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ON BELONGING OF ENTIRE DIRICHLET SERIES TO A MODIFIED GENERALIZED CONVERGENCE CLASS

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For entire Dirichlet series $F(s) = \sum_{n=0}^{+\infty} a_n e^{s\lambda_n}$ we found conditions on a_n, λ_n and on positive functions α and β continuous increasing to $+\infty$ on $[0, +\infty)$ are found, under which the condition $\int_{\sigma_0}^{+\infty} \frac{1}{\beta(\sigma)}\alpha\left(\frac{1}{\sigma}\ln M(\sigma, F)\right) d\sigma < +\infty$ is equivalent to the condition

$$\sum_{n=1}^{+\infty}(\alpha(\lambda_n) - \alpha(\lambda_{n-1}))\beta_1\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right) < +\infty,$$

where $\beta_1(x) = \int_x^{+\infty}\frac{dt}{\beta(t)}$, and $M(\sigma, F) = \sup\{|F(\sigma + it)|\colon t \in \mathbb{R}\}$.

1. Introduction. G. Valiron [1, p.18] proved that if an entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ has the order $\varrho \in (0, +\infty)$ and belongs to the convergence class, that is

$$\int_1^{+\infty} \frac{\ln M_f(r)}{r^{\varrho+1}} < +\infty, \quad M_f(r) := \max\{|f(z)|\colon |z| = r\},$$

then

$$\sum_{n=1}^{+\infty} |a_n|^{\varrho/n} < +\infty.$$

This result was generalized in [2, 3] for the case of entire (absolutely convergent in $\mathbb{C}$) Dirichlet series of the form

$$F(s) = a_0 + \sum_{n=1}^{+\infty} a_n e^{s\lambda_n}, \quad 0 < \lambda_n \uparrow +\infty \ (1 \le n \uparrow +\infty). \tag{1}$$

It was proved ([3, Theorem 2]) that if entire Dirichlet series (1) has R-order $\varrho_R = \varrho \in (0, +\infty)$ and $\ln n = O(\lambda_n)$ as $n \to +\infty$, then in order that relation

$$\int_0^{+\infty} \frac{\ln M(\sigma, F)}{r^{\varrho+1}} < +\infty, \quad M(\sigma, F) := \sup\{|F(\sigma + it)|\colon t \in \mathbb{R}\}, \tag{2}$$

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be satisfied it is necessary and in the case when $\varkappa_n(F) := \frac{\ln|a_n|-\ln|a_{n+1}|}{\lambda_{n+1}-\lambda_n} \nearrow +\infty$ as $n\to+\infty$ it is sufficient that

$$\sum_{n=1}^{+\infty}(\lambda_n-\lambda_{n-1})|a_n|^{\varrho/\lambda_n}<+\infty.$$

In [2], this result was proved by additional assumption that the sequence (λ_n) has a positive finite step, that is $0<h\le\lambda_{n+1}-\lambda_n\le H<+\infty$ for $n\ge 0$.

In papers [3,4], we also proved the results about belonging of Dirichlet series to so-called the *generalized convergence* $\alpha\beta$*-class*. To give its definition we denote, as in [5], by L the class of continuous nonnegative on $(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\ge 0$ for $x\le x_0$ and $\alpha(x)\uparrow+\infty$ as $x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$.

The following theorem is proved in [4].

Theorem 1 ([4]). *Let α be a concave function on $[x_0,+\infty)$ and $\alpha(e^x)\in L^0$, and a function $\beta\in L^0$ satisfies the conditions $x\beta'(x)/\beta(x)\ge h>0$ for $x\ge x_0$ and*

$$\int_{x_0}^{+\infty}\frac{\alpha(x)}{\beta(x)}dx<+\infty. \tag{3}$$

Suppose that the exponents λ_n of entire Dirichlet series (1) *satisfies the condition $\ln n = o(\lambda_n\beta^{-1}(\alpha(\lambda_n)))$ as $n\to+\infty$. In order that*

$$\int_{\sigma_0}^{+\infty}\frac{\alpha(\ln M(\sigma,F))}{\beta(\sigma)}d\sigma<+\infty, \tag{4}$$

it is necessary and in the case, when $\varkappa_n(F)\nearrow+\infty,\ n\to+\infty$, it is sufficient that

$$\sum_{n=1}^{+\infty}(\alpha(\lambda_n)-\alpha(\lambda_{n-1}))\beta_1\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)<+\infty,\quad \beta_1(x)=\int_x^{+\infty}\frac{dt}{\beta(t)}.$$

