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A. Z. Mokhonko, A. A. Mokhonko

FAST GROWING MEROMORPHIC SOLUTIONS OF THE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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Systems of linear differential equations that allow for dimension decrease are considered. Growth estimates for meromorphic vector-solutions are obtained. An essentially new feature is that there are no additional constraints for the growth order of the system coefficients.

Let M be the field of meromorphic in $\mathbb{C}$ functions, let $\mathcal{E}$ be the ring of entire functions, $\mathcal{E}\subset M$. Consider the system

$$\frac{dw_j}{dz}=\sum_{k=1}^{n} a_{j,k}w_k, \quad a_{j,k}\in\mathcal{E}, \quad j=1,\dots,n. \tag{1}$$

According to [1, Chapter 1, § 5], every vector-solution $W(z)=(w_1(z),\dots,w_n(z)),\ \ z\in\mathbb{C}$, of the system (1) has components $w_j\in\mathcal{E},\ j=1,\dots,n$. Applications of the Nevanlinna theory to analytic theory of differential equations are widely known, see [2]–[4]. In particular in the proof of Theorem 1 we follow the approach from [2].

Let A be the coefficients matrix of the system (1):

$$A=B_0(z)=\begin{pmatrix} s_1 & p_1 & 0 & \dots & 0\\ a_{2,1} & s_2 & p_2 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & p_{n-1}\\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & s_n \end{pmatrix}, \quad a_{j,k},\ s_j,\ p_i\in\mathcal{E}. \tag{2}$$

In [2] the properties of vector-solutions of the system (1), (2) were studied. Here the coefficients $a_{j,k},\ s_j,\ p_i$, were entire functions of finite growth rate. In this paper a significantly new feature is that we do not pose any restrictions on the growth rate of the coefficients and solutions. The scale from [4] is used in Theorem 1 to measure an arbitrarily growth rate of positive functions.

The major idea that was used in the proof by [2] was to decrease the system dimension. This transformation leads to the system with meromorphic coefficients and meromorphic components of a vector-solution (see (42), (43)). In Theorem 2 we obtain the estimates for

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the growth rate of meromorphic vector-solutions for the system of linear differential equations with meromorphic coefficients.

Let us use the standard notations of the theory of meromorphic functions [6]. Landau symbols $O(\ldots)$, $o(\ldots)$ are used in this article at $r \to +\infty$. Growth rate of $f \in M$ is described by Nevanlinna characteristics $m(r, f)$, $T(r, f)$; remind

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\varphi})| d\varphi, \quad \ln^+ x = \max(\ln x, 0), \quad x \geqslant 0.$$

If f is an entire function then $T(r, f) = m(r, f)$. Let us denote $D(r, f)$ to be any of the characteristics $T(r, f),\ m(r, f)$. If $f,\ g \in M$, then [6, pp. 44, 45]

$$D(r, f + g) \leqslant D(r, f) + D(r, g) + \ln 2,$$
$$D(r, f \cdot g) \leqslant D(r, f) + D(r, g), \quad T(r, \frac{f}{g}) \leqslant T(r, f) + T(r, g) + O(1). \tag{3}$$

As E let us denote some sets of intervals on $[0, +\infty)$ with a finite sum of lengths ($\operatorname{mes} E < +\infty$). A function $f \in M$ has a finite growth order $\rho[f]$ if

$$\rho = \rho[f] = \limsup_{r \to +\infty} \frac{\ln T(r, f)}{\ln r} < +\infty. \tag{4}$$

If $f \in M$ then the following relations are known to be true ([6, pp. 122, 125, 131])

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\ln r), \text{ if } \rho[f] < +\infty, \ k = 1, 2, \ldots\ ; \tag{5}$$

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\ln^+ T(r, f) + \ln r), \ r \notin E, \text{ if } \rho[f] = +\infty, \ k = 1, 2, \ldots \tag{6}$$

If $F(f_1, \ldots, f_n)$ is a rational function of $f_j \in M$, $\deg_{f_j} F = k_j$, $j = 1, \ldots, n$, then ([7])

$$T(r, F(f_1, \ldots, f_n)) \leqslant \sum_{j=1,\ldots,n} k_j T(r, f_j) + O(1); \tag{7}$$

if $R(f_1, \ldots, f_n)$ is a polynomial in $f_j \in M$, $\deg_{f_j} R = k_j$, $j = 1, \ldots, n$, then

$$m(r, R(f_1, \ldots, f_n)) \leqslant \sum_{j=1,\ldots,n} k_j m(r, f_j) + O(1). \tag{8}$$

If $F(z) = \frac{P(z,f(z))}{Q(z,f(z))} = \frac{a_{1t}f^t + \ldots + a_{11}f + a_{10}}{a_{2m}f^m + \ldots + a_{21}f + a_{20}}$, where $f,\ a_{ij} \in M$; $a_{1t}, a_{2m} \not\equiv 0$; $d = \max(m, t)$ and $P(z, w), Q(z, w)$ are relatively prime as polynomials in w over the field M then ([8])

$$T(r, F) = d\,T(r, f) + O\Big(\sum_{i,j} T(r, a_{ij})\Big). \tag{9}$$

Let $W(z) = (w_1(z), \ldots, w_n(z))$, $w_j \in M$, $j = 1, \ldots, n$. Denote

$$T(r, W) = \max_{j=1,\ldots,n} T(r, w_j). \tag{10}$$

If the system (1), (2) has transcendental coefficients, then the components of its vector-solutions $W(z) = (w_1(z), \dots, w_n(z))$ can be entire functions of infinite growth order $\rho[w_j]$ (see (4)). There are several scales for measuring growth order of the functions with the infinite growth rate. In the paper [9] for growth rate of linear differential equations solutions p-th iteration order $\rho_p(f)$ was used. In the article [10] $[p,q]$-order $\sigma_{[p,q]}(f)$ was applied. The definitions of these orders do not describe an arbitrary growth rate. This means that there exists a function $f \in \mathcal{E}$ that has an infinite $[p,q]$-rate and p-th iteration order for arbitrary $p \in \mathbb{N}$. There is no such a drawback in the scale proposed in [11] and adopted for various applications in [4]. As Φ let us denote the class of positive unbounded non-decreasing functions $\varphi : (0,+\infty) \to (0,+\infty)$ such that $\varphi(e^t)$ is slowly growing

$$\forall c > 0 : \ \frac{\varphi(e^{ct})}{\varphi(e^t)} \to 1, \quad t \to +\infty. \tag{11}$$

Thus if $f \in M,\ \varphi \in \Phi$ then the growth orders are defined as:

$$\sigma^0_\varphi[f] = \limsup_{r\to+\infty} \frac{\varphi(e^{T(r,f)})}{\ln r}, \quad \sigma^1_\varphi[f] = \limsup_{r\to+\infty} \frac{\varphi(T(r,f))}{\ln r}. \tag{12}$$

From (11) it follows $\forall c > 0 : \varphi((e^t)^c) = (1+o(1))\varphi(e^t),\ t \to +\infty$; if we denote $x = e^t$ then the previous iimplies

$$\forall \varphi \in \Phi \quad \forall c > 0 : \ \ \varphi(x^c) = (1+o(1))\varphi(x), \ \ x > x_0. \tag{13}$$

For the functions $\varphi \in \Phi$ it holds ([4])

$$\forall \varphi \in \Phi \quad \forall m > 0 \quad \forall k \geqslant 0 : \quad \frac{\varphi^{-1}(\ln x^m)}{x^k} \to +\infty.$$

In particular, $\forall \varphi \in \Phi\ \ \forall m > 0 : \ \ x < \varphi^{-1}(\ln x^m), \ \ x > x_0$. Thus

$$\forall \varphi \in \Phi \ \ \forall m > 0 : \ \ \ln x < \ln \varphi^{-1}(\ln x^m), \ \ x > x_0. \tag{14}$$

Due to the result of Filevych ([12]) we have:

$$(\forall f \in \mathcal{E},\ \rho[f] = +\infty)\ (\exists \varphi \in \Phi) : \ \ \sigma^0_\varphi[f] = 1. \tag{15}$$

This means that the function f has a finite positive growth order $\sigma^0_\varphi[f]$. This statement allows estimating the growth order of vector-solutions of the fundamental system of solutions of (1), (2) via the growth order of its coefficients.

