УДК 517.98

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# UNIQUE RANGE SETS FOR POWERS OF MEROMORPHIC FUNCTIONS

S. Mallick, D. Sarkar. *Unique range sets for powers of meromorphic functions*, Mat. Stud. **50** (2018), 143–157.

The prime concern of the paper is to deal with the notion of the unique range set for powers of meromorphic functions. As a consequence, we show that the lower bound of URSM (URSE) and URSM-IM (URSE-IM) can be significantly reduced up to cardinality 4 (4) from 11 (7) and 17 (10) respectively, for a class of power of meromorphic (entire) functions. Various applications of our main result also improve and generalize different results of Khoai-An-Lai (Internat. J. Math., 2018), Yi (Nagoya Math. J., 1995) and (J. Shandong Univ. Nat. Sci., 1998). Moreover, on the basis of some new notions introduced in the paper, our main result and its applications, we have partially reduced the known Gross Problem to a more narrow formulation and posed a number of open problems in the last section to unveil the least cardinality problem of unique range sets.

**1. Introduction.** The notion of the unique range set is one of the most important concepts of the uniqueness theory. We know that for two arbitrary non-constant meromorphic functions $f$, $g$ and a set $S \subseteq \mathbb{C} \cup \{\infty\}$ if $f^{-1}(S) = g^{-1}(S)$, where each element of $f^{-1}(S)$ and $g^{-1}(S)$ is counted according to its multiplicity, then we say $f$ and $g$ share the set $S$ counting multiplicities or CM in brief. If $f^{-1}(S) = g^{-1}(S)$, where we ignore the multiplicities of each element of $f^{-1}(S)$ and $g^{-1}(S)$, then we say $f$ and $g$ share the set $S$ ignoring multiplicities or IM in brief. Obviously, CM and IM stands for counting multiplicities and ignoring multiplicities respectively. Further, if $f^{-1}(S) = g^{-1}(S)$ CM implies $f \equiv g$, then $S$ is called a *unique range set* for meromorphic (entire) functions or URSM(URSE) in brief and if $f^{-1}(S) = g^{-1}(S)$ IM implies $f \equiv g$, then $S$ is called a *unique range set* for meromorphic (entire) functions ignoring multiplicities or URSM-IM(URSE-IM) or RURSM(RURSE) in brief. These notions of unique range sets were basically generated in order to answer the known **"Gross Problem"**.

**Gross Problem ([10]).** *Can one find two (or possibly even one) finite sets $S_j$ $(j = 1, 2)$ such that any two entire functions $f$ and $g$ satisfying $f^{-1}(S_j) = g^{-1}(S_j)$ CM for $j = 1, 2$ must be identical? If "yes", then how small can they be?*

After that, another three parallel problems were also posed by various authors in different papers [22, 6, 7] and likewise the notions of URSE-IM, URSM, URSM-IM were appeared. Consequently, a number of results [11, 17, 18, 21, 22, 23, 8, 7, 6, 9, 4, 3, 5] have been obtained by different authors in due course of time. On this occasion, we must make one point clear that the answers to the first part of the Gross Problem or to the parallel problems were

2010 *Mathematics Subject Classification:* 30D20, 30D30, 30D35.
*Keywords:* meromorphic function; entire function; uniqueness; unique range set; reduced unique range set.
doi:10.15330/ms.50.2.143-157

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settled in 80's and 90's but throughout these years the research has been going on mainly to answer to the second part of the same. Pertinent to this, during the last four decades the following observations have been made for unique range sets by various authors in different time.

1. Finite unique range sets are generated by the zeros of polynomials of some special kind [17].
2. The least cardinality of any unique range set for meromorphic(entire) functions must be greater than 5(4) [19, see page 527, Theorem 10.72 and page 517, Theorem 10.59].
3. The least cardinality obtained so far of any URSM(URSE) [9, 4, 5] is 11(7).
4. The least cardinality obtained so far of any URSM-IM(URSE-IM) [6, 9, 4] is 17(10).

In view of [2], [3] and [4], it is clear that a reduction of the existing lower bound of cardinalities of unique range sets has become an intriguing challenge for the researchers as there exist no counterexample proving the least cardinality of a URSM(URSE) and URSM-IM(URSE-IM) to be 11(7) and 17(10), respectively. In the meantime, Banerjee-Lahiri [4] propose an approach to improve all the existing results of URSM(URSE) by introducing the concept of URSMk(URSEk) called as URSM(URSE) with weight $k$. For that they took help of the notion of weghted sharing of values and sets [15, 16]. Below we recall these definitions but for the sake of our convenience we formulate the same definitions in a different style.

**Definition 1** ([15, 16])**.** Let $k$ be a nonnegative integer or infinity. For two non-constant meromorphic functions $f, g$ and $a \in \mathbb{C} \cup \{\infty\}$ we denote by $f^{-1}(a, k)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $f^{-1}(a, k) = g^{-1}(a, k)$, we say that $f$, $g$ *share the value $a$ with weight $k$*.

We write $f$, $g$ share $(a, k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a, k)$, then $f$, $g$ share $(a, p)$ for any integer $p$, $0 \leq p < k$. Also we note that $f$, $g$ share a value $a$ IM or CM if and only if $f$, $g$ share $(a, 0)$ or $(a, \infty)$, respectively.

**Definition 2** ([15, 16])**.** For a non-constant meromorphic function $f$ and $S \subseteq \mathbb{C} \cup \{\infty\}$ we define $f^{-1}(S, k) = \bigcup_{a \in S} f^{-1}(a, k)$, where $k$ is a non-negative integer and $a \in S$.

Then, for two non-constant meromorphic functions $f$ and $g$, by $f^{-1}(S, k) = g^{-1}(S, k)$ we mean that $f$ and $g$ share the set $S$ with weight $k$. Clearly, $f^{-1}(S, \infty) = g^{-1}(S, \infty)$ means $f^{-1}(S) = g^{-1}(S)$ CM and $f^{-1}(S, 0) = g^{-1}(S, 0)$ means $f^{-1}(S) = g^{-1}(S)$ IM.

**Definition 3** ([4])**.** A set $S \subseteq \mathbb{C} \cup \{\infty\}$ is called a *unique range set for meromorphic (entire) functions with weight* $k$ if for any two non-constant meromorphic (entire) functions $f$ and $g$, $f^{-1}(S, k) = g^{-1}(S, k)$ implies $f \equiv g$. We write S is URSMk (URSEk) in short.