Besides the generalized convergence $\alpha\beta$-class defined by condition (4) for entire Dirichlet series we also define the *modified generalized convergence* $\alpha\beta$*-class* by the condition

$$(\exists\sigma_0>0)\colon\quad \int_{\sigma_0}^{+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\ln M(\sigma,F)}{\sigma}\right)d\sigma<+\infty \tag{5}$$

for $\alpha\in L$ and $\beta\in L$. Here we obtain an analog of Theorem 1 for the modified generalized convergence $\alpha\beta$-class.

2. Modified generalized convergence $\alpha\beta$-class in terms of maximal term. For entire Dirichlet series (1) let $\mu(\sigma,F)=\max\{|a_n|\exp\{\sigma\lambda_n\}\colon n\ge 0\}$ be its maximal term and $\nu(\sigma)=\max\{n\colon |a_n|\exp\{\sigma\lambda_n\}=\mu(\sigma,F)\}$ be its central index.

At first, we investigate conditions under which restriction (5) is equivalent to the condition

$$(\exists\sigma_0>0)\colon\quad \int_{\sigma_0}^{+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\ln \mu(\sigma,F)}{\sigma}\right)d\sigma<+\infty. \tag{6}$$

We need the following properties of the functions from L^0 proved in [6, p. 25] and [7].

Lemma 1 ([6, 7]). *Let* $\alpha \in L$ *and* $B(\varepsilon) = \varlimsup_{x\to+\infty} \frac{\alpha((1+\varepsilon)x)}{\alpha(x)}$, $\varepsilon > 0$. *In order that* $\alpha \in L^0$, *it is necessary and sufficient that* $B(\varepsilon) \to 1$ *as* $\varepsilon \to 0$.

If $\alpha \in L^0$ *then* α *is RO-increasing* ([8]), *i.e. for every* $h \in [1, a]$, $1 < a < +\infty$, *and all* $x \geq x_0$ *the inequality* $\frac{\alpha(hx)}{\alpha(x)} \leq M(a) < +\infty$ *holds.*

Using this lemma we prove the following theorem.

Theorem 2. *Let* $\alpha \in L^0$ *and* $\beta(x) = x\gamma(x)$, $\gamma \in L^0$. *If*

$$\ln n = O\left(\lambda_n \gamma^{-1}(\alpha(\lambda_n))\right), \quad n \to +\infty, \tag{7}$$

then conditions (5) *and* (6) *are equivalent.*

Proof. In view of Cauchy's inequality $\mu(\sigma, F) \leq M(\sigma, f)$ from (5) we have (6). On the other hand (see [9, p.182] and [6, p.17]),

$$\ln \mu(\sigma, F) = \ln \mu(\sigma_0, F) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t,F)} dt. \tag{8}$$

Since $\lambda_{\nu(\sigma,F)}$ is nondecreasing function, the function $\frac{1}{\sigma}\int_{\sigma_0}^{\sigma} \lambda_{\nu(t,F)} dt$ ($0 \leq \sigma_0 \leq \sigma < +\infty$) is also nondecreasing. Therefore, in view of the condition $\alpha \in L^0$ from (8) we have

$$\alpha\left(\frac{\ln \mu(\sigma, F)}{\sigma}\right) = (1 + o(1))\alpha\left(\frac{1}{\sigma}\int_{\sigma_0}^{\sigma} \lambda_{\nu(t,F)} dt\right), \quad \sigma \to +\infty,$$

and in view of (8) for every $\varepsilon > 0$ and for all $\sigma \geq \sigma_0(\varepsilon)$

$$\varepsilon > \int_{\sigma}^{+\infty} \frac{1}{\beta(t})\alpha\left(\frac{\ln \mu(t, F)}{t}\right) dt \geq (1 + o(1))\alpha\left(\frac{1}{\sigma}\int_{\sigma_0}^{\sigma} \lambda_{\nu(t,F)} dt\right) \int_{\sigma}^{+\infty} \frac{dx}{x\gamma(x)} \geq$$