If $\sigma^1_\varphi[f] = \sigma < +\infty$ then taking into account (12) we have $\forall \varepsilon > 0 : \ \ \varphi(T(r,f)) < \ln r^{\sigma+\varepsilon},\ \ r > r_0$. Then

$$\sigma^1_\varphi[f] = \sigma \ \ \Rightarrow \ \ T(r,f) < \varphi^{-1}(\ln r^{\sigma+\varepsilon}), \ \ \varepsilon > 0, \ \ r > r_0. \tag{16}$$

If $g \in M$ and $\sigma^0_\varphi[g] = \alpha < +\infty$ then by taking into account (12) we obtain $\forall \varepsilon > 0 :$ $\varphi(e^{T(r,g)}) < \ln r^{\alpha+\varepsilon}, \ \ r > r_0$. Thus

$$\sigma^0_\varphi[g] = \alpha \ \ \Rightarrow \ \ T(r,g) < \ln \varphi^{-1}(\ln r^{\alpha+\varepsilon}), \ \ \varepsilon > 0, \ \ r > r_0. \tag{17}$$

Denote, see (2)$(j = 1, \ldots, n;\ \ t = 1, \ldots, n - j + 1)$

$$d_{jt}(A) = \begin{vmatrix} s_t & p_t & 0 & \ldots & 0 \\ a_{t+1,t} & s_{t+1} & p_{t+1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{t+j-2,t} & a_{t+j-2,t+1} & a_{t+j-2,t+2} & \cdots & p_{t+j-2} \\ a_{t+j-1,t} & a_{t+j-1,t+1} & a_{t+j-1,t+2} & \cdots & s_{t+j-1} \end{vmatrix}, \tag{18}$$

$$d_{0,t} \equiv 1, \qquad H_j(A) = \sum_{t=1}^{n+1-j} d_{j,t}(A). \tag{19}$$

The main result of this article is the following

Theorem 1. *Let the system (1), (2) be such that all coefficients $a_{j,k}$, s_j, $p_i \in \mathcal{E}$, and $m \in \{0, 1, \cdots, n-1\}$*

$$\sigma^0_\varphi[H_{n-m}(A)] > \sigma^0_\varphi[d_{jt}(A)], \ \ j = 1, 2, \ldots, n - m - 1;\ \ t = 1, \ldots, n - j + 1. \tag{20}$$

Then there exist no $m+1$ linear independent meromorphic vector-solutions $W_k(z) = (w_{k1}(z), \ldots, w_{kn}(z))$, of the system (1), (2) such that

$$\sigma^1_\varphi[W_k] < \sigma^0_\varphi[H_{n-m}(A)], \quad k = 0, 1, \ldots, m. \tag{21}$$

The following Theorem 2 is similar to Theorem 1, though they do not follow one from another. If in the system (1), (2) the coefficients $a_{j,k}$, s_j, $p_i \in M$ and P is the set of poles of all coefficients, then according to [1, Chapter 1, §5] every vector-solution has components, that are analytic functions in $\mathbb{C} \setminus P$. We are interested in vector-solutions $W(z) = (w_1(z), \ldots, w_n(z))$ with components $w_j \in M,\ j = 1, \ldots, n$.

Theorem 2. *Let the system (1), (2) be such that all coefficients $a_{j,k}$, s_j, $p_i \in M$, and $(m \in \{0, 1, \cdots, n-1\}, j = 1, 2, \cdots, n - m - 1)$*

$$m(r, d_{jt}(A)) = o(m(r, H_{n-m}(A))), \ \ r \notin E; \quad t = 1, \ldots, n - j + 1. \tag{22}$$

Then there exists no $m + 1$ linear independent meromorphic vector-solutions $W_k(z) = (w_{k1}(z), \ldots, w_{kn}(z))$, $k = 0, 1, \ldots, m$, of the system (1), (2) such that $\ln(r \cdot T(r, W_k)) = o(m(r, H_{n-m}(A)))$, $r \notin E$; (whose growth rate is restricted by growth rate of the coefficients).

Remark 1. If we apply more precise estimates of logarithmic derivative (5) for important sub-classes of meromorphic functions then the following can be obtained: if the coefficients of the system (1), (2) are such that

$$\begin{gathered} m(r, d_{jt}(A)) = O(\ln r), \ \ j = 1, 2, \ldots, n - m - 1;\ \ t = 1, \ldots, n - j + 1; \\ m(r, H_{n-m}(A)) \neq O(\ln r), \end{gathered} \tag{23}$$

then the system has no more than m linearly-independent meromorphic vector-solutions $W_k,\ k = 1, 2, \ldots, m$ of finite growth order. The relations (23) hold true if e.g. $d_{jt}(A)$ are any rational functions and $H_{n-m}(A)$ is transcendent function. In fact, a transcendent function grows faster than any rational function [6, pp. 49, 50].

Example 1. Consider the system $w_1' = w_2,\ w_2' = e^{2z}w_1 + w_2$. The matrix of the system is $A = \begin{pmatrix} 0 & 1 \\ \exp 2z & 1 \end{pmatrix}$, $d_{11}(A) = 0,\ d_{12} = 1;\ H_2(A) = -e^{2z},\ m(r, H_2(A)) = 2m(r, e^z) = \frac{2r}{\pi}$ ([13, p. 25]). We have: $0 = m(r, d_{11}(A)) = m(r, d_{12}(A)) = o(m(r, H_{2-0}(A)))$. In this example $n = 2$, $m = 0$. Thus from Theorem 2 it follows that the system does not have meromorphic vector-solutions W such that $\ln^+ T(r, W) + \ln r = o(m(r, H_{2-0}(A))),\ r \notin E$. This system has two linearly-independent meromorphic vector-solutions $W_1 = \left(e^{e^z},\ e^z e^{e^z}\right), W_2 = \left(e^{-e^z},\ -e^z e^{-e^z}\right)$. For entire function $\exp\exp z$ ([13, p. 26]) $T(r, e^{e^z}) = m(r, e^{e^z}) \sim \frac{e^r}{(2\pi^3 r)^{1/2}},\ r \to +\infty$. Taking into account (9) it follows $T(r, e^z e^{-e^z}) = T(r, e^{e^z}) + O(T(r, e^z)) \sim \frac{e^r}{(2\pi^3 r)^{1/2}},\ r \to +\infty$. Thus keeping in mind the definition $W_1,\ W_2$, we obtain $T(r, W_j) \sim \frac{e^r}{(2\pi^3 r)^{1/2}},\ r \to +\infty,\ j = 1, 2$. Thus $r \sim \ln(r \cdot T(r, W_j)) \neq o(m(r, H_2(A))),\ r \to +\infty$ because $m(r, H_2(A)) \sim \frac{2r}{\pi},\ r \to +\infty$.

Example 2. The system $w_1' = w_2,\ w_2' = w_2(1 + e^z)$ has the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 1+e^z \end{pmatrix}$; $H_1 = H_{2-1}(A) = 1 + e^z;\ n = 2,\ m = 1;\ \ m(r, H_1(A)) = m(r, e^z + 1) = m(r, e^z) + O(1) \sim \frac{r}{\pi}$, $r \to +\infty$. A fundamental system consists of two linearly independent meromorphic vector-solutions. According to Theorem 2 this fundamental system has no more than one meromorphic vector-solution W such that $\ln^+ T(r, W) + \ln r = o(m(r, H_1(A))),\ r \notin E$. This solution is $W_1 = (1,\ 0)$. The second linearly independent solution of the fundmental system is $W_2 = \left(e^{e^z},\ e^z e^{e^z}\right)$. Similar to Example 1 $T(r, W_2) \sim \frac{e^r}{(2\pi^3 r)^{1/2}},\ \ \ln^+ T(r, W_2) + \ln r \sim r,\ r \to +\infty$. Thus $r \sim \ln(r \cdot T(r, W_2)) \neq o(m(r, H_1(A))),\ r \to +\infty$, because $m(r, H_1(A)) \sim \frac{r}{\pi}$, $r \to +\infty$.