Clearly URSMk(URSEk) is a gradation between URSM(URSE) and URSM-IM(URSE-IM), where $k$ is a non-negative integer. For $k = \infty$ and $k = 0$, URSMk(URSEk) coincides with URSM(URSE) and URSM-IM(URSE-IM), respectively and for other values of $k$ it runs between these two. Thus every URSM-IM(URSE-IM) is a URSMk(URSEk) and every URSMk(URSEk) is a URSM(URSE).

But this notion of URSMk(URSEk) only helps us to relax the nature of sharing the unique range sets not to diminish the lower bound of its cardinality. So the problem of least cardinality of unique range sets remains unsolved at this stage. Under such circumstances, naturally the following question arises in the minds of the researchers.

**Question 1.** *Does there exist any URSM(URSE) or URSM-IM(URSE-IM) of cardinality less than* 11(7) *or* 17(10), *respectively, even for any special class of meromorphic(entire) functions?*

In this paper, we answer to this fundamental question affirmatively. Apropos of this, we define the following concepts.

**Definition 4.** Suppose $M(\mathbb{C})$ denotes the set of all meromorphic functions defined on $\mathbb{C}$. We define $M^d(\mathbb{C})$ to be the collection of all such meromorphic functions which are powers of some meromorphic functions of power at least $d$, where $d$ is a positive integer. Therefore we will write $M^d(\mathbb{C}) = \{f^{d+r} \mid d \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\} \text{ and } f \in M(\mathbb{C})\}$.

Clearly, $M^p(\mathbb{C}) \subset M^s(\mathbb{C}) \subset M^1(\mathbb{C}) = M(\mathbb{C})$ whenever $p > s > 1$.

Similar notions can be defined for entire functions and be denoted by $E(\mathbb{C})$ and $E^d(\mathbb{C})$. In that case also we would have $E^p(\mathbb{C}) \subset E^s(\mathbb{C}) \subset E^1(\mathbb{C}) = E(\mathbb{C})$ whenever $p > s > 1$.

**Definition 5.** Let $f, g \in M^d(\mathbb{C})$ be non-constant and $S \subseteq \mathbb{C} \cup \{\infty\}$. If $f^{-1}(S,\infty) = g^{-1}(S,\infty)$ implies $f \equiv g$, then $S$ is said to be a *unique range set for the power of meromorphic (entire) functions with power at least* $d$ or $URSP^dM$ ($URSP^dE$) in brief.
Similarly if $f^{-1}(S,0) = g^{-1}(S,0)$ implies $f \equiv g$, then $S$ is said to be a *unique range set for the power of meromorphic(entire) functions with power at least d ignoring multiplicities or* $URSP^dM$-IM ($URSP^dE$)-IM in brief.

Since $M^p(\mathbb{C}) \subset M^s(\mathbb{C}) \subset M^1(\mathbb{C}) = M(\mathbb{C})$ whenever $p > s > 1$, so we must have $URSM = URSP^1M$ is a $URSP^tM$ for $t \geq 1$ and every $URSP^sM$ is a $URSP^pM$.

Now we define the notion $URP^dMk(URP^dEk)$ as a gradation between

$$URSP^dM(URSP^dE) \text{ and } URSP^dM - IM(URSP^dE - IM).$$

**Definition 6.** Let $f, g \in M^d(\mathbb{C})$ be non-constant and $S \subseteq \mathbb{C} \cup \{\infty\}$. If $f^{-1}(S,k) = g^{-1}(S,k)$ implies $f \equiv g$, then $S$ is said to be a *unique range set with weight k for the power of meromorphic(entire) functions of power at least d or* $URSP^dMk$ ($URSP^dEk$) in brief.

Thus for $k = \infty$ and $k = 0$, $URP^dMk(URP^dEk)$ coincides with $URSP^dM(URSP^dE)$ and $URSP^dM$-IM($URSP^dM$-IM) respectively and for rest of the values of $k$ it runs between these two.

Using these definitions of $URSP^dM(URSP^dE)$, $URSP^dM$-IM($URSP^dM$-IM) and $URP^dMk(URP^dEk)$ we shall show that the lower bound of cardinalities of URSM(URSE) or URSM-IM(URSE-IM) can significantly be reduced for special class of meromorphic (entire) functions even up to cardinality 4 (4) and thus, we successfully answer to Question 1. In this process, we also find that the least cardinality of a unique range set may be less than 5 for special class of meromorphic functions. Furthermore, different applications of our main result (see below Remarks 4-5) directly improve Yi's result [20] for the least cardinality of a URSE by producing a more general URSE1 of the same cardinality and generalize the result of [23].

Very recently, a little bit of research on Question 1 was performed in [13]. They proved the following result.

**Theorem A** ([13]). *There exist the sets $S$ with 7 elements such that for arbitrary two meromorphic functions $f, g$ and for an integer $d \geq 2$, the condition $(f^d)^{-1}(S,\infty) = (g^d)^{-1}(S,\infty)$ implies $f = \xi g$, where $\xi$ is a root of unity of degree $d$.*

Obviously Theorem A shows the uniqueness of $f^d$ and $g^d$. It is clear from Definition 6 that the notion of $URSP^dMk$ is more general than the concept used in Theorem A. However, using this notion of $URSP^dMk$ as an application of our main result, we shall improve Theorem A by considering the uniqueness of $f^{d+r}$ and $g^{d+s}$, where $r, s \in \mathbb{N} \cup \{0\}$ under more weaker sharing hypothesis.

In fact, here we significantly reduce the sharing condition of the set from weight $\infty$ to weight 1.

Now let us move to the next section for the main result of the paper.