$$\geq (1 - \varepsilon)\alpha\left(\frac{\ln \mu(\sigma, F)}{\sigma}\right) \int_{\sigma}^{2\sigma} \frac{dx}{x\gamma(x)} \geq (1 - \varepsilon)\alpha\left(\frac{\ln \mu(\sigma, F)}{\sigma}\right) \frac{\ln 2}{\gamma(2\sigma)},$$

whence

$$\alpha\left(\frac{\ln \mu(\sigma, F)}{\sigma}\right) \leq \varepsilon_1 \gamma(2\sigma), \quad \varepsilon_1 = \frac{\varepsilon}{(1 - \varepsilon)\ln 2} \to 0 \ (\varepsilon \to 0),$$

for all large enough σ and, thus,

$$\ln |a_n| \leq \ln \mu(\sigma, F) - \sigma\lambda_n \leq \sigma(\alpha^{-1}(\varepsilon_1\gamma(2\sigma)) - \lambda_n)$$

for all $n \geq 0$ and for all large enough σ. We choose $\sigma = \sigma_n = \frac{1}{2}\gamma^{-1}\left(\frac{1}{\varepsilon_1}\alpha\left(\frac{\lambda_n}{2}\right)\right)$. Then

$$\ln |a_n| \leq -\frac{\lambda_n}{2}\sigma_n = -\frac{\lambda_n}{4}\gamma^{-1}\left(\frac{1}{\varepsilon_1}\alpha\left(\frac{\lambda_n}{2}\right)\right).$$

Since $\alpha \in L^0$, by Lemma 1 $\alpha(\lambda) \leq M\alpha(\lambda/2)$ and, therefore,

$$-\ln |a_n| \geq \frac{\lambda_n}{4}\gamma^{-1}\left(\frac{1}{M\varepsilon_1}\alpha(\lambda_n)\right),$$

i. e.

$$\varlimsup_{n\to+\infty}\frac{\ln n}{-\ln|a_n|}\le 4\varlimsup_{n\to+\infty}\left(\frac{\ln n}{\lambda_n\gamma^{-1}(\alpha(\lambda_n))}\frac{\lambda_n\gamma^{-1}(\alpha(\lambda_n))}{\lambda_n\gamma^{-1}\left(\frac{1}{M\varepsilon_1}\alpha\left(\lambda_n\right)\right)}\right)\le$$
$$\le 4\varlimsup_{n\to+\infty}\frac{\ln n}{\lambda_n\gamma^{-1}(\alpha(\lambda_n))}\varlimsup_{x\to+\infty}\frac{\gamma^{-1}(M\varepsilon_1 x)}{\gamma^{-1}(x)}.$$

We put $A(\varepsilon)=\varlimsup\limits_{x\to+\infty}\frac{\gamma^{-1}(\varepsilon x)}{\gamma^{-1}(x)}$ and prove that $A(\varepsilon)\to 0$ as $\varepsilon\to 0$. Indeed, the function $A(\varepsilon)$ does not decrease on $(0,\varepsilon_0]$. Therefore, if $A(\varepsilon)\not\to 0$ as $\varepsilon\to 0$ then $A(\varepsilon)\ge h>0$ and there exists an increasing to $+\infty$ sequence (x_k) such that $\frac{\gamma^{-1}(\varepsilon x_k)}{\gamma^{-1}(x_k)}\ge h/2$, i. e. by Lemma 1 $\varepsilon x_k\ge\gamma\left(h\gamma^{-1}(x_k)/2\right)\ge x_k/M$. It is impossible in view of the arbitrariness of ε. Thus, in view of (7) we have $\varlimsup\limits_{n\to+\infty}\frac{\ln n}{-\ln|a_n|}=0$. In [6, p. 23] it is proved that if $\varlimsup\limits_{n\to+\infty}\frac{\ln n}{-\ln|a_n|}=h_0<1$ then for every $\varepsilon\in(0,\,1-h_0)$ there exists $A_0(\varepsilon)\in(0,\,+\infty)$ such that for all $\sigma>0$ the inequality

$$M(\sigma,F)\le A_0(\varepsilon)\mu\left(\frac{\sigma}{1-h_0-\varepsilon},F\right).$$