Let us consider the vector $h(z) = (h_1, h_2, \ldots, h_n)$ where $h_j \in M$. Denote

$$Q_0(A, h) \equiv 1,\ \ Q_k(A, h) = \begin{vmatrix} s_1 - h_1 & p_1 & 0 & \ldots & 0 \\ a_{2,1} & s_2 - h_2 & p_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \ldots & p_{k-1} \\ a_{k,1} & a_{k,2} & a_{k,3} & \ldots & s_k - h_k \end{vmatrix}, \tag{24}$$

$k = 1, 2, \ldots, n$. By using (24) we have ($d_{1,k} = s_k$)

$$Q_k = -h_k Q_{k-1} + \begin{vmatrix} s_1 - h_1 & p_1 & 0 & \ldots & 0 & 0 \\ a_{2,1} & s_2 - h_2 & p_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \ldots & s_{k-1} - h_{k-1} & p_{k-1} \\ a_{k,1} & a_{k,2} & a_{k,3} & \ldots & a_{k,k-1} & s_k \end{vmatrix} =$$

$$= -h_k Q_{k-1} - Q_{k-2} h_{k-1} d_{1,k} + \begin{vmatrix} s_1 - h_1 & p_1 & 0 & \ldots & 0 & 0 \\ a_{2,1} & s_2 - h_2 & p_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \ldots & s_{k-1} & p_{k-1} \\ a_{k,1} & a_{k,2} & a_{k,3} & \ldots & a_{k,k-1} & s_k \end{vmatrix} =$$

$$= \ldots = d_{k,1}(A) - \sum_{i=0}^{k-1} Q_i(A, h) h_{i+1} d_{k-i-1,i+2}(A),\ \ d_{0,k+1}(A) = 1. \tag{25}$$

Lemma 1. *The determinant $Q_k(A, h)$ can be represented as*

$$Q_k(A, h) = d_{k1}(A) - d_{k-1,1}(A) h_k + \sum_{j=0}^{k-2} d_{j1}(A) P_{jk},\ \ k = 1, 2, \ldots, n, \tag{26}$$

where $h = (h_1, h_2 \ldots, h_n)$; P_{jk} is a polynomial in functions h_t and $d_{\nu s}(A)$, $j+1 \leqslant t \leqslant k$, $j+2 \leqslant s \leqslant k$, $\nu < k$, of degree at most 1 for every h_t, $d_{\nu s}$.

Proof of Lemma 1. Taking into account the definitions (24), (18) we have ($d_{01} = 1$) $Q_1(A,h) = s_1 - h_1 = d_{11} - d_{01}h_1$, $Q_2(A,h) = d_{21} - d_{11}h_2 - d_{01}(d_{12}h_1 - h_1h_2) = d_{21} - d_{11}h_2 - d_{01}P_{02}$, $Q_3(A,h) = d_{31} - d_{22}h_1 - h_2Q_1(A,h)d_{13} - h_3Q_2(A,h) = d_{31} - d_{21}h_3 + d_{11}(h_2h_3 - h_2d_{13}) + d_{01}(d_{13}h_1h_2 - h_1d_{22} + d_{12}h_1h_3 - h_1h_2h_3) = d_{31} - d_{21}h_3 + d_{11}P_{13} + d_{01}P_{03}$. The assumptions of the lemma preconditions for the polynomials P_{02}, P_{13}, P_{03} hold true.

Assume that the statement of the Lemma are proved for all Q_i, $i = 1, \ldots, k-1$. Let us prove it for Q_k, $k \geqslant 4$. By substituting into (25) the decompositions Q_i of the form (26), after simple transformation we obtain ($d_{0,k+1} = 1$)($k \geqslant 4$)

$$\begin{gathered}
Q_k = d_{k1} - h_k\Big(d_{k-1,1} - d_{k-2,1}h_{k-1} + \sum_{j=0}^{k-3} d_{j1}P_{j,k-1}\Big) - Q_1h_2d_{k-2,3} - Q_0h_1d_{k-1,2} - \\
- \sum_{i=2}^{k-2}\Big(d_{i1} - d_{i-1,1}h_i + \sum_{j=0}^{i-2} d_{j1}P_{j,i}\Big)h_{i+1}d_{k-i-1,i+2} = d_{k1} - h_kd_{k-1,1} - \\
-Q_1h_2d_{k-2,3} - Q_0h_1d_{k-1,2} - \sum_{i=2}^{k-2}\sum_{j=0}^{i-2} d_{j1}P_{j,i}h_{i+1}d_{k-i-1,i+2} - \sum\nolimits_1 - \sum\nolimits_2 + \sum\nolimits_3; \\
\sum_1 = \sum_{j=0}^{k-3} d_{j1}P_{j,k-1}h_k - d_{k-2,1}h_{k-1}h_k \overset{\text{def}}{=} \sum_{j=0}^{k-2} d_{j1}P^1_{j,k}; \\
\sum_2 = \sum_{i=2}^{k-2} d_{i1}h_{i+1}d_{k-i-1,i+2} \overset{\text{def}}{=} \sum_{i=2}^{k-2} d_{i1}P^2_{i,k}; \\
\sum_3 = \sum_{i=2}^{k-2} d_{i-1,1}h_ih_{i+1}d_{k-i-1,i+2} \overset{\text{def}}{=} \sum_{i=2}^{k-2} d_{i-1,1}P^3_{i-1,k}; \\
Q_1h_2d_{k-2,3} = d_{11}h_2d_{k-2,3} - d_{01}h_1h_2d_{k-2,3} \overset{\text{def}}{=} d_{11}P^4_{1,k} + d_{01}P^4_{0,k}; \\
Q_0h_1d_{k-1,2} = d_{01}h_1d_{k-1,2} \overset{\text{def}}{=} d_{01}P^5_{0,k}, \quad d_{01} = 1, \quad Q_0 = 1; \\
\sum_{i=2}^{k-2}\sum_{j=0}^{i-2} d_{j1}P_{j,i}h_{i+1}d_{k-i-1,i+2} = \sum_{j=0}^{k-4} d_{j1}\sum_{i=j+2}^{k-2} P_{j,i}h_{i+1}d_{k-i-1,i+2}.
\end{gathered} \tag{27}$$

From induction hypothesis about polynomial properties $P_{j,k-1}$ and the definitions of the polynomials $P^s_{j,k}$, $s = 1, 2, \ldots, 5$, $j = 0, 1, \ldots, k-2$, it follows that $P^s_{j,k}$ are some polynomials in h_t and $d_{\nu s}$, $j+1 \leqslant t \leqslant k$, $j+2 \leqslant t \leqslant k$, $\nu < k$ of degree no more than 1 in every h_t, $d_{\nu s}$. By grouping the summands that contain d_{j1}, $j = 0, 1, \ldots, k-2$ we obtain

$$\sum_1 + \sum_2 - \sum_3 - Q_1h_2d_{k-2,3} - Q_0h_1d_{k-2,2} \overset{\text{def}}{=} \sum_{j=0}^{k-2} d_{j1}P^*_{j,k}\,, \tag{28}$$

where $P^*_{j,k}$ are some polynomials in h_t and $d_{\nu s}$, $j+1 \leqslant t \leqslant k$, $j+2 \leqslant t \leqslant k$, $\nu < k$ of degree no more than 1 on every h_t, $d_{\nu s}$. After substitution of (27), (28) into the expression for Q_k and grouping the summands that contain d_{j1}, $j = 0, 1, \ldots, k-2$ we obtain (26). □