**2. Main Result.** Let

$$Q(z) = az^n + bz^{2m} + cz^m + 1, \tag{1}$$

where $n, m \in \mathbb{N}$ and $a, b, c \in \mathbb{C}\backslash\{0\}$ be such that $n > 2m$, $\gcd(n, m) = 1$, $\frac{c^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$ and $a \neq -\frac{(2mbe_i^{2m}+cme_i^m)}{ne_i^n} (= \gamma_i)$ with $e_i$ to be the roots of the equation

$$z^m = -\frac{2n}{(n-m)c}. \tag{2}$$

Now we show that $Q(z)$ has only simple zeros. We have $Q^{'}(z) = naz^{n-1} + 2mbz^{2m-1} + cmz^{m-1}$. Therefore, zeros of $Q^{'}(z)$ are roots of $naz^{n-1} + 2mbz^{2m-1} + cmz^{m-1} = 0$. Clearly, for any zero 's' of $Q^{'}(z)$ we have $nas^{n-1} + 2mbs^{2m-1} + cms^{m-1} = 0$; i.e., $nas^n + 2mbs^{2m} + cms^m = 0$; i.e., $as^n = -\frac{(2mbs^{2m}+cms^m)}{n}$. Now for $s = 0$ $Q(0) = 1 \neq 0$ and for $s \neq 0$

$$Q(s) = -\frac{(2mbs^{2m} + cms^m)}{n} + bs^{2m} + cs^m + 1 = \frac{(n-2m)bs^{2m} + (n-m)cs^m + n}{n} =$$
$$= \frac{c^2(n-m)^2 s^{2m} + 4cn(n-m)s^m + 4n^2}{4n^2} = \frac{(c(n-m)s^m + 2n)^2}{4n^2}.$$

So, '$s$' is a zero of $Q(z)$, if $s^m = \frac{-2n}{(n-m)c}$; i.e., if $s \in \{e_1, e_2, \ldots, e_m\}$. But then we would have $ae_i^n = -\frac{(2mbe_i^{2m}+cme_i^m)}{n}$ for $i \in \{1, 2, \ldots, m\}$, which is a contradiction as $a \neq \gamma_i = -\frac{(2mbe_i^{2m}+cme_i^m)}{ne_i^n}$. Hence, $Q(z)$ has only simple zeros. Let $P(z) = bz^{2m} + cz^m + 1$. Now we show that $P(z)$ has only simple zeros. We have

$$P^{'}(z) = mz^{m-1}(2bz^m + c).$$

Therefore the roots of $P^{'}(z)$ are 0 and $\xi_i$ for $i = 1, 2, \ldots, m$, where $\xi_i$ are the roots of the equation $2bz^m + c = 0$. Clearly $P(0) = 1 \neq 0$ and $P(\xi_i) = b\xi_i^{2m} + c\xi_i^m + 1 = b(\frac{-c}{2b})^2 + c(\frac{-c}{2b}) + 1 = \frac{4b-c^2}{4b} \neq 0$ as $\frac{c^2}{4b} = \frac{n(n-2m)}{(n-m)^2} \neq 1$. Hence let us assume $\alpha_i$ for $i = 1, 2, \ldots, 2m$ be the distinct zeros of $P(z)$.

**Theorem 1.** *Let $Q(z)$ be given by (1) and $S = \{z \colon Q(z) = 0\}$. Then*

*(i) $S$ is a $URSP^dM$-IM ($URSP^dE$-IM) for*
$n > \max\left[m + 2 + \frac{2}{d}, 2m + \frac{14}{d}\right] \left(\max\left[m + 2, 2m + \frac{7}{d}\right]\right)$;

*(ii)* *$S$ is a $URSP^dM1$ ($URSP^dE1$) for*
$n > \max\left[m+2+\frac{2}{d}, 2m+\frac{9}{d}\right] \left(\max\left[m+2, 2m+\frac{9}{2d}\right]\right)$*;*

*(iii)* *$S$ is a $URSP^dM2$ ($URSP^dE2$) for*
$n > \max\left[m+2+\frac{2}{d}, 2m+\frac{8}{d}\right] \left(\max\left[m+2, 2m+\frac{4}{d}\right]\right)$*.*

**Remark 1.** For $m = 1$, Part [i] Theorem 1 clearly implies the following:

(a) $S$ is a $URSP^1M0$ ($URSP^1E0$) or URSM-IM (URSE-IM) for $n \geq 17$ (10);
(b) $S$ is a $URSP^2M0$ ($URSP^2E0$) for $n \geq 10$ (6);
(c) $S$ is a $URSP^3M0$ ($URSP^3E0$) for $n \geq 7$ (5);
(d) $S$ is a $URSP^4M0$ ($URSP^4E0$) for $n \geq 6$ (4);
(e) $S$ is a $URSP^5M0$ ($URSP^5E0$) for $n \geq 5$ (4);
(f) $S$ is a $URSP^8M0$ ($URSP^8E0$) for $n \geq 4$ (4).

**Remark 2.** For $m = 1$, Part [ii] Theorem 1 clearly implies the following:

(a) $S$ is a $URSP^1M1$ ($URSP^1E1$) or $URSM1$ (URSE1) for $n \geq 12$ (7);
(b) $S$ is a $URSP^2M1$ ($URSP^2E1$) for $n \geq 7$ (5);
(c) $S$ is a $URSP^3M1$ ($URSP^3E1$) for $n \geq 6$ (4);
(d) $S$ is a $URSP^4M1$ ($URSP^4E1$) for $n \geq 5$ (4);
(e) $S$ is a $URSP^5M1$ ($URSP^5E1$) for $n \geq 4$ (4).

**Remark 3.** For $m = 1$, Part [iii] Theorem 1 clearly implies the following:

(a) $S$ is a $URSP^1M2$ ($URSP^1E2$) or $URSM2$ ($URSE2$) for $n \geq 11$ (7);
(b) $S$ is a $URSP^3M2$ ($URSP^3E2$) for $n \geq 5$ (4).

**Remark 4.** Item (a) of Remark 2 shows that for $m = 1$, $S$ is a $URS^1E1$ or URSE1 for $n \geq 7$ which directly improves Yi's result of [20] by relaxing the nature of sharing the URSE.

**Remark 5.** In [23], Yi proved that the following set is a URSM-IM for $n \geq 17$.

$${}^{a_1}S_n^{b_1} = \{z\colon a_1 z^n - n(n-1)z^2 + 2n(n-2)b_1 z - (n-1)(n-2)b_1^2 = 0\},$$

where $a_1, b_1 \in \mathbb{C} - \{0\}$ and $a_1 b_1^{n-2} \neq 2$. Observe that,

$$\begin{aligned} {}^{a_1}S_n^{b_1} &= \{z\colon a_1 z^n - n(n-1)z^2 + 2n(n-2)b_1 z - (n-1)(n-2)b_1^2 = 0\} \\ &= \left\{z\colon \frac{-a_1}{(n-1)(n-2)b_1^2} z^n + \frac{n}{(n-2)b_1^2} z^2 - \frac{2n}{(n-1)b_1} z + 1 = 0\right\} = \\ &= \{z\colon az^n + bz^{2m} + cz^m + 1 = 0\}, \end{aligned}$$

where $a = \frac{-a_1}{(n-1)(n-2)b_1^2}$, $b = \frac{n}{(n-2)b_1^2}$, $c = -\frac{2n}{(n-1)b_1}$ and $m = 1$. Further, we have $\frac{c^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$ and $\gcd(n, m) = 1$. Now the roots of $z^m = -\frac{2n}{(n-m)c}$; are $e_i = e_1 = -\frac{2n}{(n-1)c} = b_1$ and hence

$$\gamma_i = -\frac{(2mbe_i^{2m} + cme_i^m)}{ne_i^n} = -\frac{(2bb_1^2 + cb_1)}{nb_1^n} = -\frac{\frac{2n}{(n-2)} - \frac{2n}{(n-1)}}{nb_1^n} = -\frac{2}{(n-1)(n-2)b_1^n} = \gamma_1.$$

