(\cdot) + (x(\cdot) - x_0(\cdot))H(\cdot - z_{n+1})\big)}{\partial z_1 \partial z_2 \ldots \partial z_{n+1}} \times$$
$$\times \prod_{i=1}^{n} [x(z_i) - x_{i-1}(z_i)]\, dz_{n+1} dz_n \ldots dz_1. \tag{10}$$

Thus, in accordance with the theorem we state that the following representation is true

$$F(x(\cdot)) = P_n^T(x(\cdot)) + R_n^T(x(\cdot)),$$

where the interpolant $P_n^T(x(\cdot))$ is determined by formulas (8)—(9), and the remainder $R_n^T(x(\cdot))$ is determined by formula (10). □

Example. Consider the functional

$$F(x(\cdot)) = \left(\int_0^1 x(t)\, dt\right)^3,$$

$x(t) \in Q[0, 1]$. Let us construct $P_2^T(x(\cdot))$. We have

$$P_2^T(x(\cdot)) = F(x_0(\cdot)) + 3\left(\int_0^1 x_0(\cdot)\, ds\right)^2 \int_0^1 (x(z_1) - x_0(z_1))\, dz_1 +$$

$$+6\int_0^1 x_0(\cdot)\,ds\int_0^1 (x(z_1)-x_0(z_1))\int_{z_1}^1 (x(z_2)-x_0(z_2))\,dz_2dz_1.$$

This is a Taylor type interpolation polynomial, which fulfills the interpolation conditions

$$P_n^T(x_0(\cdot)) = F(x_0(\cdot)),$$

$$P_n^{T'}(x_0(\cdot))H(\cdot-\xi_1) = F'(x_0(\cdot))H(\cdot-\xi_1),$$
$$P_n^{T''}(x_0(\cdot))H(\cdot-\xi_1)H(\cdot-\xi_2) = F''(x_0(\cdot))H(\cdot-\xi_1)H(\cdot-\xi_2).$$

Remark 1. Since Taylor formula (8) is deduced from the Newton formula by means of passing to the limit, the Taylor type interpolant $P_n^T(x_0(\cdot))$ is also unique and invariant with respect to the polynomials of the same power.

Remark 2. In paper [2] the author proposes the construction of Newton type functional polynomial, which does not require fulfillment of the substitution rule that is achieved by means of extending of the polynomial class in which the interpolant is being searched.

Therefore, if $n=2$ then the interpolating polynomial should be found in the form

$$P_2^{mN}(x(\cdot)) = F(x_0(\cdot)) + \int_0^1 K_1(z_1)[x(z_1)-x_0(z_1)]\,dz_1 + p_2^{mN}(x(\cdot)), \tag{11}$$

where

$$p_2^{mN}(x(\cdot)) = p_{2,0}(x(\cdot)) + p_{2,1}(x(\cdot)) = \int_0^1\int_{z_1}^1 K_2(\boldsymbol{z}^2)\prod_{i=1}^{2}[x(z_i)-x_{i-1}(z_i)]\,dz_2dz_1+$$
$$+\int_0^1 K_{2,1}(z_1)\prod_{i=1}^{2}[x(z_1)-x_{i-1(z_1)}]\,dz_1,$$
$$K_p(\boldsymbol{z}^p) = (-1)^p\prod_{i=1}^{p}[x_i(z_i)-x_{i-1}(z_i)]^{-1}\frac{\partial^p}{\partial z_1\ldots\partial z_p}F\big(x^p,(\cdot,\boldsymbol{z}^p)\big),\quad p=1,2\ ,$$
$$K_{21} = -[x_2(z_1)-x_0(z_1)]^{-1}[x_2(z_1)-x_1(z_1)]^{-1}\left\{\frac{d}{dz_1}F\big(x^2(\cdot,\boldsymbol{z}^2)\big)\Big|_{z_2=z_1}-\right.$$
$$\left.-\frac{x_2(z_1)-x_0(z_1)}{x_1(z_1)-x_0(z_1)}\frac{\partial F\big(x^2(\cdot,\boldsymbol{z}^2)\big)}{\partial z_1}\Big|_{z_2=z_1}\right\},$$

Polynomial (11) is interpolating on the continual set of nodes $x^2(z,\boldsymbol{\xi}^2)$, $0\le\xi_1\le\xi_2\le 1$ if and only if its kernels are determined by the mentioned formulas. It is easy to check that it is not possible to construct an interpolation polynomial with double node $x_0(\cdot)$ on its basis.

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