Proof of Theorem 1. The case $m = n-1$ will be considered in a more general form. Consider the system (1) with the coefficients $a_{kj} \in \mathcal{E}$ (condition (2) may not be satisfied). Let $\mathrm{Sp}A = a_{11} + a_{22} + \ldots + a_{nn} = H_1(A)$ be the trace of matrix A of the system (1). Let us prove that the system (1) does not have $m+1 = n$ linearly independent vector-solutions $W_k = (w_{k1}, \ldots, w_{kn})$ such that

$$\sigma^0_\varphi[\mathrm{Sp}A] > \sigma^1_\varphi[W_k] \overset{\text{def}}{=} \sigma_k, \quad k = 1, \ldots, n. \tag{29}$$

Assume that there exist n linearly independent vector-solutions W_k for which (29) holds true. Then $W_k,\ k = 1, \ldots, n$, is the fundamental system of solutions for (1) with the determinant $D(z)$ satisfying the equality $\frac{D'(z)}{D(z)} = \mathrm{Sp}A$. Then

$$m(r, \mathrm{Sp}A) = m\left(r, \frac{D'(z)}{D(z)}\right) \overset{(6)}{=} O(\ln^+ T(r, D) + \ln r),\quad r \notin E. \tag{30}$$

Taking into account the definition of $D(z)$ and the estimate (7) we obtain:

$$T(r, D) \overset{(10)}{\leqslant} \sum_{1 \leqslant k,j \leqslant n} T(r, w_{k,j}) + O(1) \leqslant n^2 \max_{k=1,\ldots,n} T(r, W_k) + O(1). \tag{31}$$

From (29), (16) we have $\varphi(T(r, W_k)) < \ln r^{\sigma_k+\varepsilon},\ \varepsilon > 0$. Let $\sigma = \max \sigma_k,\ k = 1, \ldots, n$. Then $T(r, W_k) < \varphi^{-1}(\ln r^{\sigma+\varepsilon}),\quad \varepsilon > 0,\quad k = 1, \ldots, n$, and by taking into account (31), $T(r, D) = O(\varphi^{-1}(\ln r^{\sigma+\varepsilon})),\ \varepsilon > 0$. Then from (30) it follows ($K = \mathrm{const} > 0$)

$$m(r, \mathrm{Sp}A) < K(\ln^+ T(r, D) + \ln r) \overset{(14)}{<} 2K(\ln \varphi^{-1}(\ln r^{\sigma+\varepsilon})),\ r \notin E. \tag{32}$$

If $r > \mathrm{mes}\, E$ then $\exists r_1 \in [r,\, 2r] \setminus E$. Since the function $\mathrm{Sp}A$ is entire then the functions $m(r, \mathrm{Sp}A),\ \ \ln \varphi^{-1}(\ln r^{\sigma+\varepsilon})$ are increasing. So we have:

$$\begin{gathered} m(r, \mathrm{Sp}A) \leqslant m(r_1, \mathrm{Sp}A) < 2K(\ln \varphi^{-1}(\ln r_1^{\sigma+\varepsilon})) \leqslant \\ \leqslant 2K(\ln \varphi^{-1}(\ln (2r)^{\sigma+\varepsilon})) < 2K(\ln \varphi^{-1}(\ln r^{\sigma+2\varepsilon})),\quad r > r_0,\quad \varepsilon > 0. \end{gathered} \tag{33}$$

Therefore $m(r, \mathrm{Sp}A) = O(\ln \varphi^{-1}(r^{\sigma+2\varepsilon})),\quad r > r_0,\quad \varepsilon > 0$. So

$$\begin{gathered} \varphi\big(e^{m(r,\mathrm{Sp}A)}\big) = \varphi\big(e^{O(\ln \varphi^{-1}(\ln r^{\sigma+2\varepsilon}))}\big) \overset{(13)}{<} (1 + o(1))\varphi\big(e^{\ln \varphi^{-1}(\ln r^{\sigma+2\varepsilon})}\big) = \\ = (1 + o(1)) \ln r^{\sigma+2\varepsilon},\quad r > r_0. \end{gathered}$$

By taking into account the definition $\sigma^0_\varphi[\mathrm{Sp}A]$ and the fact that for the entire function $m(r, \mathrm{Sp}A) = T(r, \mathrm{Sp}A)$ we obtain $\sigma^0_\varphi[\mathrm{Sp}A] \leqslant \sigma + 2\varepsilon$. Thus $\sigma^0_\varphi[\mathrm{Sp}A] \leqslant \sigma = \max \sigma_k,\ k = 1, \ldots, n$, which contradicts (29).

Suppose now that in (2) all $p_j \not\equiv 0,\ j = 1, \ldots, n-1$. If $W = (w_1, \ldots, w_n)$ is a non-trivial meromorphic vector-solution of the system (1), (2) then from matrix (2) structure it follows that $w_1 \not\equiv 0$.

Let $m = 0$. Then $n - m = n$,

$$H_{n-m}(A) = H_n(A) = d_{n1}(A). \tag{34}$$

Assume that there exists a non-trivial meromorphic solution $W = (w_1, \ldots, w_n)$ of the system (1), (2) such that

$$\sigma \overset{\mathrm{def}}{=} \sigma^1_\varphi[W] < \sigma^0_\varphi[H_n(A)] \overset{\mathrm{def}}{=} \alpha. \tag{35}$$

Let us rewrite the system (1), (2) as:

$$\begin{aligned} &w_1(s_1 - w_1'/w_1) + p_1 w_2 = 0, \\ &w_1 a_{21} + w_2(s_2 - w_2'/w_2) + w_3 p_2 = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &w_1 a_{n1} + \ldots + w_{n-1} a_{n,n-1} + w_n(s_n - w_n'/w_n) = 0. \end{aligned} \tag{36}$$

This system has a non-trivial solution $W(z)$. So (24) $Q_n(A,h_0)\equiv 0$, where $h_0=(h_{01},\ldots,h_{0n})$, $h_{0j}=w_j'/w_j,\ j=1,\ldots,n$. From Lemma 1 it follows

$$0\equiv Q_n(A,h_0)\overset{(26)}{=}d_{n1}(A)-d_{n-1,1}(A)h_{0n}+\sum_{j=0}^{n-2}d_{j1}(A)P_{jn}, \tag{37}$$

where P_{jn} are some polynomials in functions $h_{0t}=w_t'/w_t,\ j+1\leqslant t\leqslant n$, and $d_{\nu s}(A),\ \nu<n,\ j+2\leqslant s\leqslant n$, of degree no more than 1 in every of h_{0t}, and $d_{\nu s}$. Thus taking into account (34) we have

$$H_n(A)=d_{n-1,1}(A)h_{0n}-\sum_{j=0}^{n-2}d_{j1}(A)P_{jn}.$$

From this equality and properties of the polynomials P_{jn} it follows:

$$\begin{aligned}m(r,H_n(A))&\overset{(8)}{\leqslant}\sum_{j=1}^{n}m\left(r,\frac{w_j'}{w_j}\right)+\sum_{\substack{1\leqslant j\leqslant n-1,\\1\leqslant t\leqslant n-j+1}}m(r,d_{jt})+O(1)\overset{(6)}{\leqslant}\\&\leqslant O\Big(\sum_{j=1}^{n}\ln^+T(r,w_j)+\ln r\Big)+\sum_{\substack{1\leqslant j\leqslant n-1,\\1\leqslant t\leqslant n-j+1}}m(r,d_{jt}),\quad r\notin E.\end{aligned}\tag{38}$$

Inequality (20) for $m=0$ implies

$$\beta_{jt}\overset{\text{def}}{=}\sigma_\varphi^0[d_{jt}]<\sigma_\varphi^0[H_n(A)]\overset{\text{def}}{=}\alpha,\quad j=1,\ldots,n-1;\quad t=1,\ldots,n-j+1. \tag{39}$$

Let us denote $\max\beta_{jt}\overset{\text{def}}{=}\beta\overset{(39)}{<}\alpha$. For the entire function $d_{jt}(A)$ the following equality $T(r,d_{jt})=m(r,d_{jt})$ holds. Keeping in mind (17), (39) this gives us:

$$m(r,d_{jt})<\ln\varphi^{-1}(\ln r^{\beta+\varepsilon}),\quad \varepsilon>0. \tag{40}$$