Now $a_1 b_1^{n-2} \neq 2$ implies $a = \frac{-a_1}{(n-1)(n-2)b_1^2} = -\frac{a_1 b_1^{n-2}}{(n-1)(n-2)b_1^n} \neq -\frac{2}{(n-1)(n-2)b_1^n} = \gamma_1 = \gamma_i$. Therefore all the conditions of *Theorem 1* are satisfied. Hence ${}^{a_1}S_n^{b_1}$ is a URSM-IM for $n > \max\left[m+2+\frac{2}{d}, 2m+\frac{14}{d}\right] = \max[5, 16]$; i.e., for $n \geq 17$ according to Theorem 1. So, Part [i] of Theorem 1 is clearly a generalization of the above mentioned result in [23].

**Remark 6.** Note that Part [ii], Part [iii] and all the consequences of Theorem 1 such as Remark 1–3 are also applicable to ${}^{a_1}S_n^{b_1}$.

**Remark 7.** Obviously item (b) of Remark 2 directly improves Theorem A.

As an application of Theorem 1 with the help of the above remarks we can find out uncountably many sets for the uniqueness of the functions $f^{d+r}$ and $g^{d+s}$, where $d \in \mathbb{N}$, $r, s \in \mathbb{N} \cup \{0\}$ and $f, g$ be two arbitrary non-constant meromorphic functions. Below we provide some of them with least possible cardinality.

**Example 1.** Let us recall the set ${}^{a_1}S_n^{b_1}$ as defined in Remark 5 for $n = 4$, $a_1 = b_1 = 1$; i.e., ${}^1S_4^1 = \{z\colon z^4 - 12z^2 + 16z - 6 = 0\}$. Now, if $f^{8+r}$ and $g^{8+s}$ share the set ${}^1S_4^1$ ignoring multiplicity, then $f^{8+r} = g^{8+s}$.

**Example 2.** If $f^{5+r}$ and $g^{5+s}$ share the set ${}^1S_4^1 = \{z\colon z^4 - 12z^2 + 16z - 6 = 0\}$ with weight 1, then $f^{5+r} = g^{5+s}$.

**Example 3.** If $f^{3+r}$ and $g^{3+s}$ share the set ${}^1S_5^1 = \{z\colon z^5 - 20z^2 + 30z - 12 = 0\}$ with weight 2, then $f^{3+r} = g^{3+s}$.

**Example 4.** If $f$ and $g$ are entire functions and $f^{4+r}$, $g^{4+s}$ share the set ${}^1S_4^1$ ignoring multiplicity, then $f^{4+r} = g^{4+s}$.

**Example 5.** Similarly as Example 4, if $f$ and $g$ are entire functions and $f^{3+r}$, $g^{3+s}$ share the set ${}^1S_4^1$ with weight 1, then $f^{3+r} = g^{3+s}$.

**Example 6.** Again suppose that $f$ and $g$ are entire functions and $f^{2+r}$, $g^{2+s}$ share the set ${}^1S_5^1$ with weight 1. Then $f^{2+r} = g^{2+s}$.

**3. Lemmas.** From now on, we shall use standard notations and definitions of the Nevanlinna Theory throughout the paper. So, for the sake of convenience of our reader we refer them to follow [12, 19]. Besides that, we need the following definitions to proceed further.

**Definition 7** ([14])**.** For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple $a$-points of $f$. For a positive integer $m$ we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those $a$-points of $f$ whose multiplicities are not greater(less) than $m$, where each $a$-point is counted according to its multiplicity. $\overline{N}(r, a; f \mid\leq m)(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the $a$-points of $f$ we ignore the multiplicities. Also, $N(r, a; f \mid< m), N(r, a; f \mid> m), \overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ are defined analogously.

**Definition 8** ([24])**.** Let $f$ and $g$ be two non-constant meromorphic functions such that $f$ and $g$ share $(a, 0)$, where $a \in \mathbb{C} \cup \{\infty\}$. Let $z_0$ be an $a$-point of $f$ with multiplicity $p$, an $a$-point of $g$ with multiplicity $q$. We denote by $\overline{N}_L(r, a; f)$ ($\overline{N}_L(r, a; g)$) the reduced counting function of those $a$-points of $f$ and $g$ where $p > q$ ($q > p$), by $N_E^{1)}(r, a; f)$ the counting function of those $a$-points of $f$ and $g$ where $p = q = 1$. Clearly, when $f$ and $g$ share $(a, m)$, $m \geq 1$, then $N_E^{1)}(r, a; f) = N(r, a; f \mid= 1)$.

**Definition 9** ([15, 16])**.** Let $f$, $g$ share $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those $a$-points of $f$ whose multiplicities differ from the multiplicities of the corresponding $a$-points of $g$.

Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Throughout the paper for two meromorphic functions $f$, $g$ and $a \in \mathbb{C} \cup \{\infty\}$, any type of counting functions of $a$ points of $f$ or $g$ starting with $N$ implies we count the corresponding $a$ points according to its multiplicities and starting with $\overline{N}$ implies we ignore the multiplicities of the corresponding $a$ points.

Let

$$R(z) = -\frac{az^n}{bz^{2m} + cz^m + 1} = -\frac{az^n}{b\prod_{i=1}^{2m}(z-\alpha_i)}, \tag{3}$$

where $a, b, c, n, m$ are exactly same as defined in (1) and $\alpha_i$'s are the distinct roots of the equation $bz^{2m} + cz^m + 1 = 0$. Then $R(z) - 1 = -Q(z)/b\prod_{i=1}^{2m}(z-\alpha_i)$. Now we define

$$f_1^{d+r} = f,\ g_1^{d+s} = g, \tag{4}$$

$$F = R(f),\ G = R(g), \tag{5}$$

where $f_1$, $g_1$ be two arbitrary non-constant meromorphic functions with $d \in \mathbb{N}$ and $r, s \in \mathbb{N} \cup \{0\}$ and $R(z)$ is given by (3). Clearly $f, g \in M^d(\mathbb{C})$. Henceforth, we shall denote by $H$ the following function: $H = (\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F-1}) - (\frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G-1})$.

**Lemma 1** ([25])**.** *If $F$, $G$ are two non-constant meromorphic functions such that they share $(1,0)$ and $H \not\equiv 0$, then $N_E^{1)}(r,1;F) \le N(r,H) + S(r,F) + S(r,G)$.*

**Lemma 2** ([2])**.** *Let $f$ and $g$ be two meromorphic functions sharing $(1,l)$, where $0 \le l < \infty$. Then*

$$\overline{N}(r,1;f) + \overline{N}(r,1;g) - N_E^{1)}(r,1;f) + \left(l - \frac{1}{2}\right)\overline{N}_*(r,1;f,g) \le \frac{1}{2}\left[N(r,1;f) + N(r,1;g)\right].$$

**Lemma 3.** *Let $F$ and $G$ be given by (5) and $H \not\equiv 0$. If $F$, $G$ share $(1,0)$, then*

$$N(r,H) \le \overline{N}(r,0;f) + \sum_{i=1}^{m}\overline{N}(r,e_i;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \sum_{i=1}^{m}\overline{N}(r,e_i;g) +$$
$$+\overline{N}(r,\infty;g) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;f^{'}) + \overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g),$$

*where $e_i$'s are the distinct roots of $z^m = -\frac{2n}{(n-m)c}$ and $\overline{N}_0(r,0;f^{'})$ denotes the reduced counting function corresponding to the zeros of $f^{'}$ which are not the zeros of $f\prod_{i=1}^{m}(f - e_i)(F-1)$. $\overline{N}_0(r,0;g^{'})$ is defined similarly.*

*Proof.* From (5) and (3) we have $F = -\frac{af^n}{b\prod_{i=1}^{2m}(f-\alpha_i)}$, $\quad G = -\frac{ag^n}{b\prod_{i=1}^{2m}(g-\alpha_i)}$. Now using $\frac{c^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$, we get

$$F^{'} = \frac{(n-2m)af^{n-1}\prod_{i=1}^{m}(f-e_i)^2 f^{'}}{b\prod_{i=1}^{2m}(f-\alpha_i)^2}. \tag{6}$$

Therefore,

$$\log(F^{'}) = (n-1)\log(f) + 2\sum_{i=1}^{m}\log(f-e_i) + \log(f^{'}) - 2\sum_{i=1}^{2m}\log(f-\alpha_i) +$$

$$+\log(n-2m)a-\log b \implies \frac{F''}{F'}=(n-1)\frac{f'}{f}+2\sum_{i=1}^{m}\frac{f'}{f-e_i}+\frac{f''}{f'}-2\sum_{i=1}^{2m}\frac{f'}{f-\alpha_i}. \tag{7}$$

Similarly,

$$F-1=\frac{a\prod_{i=1}^{n}(f-\omega_i)}{b\prod_{i=1}^{2m}(f-\alpha_i)},$$

where $\omega_i$'s are the distinct zeros of (1). Hence,

$$\log(F-1)=\sum_{i=1}^{n}\log(f-\omega_i)-\sum_{i=1}^{2m}\log(f-\alpha_i)+\log a-\log b$$

$$\implies \frac{F'}{F-1}=\sum_{i=1}^{n}\frac{f'}{f-\omega_i}-\sum_{i=1}^{2m}\frac{f'}{f-\alpha_i}. \tag{8}$$

Proceeding in the same way, we have

$$\frac{G''}{G'}=(n-1)\frac{g'}{g}+2\sum_{i=1}^{m}\frac{g'}{g-e_i}+\frac{g''}{g'}-2\sum_{i=1}^{2m}\frac{g'}{g-\alpha_i}, \tag{9}$$

$$\frac{G'}{G-1}=\sum_{i=1}^{n}\frac{g'}{g-\omega_i}-\sum_{i=1}^{2m}\frac{g'}{g-\alpha_i}. \tag{10}$$

Now using (7), (8), (9) and (10) into $H=(\frac{F''}{F'}-\frac{2F'}{F-1})-(\frac{G''}{G'}-\frac{2G'}{G-1})$; we get

$$H=\left[(n-1)\frac{f'}{f}+2\sum_{i=1}^{m}\frac{f'}{f-e_i}+\frac{f''}{f'}-2\sum_{i=1}^{n}\frac{f'}{f-\omega_i}\right]- \tag{11}$$
$$-\left[(n-1)\frac{g'}{g}+2\sum_{i=1}^{m}\frac{g'}{g-e_i}+\frac{g''}{g'}-2\sum_{i=1}^{n}\frac{g'}{g-\omega_i}\right].$$

Since $F$, $G$ share $(1,0)$, so $f$, $g$ share the set $S$ IM, where $S=\{\omega_1,\omega_2,\ldots,\omega_n\}$. From (11), it is obvious that all the poles of $H$ can come from the following sources only.

(i) Zeros of $f$ and $g$.

(ii) $e_i$ points of $f$ and $g$.

(iii) Poles of $f$ and $g$.

(iv) $\omega_i$ points of $f$ which are different in multiplicities from the corresponding $\omega_j$ points of $g$, where $i,j\in\{1,2,\ldots,n\}$ or we can say 1 points of $F$ which are different in multiplicities from the corresponding 1 points of $G$.

(v) Zeros of $f'$ and $g'$ which are not zeros of $f\prod_{i=1}^{m}(f-e_i)(F-1)$ and $g\prod_{i=1}^{m}(g-e_i)(G-1)$ respectively.