From (10), (35) it follows

$$T(r,w_j)\overset{(10)}{\leqslant}T(r,W)\overset{(35),(16)}{<}\varphi^{-1}(\ln r^{\sigma+\varepsilon}),\quad \varepsilon>0,\quad r>r_0. \tag{41}$$

From (38), (41), (14), (40) we obtain ($K=\text{const}>0$) $m(r,H_n(A))<K\cdot(\ln\varphi^{-1}(\ln r^{\sigma+\varepsilon})+$ $+\ln\varphi^{-1}(\ln r^{\beta+\varepsilon}))<2K\ln\varphi^{-1}(\ln r^{\max(\sigma,\beta)+\varepsilon}),\quad r\notin E$. Similar to (33) we obtain:

$$\begin{gathered}m(r,H_n(A))=O\left(\ln\varphi^{-1}(\ln r^{\max(\sigma,\beta)+2\varepsilon})\right),\quad r>r_0,\quad \varepsilon>0;\\ e^{m(r,H_n(A))}=e^{O(\ln\varphi^{-1}(\ln r^{\max(\sigma,\beta)+2\varepsilon}))};\quad \varphi(e^{m(r,H_n(A))})=\varphi(e^{O(\ln\varphi^{-1}(\ln r^{\max(\sigma,\beta)+2\varepsilon}))})\overset{(13)}{<}\\ <(1+o(1))\varphi(e^{\ln\varphi^{-1}(\ln r^{\max(\sigma,\beta)+2\varepsilon}})=(1+o(1))\ln r^{\max(\sigma,\beta)+2\varepsilon},\quad r>r_0.\end{gathered}$$

From this estimate, from (12) and from the fact that for entire function $m(r,H_n(A))=T(r,H_n(A))$ we conclude that $\sigma_\varphi^0[H_n(A)]\leqslant\max(\sigma,\beta)$, which contradicts (35), (39).

Let $0<m<n-1$. Suppose that there exists $m+1$ linearly-independent meromorphic vector-solutions $W_k=(w_{k1},\ldots,w_{kn}),\ k=0,\ldots,m$ of the system (1), (2) such that (21) holds. One of these $m+1$ solutions e.g. W_0 we denote by U, $W_0=U=(u_1,\ldots,u_n)=(w_{01},\ldots,w_{0n})$. Any of the remaining m meromorphic vector-solutions is denoted by $W=$

$(w_1, \dots, w_n)$. Since U is a non-trivial meromorphic vector-solution of the system (1), (2) then $u_1 \not\equiv 0$. Let us describe the transformation from the system (1) with coefficient matrix (2) of the dimension n to the system of differential equations with a coefficient matrix of the form (2) and dimension $n-1$.

For every of m meromorphic vector-solutions $W = (w_1, \dots, w_n)$ of the system (1), (2) let us assign the corresponding vector

$$V = (v_1, v_2, \dots, v_n) = \left(\frac{w_1}{u_1}, w_2 - \frac{w_1 u_2}{u_1}, \dots, w_n - \frac{w_1 u_n}{u_1}\right), \quad v_1 = \frac{w_1}{u_1} \not\equiv 0. \tag{42}$$

From (1), (2), (42) it follows that these m vectors V (42) are the solutions of the system [2, formulae (3,9)–(3,13)]

$$\begin{array}{l} v_1' = v_2 p_1/u_1, \\ v_2' = v_2(s_2 - p_1 u_2/u_1) + p_2 v_3, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ v_n' = v_2(a_{n2} - p_1 u_n/u_1) + \sum\limits_{k=3}^{n-1} a_{nk} v_k + s_n v_n, \end{array} \tag{43}$$

whose coefficients matrix has the form

$$\left(\begin{array}{c|cccc} 0 & p_1/u_1 & 0 & \dots & 0 \\ \hline 0 & & & & \\ \vdots & & B_1 & & \\ 0 & & & & \end{array}\right), \quad B_1 = \begin{pmatrix} s_2 - p_1 u_2/u_1 & p_2 & 0 & \dots & 0 \\ a_{32} - p_1 u_3/u_1 & s_3 & p_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n2} - p_1 u_n/u_1 & a_{n3} & a_{n4} & \dots & s_n \end{pmatrix}. \tag{44}$$

Lemma 2. *The following relations hold true* $(d_{0,j+2}(A) = 1)$

$$\begin{array}{c} d_{j,k-1}(B_1) = d_{jk}(A), \ k > 2, \ \ 1 \leqslant j \leqslant n-2, \\ d_{j1}(B_1) = d_{j2}(A) + \sum_{k=1}^{j} Q_k(A, h) d_{j-k,k+2}(A), \ \ j = 1, 2, \dots, n-1, \end{array} \tag{45}$$

where $h = h_0 = (u_1'/u_1, \dots, u_n'/u_n) = (w_{01}'/w_{01}, \dots, w_{0n}'/w_{0n})$.

Proof of Lemma 2. If $k > 2$, then the first of equations (45) follows from the definition of $d_{j,k-1}(B_1)$ and matrix B_1 (44). If $k = 2$ then

$$d_{j,1}(B_1) \overset{(44)}{=} d_{j2}(A) - \frac{p_1}{u_1} \begin{vmatrix} u_2 & p_2 & 0 & \dots & 0 \\ u_3 & s_3 & p_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ u_j & a_{j,3} & a_{j,4} & \dots & p_j \\ u_{j+1} & a_{j+1,3} & a_{j+1,4} & \dots & s_{j+1} \end{vmatrix} = d_{j2}(A) -$$

$$-\frac{p_1 u_2}{u_1} d_{j-1,3} + \frac{p_1 p_2}{u_1} \begin{vmatrix} u_3 & p_3 & 0 & \dots & 0 \\ u_4 & s_4 & p_4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ u_j & a_{j,4} & a_{j,5} & \dots & p_j \\ u_{j+1} & a_{j+1,4} & a_{j+1,5} & \dots & s_{j+1} \end{vmatrix} = d_{j2} - \frac{p_1 u_2}{u_1} \times$$

$$\times d_{j-1,3}+\frac{p_1p_2}{u_1}u_3d_{j-2,4}(A)-\frac{p_1p_2}{u_1}p_3\begin{vmatrix} u_4 & p_4 & 0 & \dots & 0\\ u_5 & s_5 & p_5 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ u_j & a_{j,5} & a_{j,6} & \dots & p_j\\ u_{j+1} & a_{j+1,5} & a_{j+1,6} & \dots & s_{j+1}\end{vmatrix}=$$

$$=d_{j2}(A)+\sum_{k=1}^{j}(-1)^k\frac{u_{k+1}}{u_1}p_1p_2\dots p_kd_{j-k,k+2}(A).$$

It is known [2] that $(-1)^k\frac{u_{k+1}}{u_1}p_1p_2\dots p_k = Q_k(A,h)$, $h=(u_1'/u_1,\dots,u_n'/u_n)$. So we obtain (45). □

The matrix B_1 (44) has the form (2). Taking into account (43), (42) each of m vectors

$$Y_1=(v_2,v_3\dots,v_n)\stackrel{\text{def}}{=}(v_{12},v_{13},\dots,v_{1n}) \tag{46}$$

is a solution of the system of differential equations

$$Y_1'=B_1Y_1, \tag{47}$$

whose dimension is $n-1$.