In view of (11), we can also say that $H$ has only simple poles. □

**Lemma 4.** *Let $F$ and $G$ be given by (5) and $H \not\equiv 0$. If $F$, $G$ share $(1,l)$, then*

$$\left(\frac{n}{2}-m\right)\,[T(r,f)+T(r,g)] \leq 2\left[\overline{N}(r,0;f)+\overline{N}(r,0;g)\right]+$$
$$+2\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right]-(l-\frac{3}{2})\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).$$

*Proof.* By the second fundamental theorem we get

$$(n+m)T(r,f)\leq\overline{N}(r,1;F)+\overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\sum_{i=1}^{m}\overline{N}(r,e_i;f)-N_0(r,0;f^{'})+S(r,f). \tag{12}$$

$$(n+m)T(r,g)\leq\overline{N}(r,1;G)+\overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\sum_{i=1}^{m}\overline{N}(r,e_i;g)-N_0(r,0;g^{'})+S(r,g). \tag{13}$$

Now combining (12), (13) and using Lemma 2, Lemma 1 and Lemma 3 we get

$$(n+m)\,[T(r,f)+T(r,g)] \leq \overline{N}(r,1;F)+\overline{N}(r,1;G)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+$$
$$+\overline{N}(r,0;f)+\overline{N}(r,0;g)+\sum_{i=1}^{m}\overline{N}(r,e_i;f)+\sum_{i=1}^{m}\overline{N}(r,e_i;g)-N_0(r,0;g^{'})-N_0(r,0;f^{'})+$$
$$+S(r,f)+S(r,g)\leq\frac{n}{2}\,[T(r,f)+T(r,g)]+2\left[\overline{N}(r,0;f)+\overline{N}(r,0;g)\right]+2\sum_{i=1}^{m}[\overline{N}(r,e_i;f)+$$
$$+\overline{N}(r,e_i;g)]-\left(l-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+2\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right]+S(r,f)+S(r,g),$$

which proves the lemma. □

**Lemma 5** ([5])**.** *Let $\Psi(z)=c^2(z^{n-m}-A)^2-4b(z^{n-2m}-A)(z^n-A)$, where $A,c,b\in\mathbb{C}-\{0\}$, $\frac{c^2}{4b}=\frac{n(n-2m)}{(n-m)^2}$, $\gcd(m,n)=1$, $n>2m$. If $\omega^l$ is the $m$-th root of unity for $l=0,1,\ldots,m-1$, then*
*i) $\Psi(z)$ has no multiple zero, when $A\neq\omega^l$.*
*ii) $\Psi(z)$ has exactly one multiple zero, when $A=\omega^l$ and that is of multiplicity 4.*
*In particular, when $A=1$, then the multiple zero is 1.*

**Lemma 6.** *Let $F$ and $G$ be given by (5). If $F$, $G$ share $(1,l)$, where $0\leq l<\infty$ . Then*

$$\overline{N}_*(r,1;F,G)\leq\frac{1}{l+1}[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)]+S(r,f)+S(r,g).$$

*Proof.* Using $\overline{N}_*(r,1;F,G)=\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)$, the lemma can be proved same as in the line of proof of Lemma 2.10 in [1]. □

**Lemma 7.** *Let $F$ and $G$ be defined by (5). Then $FG\neq 1$ for $n>2+\frac{2}{2md-1}$.*

*Proof.* On the contrary suppose that $FG=1$. That is

$$\frac{af^n}{b\prod_{i=1}^{2m}(f-\alpha_i)}\frac{ag^n}{b\prod_{i=1}^{2m}(g-\alpha_i)}=1, \tag{14}$$

which clearly implies each pole of $F$ is a zero of $G$ and vice-versa. Suppose $z_1$ be a pole of $f$ of multiplicity $p$ and a zero of $g$ of multiplicity $q$, then $p(n-2m)=nq$; i.e., $n(p-q)=2mp$; i.e., $p \geq \frac{n}{2m}$. Therefore $\overline{N}(r,\infty;f) \leq \frac{2m}{n} N(r,\infty;f)$. Using similar arguments we can prove that each $\alpha_i$ point of $f$ is of multiplicity at least $n$ and hence $\overline{N}(r,\alpha_i;f) \leq \frac{1}{n} N(r,\alpha_i;f)$. Furthermore, (4) clearly implies each zero of $f$ is of multiplicity at least $d$ and hence $\overline{N}(r,0;f) \leq \frac{1}{d} N(r,0;f)$. Therefore by the second fundamental theorem we can have

$$2m\, T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \sum_{i=1}^{2m} \overline{N}(r,\alpha_i;f) + S(r,f) \leq$$
$$\leq \frac{1}{d} T(r,f) + \frac{2m}{n} T(r,f) + \frac{2m}{n} T(r,f) + S(r,f),$$

which is a contradiction for $n > \frac{2d}{(d-\frac{1}{2m})} = 2 + \frac{2}{2md-1}$. □

**Remark 8.** Observe that in Lemma 7, if we consider $f$ and $g$ to be entire functions, then using the similar arguments as above with $\overline{N}(r,\infty;f) = S(r,f)$, we would have a contradiction as $n > 2m > \frac{2m}{2m-\frac{1}{d}}$.

**Lemma 8.** *Let $F$ and $G$ be defined by (5). Then $F \equiv G$ implies $f \equiv g$ for $n > m + 2 + \frac{2}{d}$.*

*Proof.* Since $F \equiv G$, we have

$$\frac{f^n}{bf^{2m} + cf^m + 1} = \frac{g^n}{bg^{2m} + cg^m + 1} \tag{15}$$

$$\Longrightarrow b\left[f^n g^{2m} - g^n f^{2m}\right] + c\left[f^n g^m - g^n f^m\right] + \left[f^n - g^n\right] = 0.$$

Substituting $h = \frac{g}{f}$ in the above equation we get

$$bf^{2m}h^{2m}(1-h^{n-2m}) + cf^m h^m (1-h^{n-m}) + (1-h^n) = 0. \tag{16}$$

If $h$ is non-constant, then from (16) we have

$$\left[f^m + \frac{c(1-h^{n-m})}{2bh^m(1-h^{n-2m})}\right]^2 = \frac{c^2(h^{n-m}-1)^2 - 4b(h^n-1)(h^{n-2m}-1)}{4b^2h^{2m}(1-h^{n-2m})^2} = \tag{17}$$
$$= \frac{\Psi(h)}{4b^2h^{2m}(1-h^{n-2m})^2}.$$

Since $\Psi(z)$ is a polynomial of degree $2n-2m$, so from Lemma 5, we can write

$$\Psi(h) = (h-1)^4 \prod_{i=1}^{2n-2m-4} (h-\beta_i),$$

where $\beta_i$'s are distinct elements of $\mathbb{C}\setminus\{0,1\}$. From (17) it is clear that each $\beta_i$ point of $h$ is of multiplicity at least 2. Further, we have

$$h = \frac{f}{g} = \frac{f_1^{d+r}}{g_1^{d+s}}. \tag{18}$$

Since (15) implies that $f$ and $g$ share 0 CM, so zeros of $h$ comes from the poles of $g$. Again poles of $g$ are the poles of $f$ and $\alpha_i$ points of $f$. Observe that from (15) we can easily verify that those poles of $g$ which are poles of $f$ must have the same multiplicity. Hence none of such poles of $g$ can belong to the zeros of $h$. Therefore zeros of $h$ are precisely those poles of $g$ which are the $\alpha_i$ points of $f$ and in view of (18) those zeros of $h$ are of multiplicity at least $d$. Resorting similar arguments to the poles of $h$ we shall obtain that each pole of $h$ is of multiplicity at least $d$. Hence using the second fundamental theorem we get

$$\begin{aligned}(2n-2m-4)T(r,h) &\leq \sum_{i=1}^{2n-2m-4} \overline{N}(r,\beta_i;h)+\overline{N}(r,0;h)+\overline{N}(r,\infty;h)+S(r,h)\\ &\leq \frac{1}{2}\sum_{i=1}^{2n-2m-4} N(r,\beta_i;h)+\frac{1}{d}N(r,0;h)+\frac{1}{d}N(r,\infty;h)+S(r,h)\leq\\ &\leq \Big(n-m-2+\frac{2}{d}\Big)T(r,h)+S(r,h),\end{aligned}$$

which is a contradiction for $n>m+2+\frac{2}{d}$. Thus $h$ is a constant. Since $f$ is non-constant, so (16) clearly implies that $h^n-1=h^{n-m}-1=h^{n-2m}-1=0$. Hence $h^n=h^{n-m}=h^{n-2m}=1$. Now as $\gcd(m,n)=1$, so $h=1$; i.e., $f=g$. □

**Remark 9.** Note that if $f$ and $g$ are entire functions in Lemma 8, then one can easily conclude that $h$ omits 0 and $\infty$ using similar arguments as above for the zeros and poles of $h$. Hence we would have $f\equiv g$ for $n>m+2$.

## 4. Proof of the Theorems.

*Proof of Theorem 1.* **[i]** Let $F$ and $G$ be defined by (5). Then $F$, $G$ share $(1,0)$.

**Case-1.** Suppose $H\not\equiv 0$. Then using Lemma 6 for $l=0$ in Lemma 4 we get

$$\begin{aligned}&(\frac{n}{2}-m)\;[T(r,f)+T(r,g)]\leq \frac{3}{2}\overline{N}_*(r,1;F,G)+2\left[\overline{N}(r,0;f)+\overline{N}(r,0;g)\right]+2[\overline{N}(r,\infty;f)+\\ &+\overline{N}(r,\infty;g)]+S(r,f)+S(r,g)\leq\left(\frac{3}{2}\right)[\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)]+\\ &+\frac{4}{d}\,[T(r,f)+T(r,g)]+S(r,f)+S(r,g)\leq\left(\frac{7}{d}\right)[T(r,f)+T(r,g)]+S(r,f)+S(r,g),\end{aligned}$$

which is a contradiction for $n>2m+\frac{14}{d}$.

**Case-2.** Suppose $H\equiv 0$. Then on integration we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{19}$$

where $A(\neq 0),B$ are complex constants. From (19), clearly we have

$$T(r,f)=T(r,g)+S(r,g). \tag{20}$$

Now we can write (19) as

$$F=\frac{(B+1)G+A-B-1}{BG+A-B}. \tag{21}$$

Hence let us consider the following subcases.

**Subcase-2.1** Let $B \neq 0$.

**Subcase-2.1.1** Let $B \neq -1$. Obviously $\frac{A-B-1}{B+1} \neq \frac{A-B}{B}$. For if $\frac{A-B-1}{B+1} = \frac{A-B}{B}$, then $A=0$, which is absurd. Therefore

$$\overline{N}(r, -\frac{A-B-1}{B+1}; G) = \overline{N}(r, 0; F). \tag{22}$$

Now using the second fundamental theorem we have

$$T(r,G) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r, -\frac{A-B-1}{B+1}; G) + S(r,G) \leq \overline{N}(r,0;g) +$$
$$+\overline{N}(r,\infty;g) + \sum_{i=1}^{2m} \overline{N}(r,\alpha_i;g) + \overline{N}(r,0;f) + S(r,G) \leq \left(\frac{2m+\frac{3}{d}}{n}\right) T(r,G) + S(r,G),$$

which is a contradiction for $n > 2m + \frac{3}{d}$.

**Subcase-2.1.2.** Let $B=-1$. Then from (21) we get

$$F = \frac{A}{-G+A+1}. \tag{23}$$

**Subcase-2.1.2.1.** Let $A+1 \neq 0$. Then $\overline{N}(r, A+1; G) = \overline{N}(r,\infty;F)$ and of course $\overline{N}(r,\infty;G) = \overline{N}(r,0;F)$. Now using the second fundamental theorem we have

$$T(r,G) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,A+1;G) + S(r,G) \leq \overline{N}(r,0;g) +$$
$$+\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \sum_{i=1}^{2m} \overline{N}(r,\alpha_i;f) + S(r,G) \leq \left(\frac{2m+\frac{3}{d}}{n}\right) T(r,G) + S(r,G),$$

which is a contradiction for $n > 2m + \frac{3}{d}$.

**Subcase-2.1.2.2.** Let $A+1=0$. Then $FG=1$. Since $n > 2m + \frac{14}{d}$, so in view of Lemma 7, this case is invalid.

**Subcase-2.2.** Suppose $B=0$ then from (21) we get

$$AF = G + A - 1. \tag{24}$$

**Subcase-2.2.1.** Let $A \neq 1$. Therefore (24) implies $\overline{N}(r,0;F) = \overline{N}(r, 1-A; G)$. Similarly proceeding like Subcase-2.1.1 we shall arrive to a contradiction for $n > 2m + \frac{3}{d}$.

**Subcase-2.2.2.** Let $A=1$ i.e., $F \equiv G$. So in view of Lemma 8, we get $f \equiv g$; i.e., $f_1^{d+r} = g_1^{d+s}$ for $n > \max\left[m+2+\frac{2}{d}, 2m+\frac{14}{d}\right]$.