By utilizing one solution $U=(u_1,\dots,u_n)$ of the previously known $m+1$ meromorphic vector-solutions of the system (1), (2) we decreased the dimension of this system by 1 and obtained the system (47), (44) that has m meromorphic vector-solutions (46). Let $Y_{11},Y_{12},\dots,Y_{1m}$ be meromorphic vector-solutions of the system (47), (44) obtained in the described above way (Y_1 is one of these solutions). Since $m+1$ meromorphic vector-solutions $W_0=U=(u_1,\dots,u_n)$, $W_j=(w_{j1},\dots,w_{jn})$, $u_1,w_{j1}\not\equiv 0$, $j=1,\cdots,m$ of the system (1), (2) are linearly independent, we obtain that m meromorphic vector-solutions $Y_{11},Y_{12},\dots,Y_{1m}$ of the system (47), (44) are also linearly independent ($Y_1=(v_{12},v_{13},\dots,v_{1n})$, $v_{12}\not\equiv 0$). From (42), (46), (10), (7) we obtain

$$T(r,Y_1)\stackrel{\text{def}}{=}\max_{j=2,3,\dots,n}T(r,v_j)\stackrel{(42),(7)}{\leqslant}\sum_{\substack{0\leqslant i\leqslant m,\\ 1\leqslant j\leqslant n}}T(r,w_{i,j})+O(1). \tag{48}$$

Then $W_i=(w_{i1},\dots,w_{in})$, $T(r,W_i)\stackrel{(10)}{=}\max_{j=1,\dots,n}T(r,w_{ij})$; $i=0,1,\dots,m$;

$$\sum_{j=1}^{n}T(r,w_{i,j})\leqslant n\max_{j=1,\dots,n}T(r,w_{i,j})=nT(r,W_i); \tag{49}$$

$$\sum_{i=0}^{m}\sum_{j=1}^{n}T(r,w_{i,j})\stackrel{(49)}{\leqslant}\sum_{i=0}^{m}nT(r,W_i)\leqslant n(m+1)\max_{i=0,1,\dots,m}T(r,W_i).$$

Thus from (48) it follows

$$\max T(r,Y_1)\stackrel{\text{def}}{=}\max_{t=1,\dots,m}T(r,Y_{1,t})=O\Big(\max_{i=0,1,\dots,m}T(r,W_i)\Big). \tag{50}$$

Under transformation (42) $m+1$ linearly-independent meromorphic vector-solutions $W_k(z)$, $k=0,1,\dots,m$ of the system (1), (2) become m linearly-independent meromorphic vector-solutions $Y_{11},Y_{12},\dots,Y_{1m}$ of the form (46) of the system (47), (44) for which the estimate (50) is valid.

By using solutions $Y_{11}, Y_{12}, \dots, Y_{1m}$ let us decrease the dimension of the matrix A another $m-1$ times and receive the systems of differential equations

$$Y_k' = B_k Y_k, \quad k = 1, 2, \dots, m, \tag{51}$$

of the dimension $n-k$ where $Y_k = (v_{k,k+1}, v_{k,k+2}, \dots, v_{k,n})$ and the matrix

$$B_k = \begin{pmatrix} s_{k+1} - p_k v_{k-1,k+1}/v_{k-1,k} & p_{k+1} & 0 & \dots & 0 \\ a_{k+2,k+1} - p_k v_{k-1,k+2}/v_{k-1,k} & s_{k+2} & p_{k+2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n,k+1} - p_k v_{k-1,n}/v_{k-1,k} & a_{n,k+2} & a_{n,k+3} & \dots & s_n \end{pmatrix}. \tag{52}$$

By applying the estimate (50) of the meromorphic vector-solution of the system (51) several times ($k = 1, \dots, m$) we finally obtain

$$\max T(r, Y_k) \overset{\text{def}}{=} \max_{t=1,2,\dots,m-k+1} T(r, Y_{k,t}) = O\Big(\max_{i=0,1,\dots,m} T(r, W_i)\Big). \tag{53}$$

For the meromorphic solution $Y_k = (v_{k,k+1}, v_{k,k+2}, \dots, v_{k,n})$ of the system (51), (52) let us put into the correspondence the vector $h_k = (h_{k,k+1}, h_{k,k+2}, \dots, h_{k,n})$, where $h_{k,k+p} = v'_{k,k+p}/v_{k,k+p}$, $p = 1, 2, \dots, n-k$; $k = 1, 2, \dots, m$. Then

$$\begin{aligned} m(r, h_{k,k+p}) = m(r, \tfrac{v'_{k,k+p}}{v_{k,k+p}}) &\overset{(6)}{=} O(\ln^+ T(r, v_{k,k+p}) + \ln r) \overset{(10)}{=} \\ = O\Big(\ln^+\Big(&\max_{t=1,2,\dots,m-k+1} T(r, Y_{k,t})\Big) + \ln r\Big) \overset{(53)}{=} \\ = O\Big(\ln^+\Big(\max_{i=0,1,\dots,m} T(r, W_i)\Big) + \ln r\Big), &\quad p = 1, \dots, n-k; \ k = 1, \dots, m, \end{aligned} \tag{54}$$

$r \notin E$. We will use the following lemma.

Lemma 3. *The following equality holds true* ($j \in \mathbb{N}$, $j \leqslant n-m$)

$$d_{j1}(B_m) = d_{j,m+1}(A) + d_{j,m}(A) + \ldots + d_{j,1}(A) + \tilde{P}_{mj}, \tag{55}$$

$\tilde{P}_{mj} = \tilde{P}_{mj}(h_{k,k+p}, d_{\nu,s}(A))$ *are polynomials in* $h_{k,k+p}$, $k = 0, 1, \dots, m-1$; $p = 1, 2, \dots, j$, *and* $d_{\nu,s}(A)$, $s = 1, 2, \dots, m+j$; $\nu \leqslant j-1$, *of degree no more than 1 in every function* $h_{k,k+p}$, $d_{\nu,s}(A)$.

Let us continue the proof of the theorem. By decreasing the dimension of the matrix A we used m meromorphic vector-solutions. Since we have assumed that there are $m+1$ such solutions of system (1), (2) then the system $Y_m' = B_m Y_m$ (see (51), (52)) has at least one more non-trivial meromorphic vector-solution $Y_m = (v_{m,m+1}, v_{m,m+2}, \dots, v_{m,n})$ for which (see (53)) the following estimate holds

$$T(r, Y_m) = O\Big(\max_{i=0,1,\dots,m} T(r, W_i)\Big). \tag{56}$$

By transforming the system $Y_m' = B_m Y_m$ to the form similar to (36) we get the system of linear homogeneous equations with the matrix $Q_{n-m}(B_m, h_m)$ (see (24)) with the non-trivial solution $Y_m = (v_{m,m+1}, v_{m,m+2}, \dots, v_{m,n})$. Thus $Q_{n-m}(B_m, h_m) \equiv 0$. Hence, taking into account (25), we obtain

$$\Big(h_m = (h_{m,m+1}; h_{m,m+2}; \dots; h_{m,m+i}; \dots; h_{m,n}), \ h_{m,m+i} = \frac{v'_{m,m+i}}{v_{m,m+i}},$$

$$i = 1, 2, \ldots, n-m; \;\; Q_0(B_m, h_m) = 1, \;\; d_{0,n-m+1}(B_m) = 1\big), \tag{57}$$

$$d_{n-m,1}(B_m) = h_{m,n} Q_{n-m-1}(B_m, h_m) + \sum_{i=0}^{n-m-2} Q_i(B_m, h_m) h_{m,m+i+1} d_{n-m-i-1,i+2}(B_m). \tag{58}$$

Let us apply in (58) to $Q_i(B_m, h_m)$, $i \leqslant n-m-1$, Lemma 1 (see (26)). To $d_{j1}(B_m)$, $j \leqslant n-m-1$, let us apply the formula (55). By taking into account that $d_{j,t}(B_m) = d_{j,m+t}(A)$ for $t \geqslant 2$ and $j = 1, 2, \ldots, n-m$ (see (52)), we obtain

$$d_{n-m,1}(B_m) \overset{(58)}{=} P(d_{\nu,s}(A), h_{k,k+p}), \tag{59}$$

where P is a polynomial of degree no more than 1 in $d_{\nu,s}(A)$, $\nu < n-m$, $s = 1, \ldots, n$ and $h_{k,k+p}$, $k = 0, 1, \ldots, m$; $p = 1, \ldots, n-m$. From (55) at $j = n-m$ it follows

$$d_{n-m,1}(B_m) = d_{n-m,m+1}(A) + d_{n-m,m}(A) + \ldots + d_{n-m,1}(A) + \tilde{P}_{m,n-m}, \tag{60}$$