**[ii]** Let $F$ and $G$ be defined by (5). Then $F$, $G$ share $(1,1)$.

**Case-1** Suppose $H \not\equiv 0$. Then using Lemma 6 for $l=1$ in Lemma 4 we get

$$\left(\frac{n}{2} - m\right) \left[T(r,f) + T(r,g)\right] \leq \frac{1}{2}\overline{N}_*(r,1;F,G) + 2\left[\overline{N}(r,0;f) + \overline{N}(r,0;g)\right] +$$
$$+2\left[\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\right] + S(r,f) + S(r,g) \leq$$
$$\leq \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left[\overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\right]$$

$$+\frac{4}{d}\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g)\leq\left(\frac{9}{2d}\right)\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g),$$

which is a contradiction for $n>2m+\frac{9}{d}$.

**Case-2** Suppose $H\equiv 0$. Then this case can be dealt same as in the line of the proof of Case-2 of Part [i] of this theorem. We omit the details.

**[iii]** Let $F$ and $G$ be defined by (5). Then $F$, $G$ share $(1,2)$.

**Case-1** Suppose $H\not\equiv 0$. Then using Lemma 6 for $l=2$ in Lemma 4 we get

$$\left(\frac{n}{2}-m\right)\left[T(r,f)+T(r,g)\right]\leq 2\left[\overline{N}(r,0;f)+\overline{N}(r,0;g)\right]+$$

$$+2\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right]+S(r,f)+S(r,g)\leq\frac{4}{d}\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g),$$

which is a contradiction for $n>2m+\frac{8}{d}$.

**Case-2.** Suppose $H\equiv 0$. This case can also be considered same as proof of Case-2 of Part [i] of this theorem. □

**5. Discussion and Some Open Problems.** Now let us have a brief discussion about the existence of a URSM between the cardinality 5 and 11. On the contrary suppose that there does not exist any URSM of cardinality 7 to 10. Then for any set of cardinality 7 to 10 there must exist two distinct meromorphic functions which share the set. But in that case, according to item (b) of Remark 2, we are sure that the functions must not be of the form like $f^p$ and $g^q$ for $p,q\geq 2$. That is, at least one of them must belong to the class of functions $M(\mathbb{C})\backslash M^2(\mathbb{C})$. So the research on sharing of any set for the class of functions $M(\mathbb{C})\backslash M^2(\mathbb{C})$ together with $M(\mathbb{C})$ demands attention to get out of this mystery of least cardinality of a URSM. Similarly for the cardinality 6, the research partially shrinks to the class of meromorphic functions $M(\mathbb{C})\backslash M^3(\mathbb{C})$. In case of URSE, the problem of least cardinality is open for 5 and 6 only. So in that case the research should partially be reduced to the class of functions $E(\mathbb{C})\backslash E^2(\mathbb{C})$, according to item (b) of Remark 2. Similar observations can also be made for URSM-IM(URSE-IM).

Note that in the examples for the non-existence of URSM (URSE) [19, see page 527, Theorem 10.72 and page 517, Theorem 10.59] of cardinality 5 (4), the researchers had to use distinct functions belong to $M(\mathbb{C})\backslash M^3(\mathbb{C})$ and $M(\mathbb{C})$ ($E(\mathbb{C})\backslash E^3(\mathbb{C})$ *and* $E(\mathbb{C})$) respectively, due to item (b) of Remark 3. Thus, in view of this, Gross Problem or the parallel problems for the unique range sets should be precise to the following problems.

**Problem 1.** *Can one find a finite set $S$ of cardinality* 5 *or* 6 *such that for two non-constant functions $f\in E(\mathbb{C})-E^2(\mathbb{C})$ and $g\in E(\mathbb{C})$ satisfying $f^{-1}(S)=g^{-1}(S)$ CM must be identical?*

**Problem 2.** *Can one find a finite set $S$ of cardinality* 7 *to* 10 *such that for two non-constant functions $f\in M(\mathbb{C})-M^2(\mathbb{C})$ and $g\in M(\mathbb{C})$ satisfying $f^{-1}(S)=g^{-1}(S)$ CM must be identical?*

**Problem 3.** *Can one find a finite set $S$ of cardinality* 6 *such that for two non-constant functions $f\in M(\mathbb{C})-M^3(\mathbb{C})$ and $g\in M(\mathbb{C})$ satisfying $f^{-1}(S)=g^{-1}(S)$ CM must be identical?*

The answers to these problems would settle the least cardinality problem of URSM (URSE) completely.

In relation to this, we also pose the following problems to fix the least cardinality problem of URSM-IM(URSE-IM) too.

**Problem 4.** *Can one find a finite set $S$ of cardinality $6$ to $9$ such that for two non-constant functions $f \in E(\mathbb{C})\backslash E^2(\mathbb{C})$ and $g \in E(\mathbb{C})$ satisfying $f^{-1}(S) = g^{-1}(S)$ IM must be identical?*

**Problem 5.** *Can one find a finite set $S$ of cardinality $5$ such that for two non-constant functions $f \in E(\mathbb{C})\backslash E^3(\mathbb{C})$ and $g \in E(\mathbb{C})$ satisfying $f^{-1}(S) = g^{-1}(S)$ IM must be identical?*

**Problem 6.** *Can one find a finite set $S$ of cardinality $10$ to $16$ such that for two non-constant functions $f \in M(\mathbb{C})\backslash M^2(\mathbb{C})$ and $g \in M(\mathbb{C})$ satisfying $f^{-1}(S) = g^{-1}(S)$ IM must be identical?*

**Problem 7.** *Can one find a finite set $S$ of cardinality $7$ to $9$ such that for two non-constant functions $f \in M(\mathbb{C})\backslash M^3(\mathbb{C})$ and $g \in M(\mathbb{C})$ satisfying $f^{-1}(S) = g^{-1}(S)$ IM must be identical?*

**Problem 8.** *Can one find a finite set $S$ of cardinality $6$ such that for two non-constant functions $f \in M(\mathbb{C})\backslash M^4(\mathbb{C})$ and $g \in M(\mathbb{C})$ satisfying $f^{-1}(S) = g^{-1}(S)$ IM must be identical?*

Finally we have the following very interesting problem.

**Problem 9.** *How small can we have a set for the uniqueness of two arbitrary functions belonging to any special class of meromorphic functions? If it's cardinality is less than $4$, then what is the set?*

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*Received 30.10.2018*
*Revised 15.12.2018*