$\tilde{P}_{m,n-m} = \tilde{P}_{m,n-m}(h_{k,k+p}, d_{\nu,s}(A))$ is the polynomial in $h_{k,k+p}$, $k = 0, 1, \ldots, m-1$; $p = 1, 2, \ldots, n-m$, and $d_{\nu,s}(A)$, $s = 1, 2, \ldots, n$; $\nu \leqslant n-m-1$, of degree no more than 1. By taking into account the definition of $H_{n-m}(A)$ (19) and also the equalities (59), (60) and properties of the polynomials $P(d_{\nu,s}(A), h_{m,m+p})$, $\tilde{P}_{m,n-m}$ we obtain

$$H_{n-m}(A) = d_{n-m,m+1}(A) + d_{n-m,m}(A) + \ldots + d_{n-m,1}(A) = R_{m,n-m}, \tag{61}$$

$R_{m,n-m} = R_{m,n-m}(h_{k,k+p}, d_{\nu,s}(A))$ is a polynomial in $h_{k,k+p}$, $k = 0, 1, \ldots, m$; $p = 1, 2, \ldots, n-m$, and $d_{\nu,s}(A)$, $s = 1, 2, \ldots, n$; $\nu \leqslant n-m-1$, of degree no more than 1 in every variable. From the equality (61) and by taking into account properties of the polynomials $R_{m,n-m}$ we obtain ($r \notin E$)

$$\begin{aligned} m(r, H_{n-m}(A)) &\overset{(8)}{\leqslant} \sum_{\substack{k=0,1,\ldots,m,\\ p=1,\ldots,n-m}} m(r, h_{k,k+p}) + \sum_{\substack{s=1,2,\ldots,n,\\ \nu\leqslant n-m-1}} m(r, d_{\nu,s}(A)) + \\ +O(1) &\overset{(54)}{=} O\big(\ln^+\big(\max_{i=0,1,\ldots,m} T(r, W_i)\big) + \ln r\big) + \sum_{\substack{s=1,2,\ldots,n,\\ \nu\leqslant n-m-1}} m(r, d_{\nu,s}(A)). \end{aligned} \tag{62}$$

From (20) we have

$$\beta_{\nu s} \overset{\text{def}}{=} \sigma_\varphi^0[d_{\nu s}] < \sigma_\varphi^0[H_{n-m}(A)] \overset{\text{def}}{=} \alpha; \quad \max \beta_{\nu s} \overset{\text{def}}{=} \beta < \alpha, \tag{63}$$

$s = 1, \ldots, n$; $\nu = 1, \ldots, n-m+1$. Similar to (40) we obtain

$$m(r, d_{\nu s}) < \ln \varphi^{-1}(\ln r^{\beta+\varepsilon}), \;\; \varepsilon > 0. \tag{64}$$

Let us denote

$$\sigma_i = \sigma_\varphi^1[W_i], \;\; i = 0, 1, \ldots, m; \;\; \sigma = \max \sigma_i \overset{(21)}{<} \sigma_\varphi^1[H_{n-m}(A)]. \tag{65}$$

Then by taking into account (16) we obtain

$$T(r, W_i) < \varphi^{-1}(\ln r^{\sigma_i+\varepsilon}) \leqslant \varphi^{-1}(\ln r^{\sigma+\varepsilon}), \;\; \varepsilon > 0, \;\; r > r_0. \tag{66}$$

From (14), (62), (64), (66) it follows ($K = \text{const} > 0$)

$$m(r, H_{n-m}(A)) < K(\ln\varphi^{-1}(\ln r^{\sigma+\varepsilon}) + \ln\varphi^{-1}(\ln r^{\beta+\varepsilon})) < 2K\ln\varphi^{-1}(\ln r^{\max(\sigma,\beta)+\varepsilon}), \quad r \notin E.$$

Similar to (33) we obtain

$$m(r, H_{n-m}(A)) = O(\ln\varphi^{-1}(\ln r^{\max(\sigma,\beta)+2\varepsilon})), \ r > r_0, \ \ \varepsilon > 0.$$

Thus

$$\varphi\left(e^{m(r,H_{n-m}(A))}\right) = \varphi\left(e^{O(\ln\varphi^{-1}(\ln r^{\max(\sigma,\beta)+2\varepsilon}))}\right) \overset{(13)}{<}$$
$$< (1+o(1))\varphi\left(e^{\ln\varphi^{-1}(\ln r^{\max(\sigma,\beta)+2\varepsilon})}\right) = (1+o(1))\ln r^{\max(\sigma,\beta)+2\varepsilon}.$$

From here and from (12) we have $\sigma^0_\varphi[H_{n-m}(A)] \leqslant \max(\sigma,\beta)$, which contradicts (63), (65).

The case where in (2) some of $p_j \equiv 0$, shall be considered in a way similar to [2]. The proof of Theorem 2 is similar to that of Theorem 1. □

Proof of Lemma 3. By taking into account (45) let us represent $d_{j1}(B_m)$ via the determinants of the matrix B_{m-1} ($B_0 = A$, $d_{0,j+2}(B_{m-1}) = 1$ (see (19), (2)))

$$d_{j1}(B_m) = d_{j2}(B_{m-1}) + Q_j(B_{m-1}, h_{m-1}) + \sum_{i=1}^{j-1} Q_i(B_{m-1}, h_{m-1}) d_{j-i,i+2}(B_{m-1}). \qquad (67)$$

By using (24) we have $Q_0(A,h) \equiv 1,\ Q_1(A,h) = s_1 - h_1 = d_{11}(A) - h_1$,

$$Q_0(B_{m-1}, h_{m-1}) \equiv 1, \quad Q_1(B_{m-1}, h_{m-1}) = d_{11}(B_{m-1}) - h_{m-1,m},$$

where $h_{m-1} = (h_{m-1,m}; h_{m-1,m+1}; \ldots; h_{m-1,n})$, $h_{m-1,m+i} = \frac{v'_{m-1,m+i}}{v_{m-1,m+i}}$, $i = 0, 1, \ldots, n-m$. Thus ($d_{0,j+1}(B_{m-1}) = 1$)

$$\begin{aligned} Q_j(B_{m-1}, h_{m-1}) &\overset{(25)}{=} d_{j,1}(B_{m-1}) - h_{m-1,m} d_{j-1,2}(B_{m-1}) - \\ &-(d_{11}(B_{m-1}) - h_{m-1,m}) h_{m-1,m+1} d_{j-2,3}(B_{m-1}) - \\ &-\sum_{i=2}^{j-1} Q_i(B_{m-1}, h_{m-1}) h_{m-1,m+i} d_{j-i-1,i+2}(B_{m-1}) \overset{(26)}{=} \\ &= d_{j,1} - h_{m-1,m} d_{j-1,2} - (d_{11} - h_{m-1,m}) h_{m-1,m+1} d_{j-2,3} - \\ &-\sum_{i=2}^{j-1}\Big(d_{i1} - d_{i-1,1} h_{m-1,m-1+i} + \sum_{t=0}^{i-2} d_{t1} P_{ti}\Big) h_{m-1,m+i} d_{j-i-1,i+2}, \end{aligned} \qquad (68)$$

where $d_{t1} = d_{t1}(B_{m-1})$; $P_{ti} = P_{ti}(h_{m-1,m-1+p}, d_{\nu,s}(B_{m-1}))$ are some polynomials in $h_{m-1,m-1+p}$ and $d_{\nu,s}(B_{m-1})$; $p = t+1, t+2, \ldots, i$; $s = t+2, t+3, \ldots, i$; $\nu \leqslant i-1$; $i = 2, 3, \ldots, j-1$; $t = 0, 1, \ldots, i-2$ of degree no more than 1 in every $h_{m-1,m-1+p}$ and $d_{\nu,s}(B_{m-1})$. By grouping in (68) the summands that contain $d_{i1} = d_{i1}(B_{m-1})$, $i = 0, 1, \ldots, j-1$, we obtain ($d_{0i} = 1$)

$$Q_j(B_{m-1}, h_{m-1}) = d_{j,1}(B_{m-1}) + \sum_{i=0}^{j-1} d_{i1}(B_{m-1}) P^*_{ij}(h_{m-1,m-1+p}, d_{\nu,s}(B_{m-1})), \qquad (69)$$

$P^*_{ij}(h_{m-1,m-1+p}, d_{\nu,s}(B_{m-1}))$ are polynomials in $h_{m-1,m-1+p}$ and $d_{\nu,s}(B_{m-1})$; $p = i+1, i+2, \ldots, j$; $s = i+2, i+3, \ldots, j$; $\nu \leqslant j-1$; $i = 0, 1, \ldots, j-1$, of degree no more than 1 on every of $h_{m-1,m-1+p}$ and $d_{\nu,s}(B_{m-1})$. By transforming the sum in the right hand side of (67) $(Q_1(B_{m-1},h_{m-1}) = d_{11} - h_{m-1,m})$:

$$\begin{aligned}\sum_{i=1}^{j-1} Q_i(B_{m-1},h_{m-1})d_{j-i,i+2}(B_{m-1}) &\overset{(26)}{=} (d_{11} - h_{m-1,m})d_{j-1,3}+\\ &+\sum_{i=2}^{j-1}\left(d_{i1} - d_{i-1,1}h_{m-1,m-1+i} + \sum_{t=0}^{i-2} d_{t1}P_{ti}\right)d_{j-i,i+2} =\\ =\sum_{i=0}^{j-1} d_{i1}(B_{m-1})P^{\star}_{ij}(h_{m-1,m-1+p},d_{\nu,s}(B_{m-1})),&\quad d_{01}(B_{m-1}) = 1,\end{aligned} \tag{70}$$

where the polynomials $P_{ti} = P_{ti}(h_{m-1,m-1+p},d_{\nu,s}(B_{m-1}))$ are the same as in (68); $P^{\star}_{ij}(h_{m-1,m-1+p},d_{\nu,s}(B_{m-1}))$ are polynomials in $h_{m-1,m-1+p}$ and $d_{\nu,s}$, of degree no more than 1 in every $h_{m-1,m-1+p}$ and $d_{\nu,s}(B_{m-1})$; $p = i+1, i+2, \ldots, j$; $s = i+2, i+3, \ldots, j+1$; $\nu \leqslant j-1$; $i = 1,\ldots,j-1$. By substituting (69), (70) into (67) and then grouping the summands with $d_{i1}(B_{m-1})$, we obtain

$$d_{j1}(B_m) = d_{j,1}(B_{m-1}) + d_{j2}(B_{m-1}) + \sum_{i=0}^{j-1} d_{i1}(B_{m-1})P_{ij}, \tag{71}$$

$P_{ij} = P_{ij}(h_{m-1,m-1+p},d_{\nu,s}(B_{m-1}))$ are polynomial in $h_{m-1,m-1+p}$ and $d_{\nu,s}(B_{m-1})$, of degree no more than 1 in every $h_{m-1,m-1+p}$ and $d_{\nu,s}(B_{m-1})$; $p = i+1, i+2, \ldots, j$; $s = i+2$, $i+3,\ldots,j+1$; $\nu \leqslant j-1$; $i = 1,2,\ldots,j-1$. But (see (52), (44), (18)) $d_{\nu,s}(B_{m-1}) = d_{\nu,m+s-1}(A)$ at $s \geqslant 2$. Thus

$$d_{j1}(B_m) = d_{j,1}(B_{m-1}) + d_{j,m+1}(A) + \sum_{i=0}^{j-1} d_{i1}(B_{m-1})P_{ij}(h_{m-1,m-1+p},d_{\nu,m+s-1}), \tag{72}$$

$P_{ij}(h_{m-1,m-1+p},d_{\nu,m+s-1}(A))$ are polynomials in $h_{m-1,m-1+p}$ and $d_{\nu,m+s-1}(A)$, of degree no more than 1 on every $h_{m-1,m-1+p}$ and $d_{\nu,m+s-1}(A)$; $p = i+1, i+2, \ldots, j$; $s = i+2$, $i+3,\ldots,j+1$; $\nu \leqslant j-1$; $i = 1,2,\ldots,j-1$.

Let us prove the formula (55). If $m = 1$ then from (72) it follows ($B_0 = A$, $h_0 = (w'_{01}/w_{01},\ldots,w'_{0n}/w_{0n})$, $h_{0,p} = w'_{0p}/w_{0p}$ (see (45)))

$$\begin{aligned}d_{i1}(B_1) &= d_{i,1}(A) + d_{i,2}(A) + \sum_{t=0}^{i-1} d_{t1}(A)P_{ti}(h_{0,p};d_{\nu,s}(A)) =\\ &= d_{i,1}(A) + d_{i,2}(A) + \tilde{P}_{1i},\quad i\in\mathbb{N},\quad i\leqslant n-1,\end{aligned} \tag{73}$$

$\tilde{P}_{1i}$ is a polynomial in $h_{0,p}$ and $d_{\nu,s}(A)$, $p = 1,2,\ldots,i$; $s = 1,2,\ldots,i+1$; $\nu < i$ of degree no more than 1 in every of the functions.

Let for every $i\in\mathbb{N}$, $i\leqslant j\leqslant n-m$, $2\leqslant m$ the following equality take place

$$d_{i1}(B_{m-1}) = d_{i,1}(A) + d_{i,2}(A) + \ldots + d_{i,m}(A) + \tilde{P}_{m-1,i},\quad i\leqslant n-m, \tag{74}$$

$\tilde{P}_{m-1,i}$ is a polynomial in h_{0p}, $h_{1,p+1},\ldots,h_{m-2,m-2+p}$ and $d_{\nu,s}(A)$; $p = 1,2,\ldots,i$; $s = 1,2$, $\ldots,i+m-1$; $\nu < i$ of degree no more than 1 in every of $h_{k,k+t}$ and $d_{\nu,s}(A)$. By substituting (74) into (72) we obtain

$$\begin{aligned}d_{j1}(B_m) &= d_{j,1}(A) + d_{j,2}(A) + \ldots + d_{j,m}(A) + d_{j,m+1}(A) + \tilde{P}_{m-1,j}+\\ &+\sum_{i=0}^{j-1}(d_{i,1}(A) + \ldots + d_{i,m}(A) + \tilde{P}_{m-1,i})P_{ij}(h_{m-1,m-1+p},d_{\nu,m+s-1}(A)) =\\ &= d_{j,1}(A) + d_{j,2}(A) + \ldots + d_{j,m+1}(A) + \tilde{P}_{m,j},\end{aligned}$$

$\tilde{P}_{m,j}$ is a polynomial in $h_{0p},\ h_{1,p+1},\dots,h_{m-1,m-1+p}$ and $d_{\nu,s}(A)$; $p=1,2,\dots,j$; $s=1,2,\dots,j+m$; $\nu<j$ of degree no more than 1 on every $h_{k,k+t}$ and $d_{\nu,s}(A)$. Here we took into account that $\tilde{P}_{m-1,i}$ contains $d_{\nu,s}(A)$ with indices $s=1,2,\dots,i+m-1$ and $P_{ij}(h_{m-1,m-1+p}, d_{\nu,m+s-1}(A))$ contain $d_{\nu,m+s-1}(A)$ with indexes $s=i+2,i+3,\dots,j+1$. Then $\tilde{P}_{m-1,i}$ includes also $h_{0p},\ h_{1,p+1},\dots,h_{m-2,m-2+p}$ at $p=1,2,\dots,i$ and $P_{ij}(h_{m-1,m-1+p},d_{\nu,m+s-1}(A))$ contain $h_{m-1,m-1+p}$ with indices $p=i+1,i+2,\dots,j$. □

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Lviv Polytechnic National University, Lviv, Ukraine
mohon@soft-hitech.com

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