holds. Since now $h_0=0$, by this statement one has $\ln M(\sigma,F)\le\ln\mu((1+\varepsilon)\sigma,F)+A_1(\varepsilon)$ for each $\varepsilon\in(0,\,1)$, all $\sigma>0$ and some $A_1(\varepsilon)<+\infty$. Hence, it follows that

$$\int\limits_{\sigma_0}^{+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\ln M(\sigma,F)}{\sigma}\right)d\sigma\le\int\limits_{\sigma_0}^{+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\ln\mu((1+\varepsilon)\sigma,F)+A_1(\varepsilon)}{\sigma}\right)d\sigma=$$
$$=\int\limits_{\sigma_0}^{+\infty}\frac{1}{\beta((1+\varepsilon)\sigma)}\alpha\left((1+\varepsilon)\frac{\ln\mu((1+\varepsilon)\sigma,F)+A_1(\varepsilon)}{(1+\varepsilon)\sigma}\right)\frac{\beta((1+\varepsilon)\sigma)}{\beta(\sigma)}d\sigma.$$

Using Lemma 1, the last inequality and (6), in view of the conditions $\alpha\in L^0$, $\beta(x)=x\gamma(x)$ and $\gamma\in L^0$ imply correlation (5).

3. Main theorem. To description the belonging of entire Dirichlet series to the modified generalized convergence class by its coefficients we need three lemmas.

Lemma 2. *If $\alpha\in L^0$ and $\beta\in L^0$ then for entire Dirichlet series* (1) *correlation* (6) *is equivalent to*

$$\int\limits_{\sigma_0}^{+\infty}\frac{\alpha(\lambda_{\nu(\sigma,F)})}{\beta(\sigma)}d\sigma<+\infty, \tag{9}$$

Indeed, using (8) we have

$$\ln\mu(\sigma,F)-\ln\mu(0,F)=\int\limits_0^{\sigma}\lambda_{\nu(t,F)}dt\le\sigma\lambda_{\nu(\sigma,F)}$$

and

$$\ln\mu(\sigma,F)-\ln\mu(0,F)\ge\int\limits_{\sigma/2}^{\sigma}\lambda_{\nu(t,F)}dt\ge\frac{1}{2}\sigma\lambda_{\nu(\sigma/2,F)},$$

whence the necessary result follows. □

Majorant of Newton for Dirichlet series (see [9, p.180–183] and [3]) also plays a part.

Lemma 3. *Let*

$$F^0(s) = \sum_{n=0}^{+\infty} a_n^0 e^{s\lambda_n} \tag{10}$$

be the majorant of Newton for entire Dirichlet series (1). *Then* $\mu(\sigma, F^0) = \mu(\sigma, F)$, $\nu(\sigma, F^0) = \nu(\sigma, F)$, $\varkappa_n(F^0) \nearrow +\infty$ *as* $n \to +\infty$ *and* $|a_n| \le a_n^0$ *for all* $n \ge 1$.

Let $p > 1$, $q = \frac{p}{p-1}$ and the function f be positive on $(A,\, B)$, $-\infty \le A < B \le +\infty$. If (λ_n^*) is a sequence of positive number and (c_n) is a sequence of numbers from $(A,\, B)$ then

$$C_n := \frac{\lambda_1^* c_1 + \cdots + \lambda_n^* c_n}{\lambda_1^* + \cdots + \lambda_n^*} \in (A,\, B).$$

It is proved [3] the following generalization of classical Hardy's inequality.

Lemma 4 ([3])**.** *Let* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If the function* $f^{1/p}$ *is convex on* $(A,\, B)$ *and the sequence* (μ_n) *is positive and nonincreasing then for each* $\omega \le +\infty$

$$\sum_{n=1}^{\omega} \mu_n \lambda_n^* f(C_n) \le q^p \sum_{n=1}^{\omega} \mu_n \lambda_n^* f(c_n). \tag{11}$$

Using Theorem 2 and Lemmas 2–4, we prove the following main theorem.

Theorem 3. *Let* $\alpha \in L^0$ *be a concave function on* $[x_0, +\infty)$ *and* $\beta(x) = x\gamma(x)$, *where* $\gamma \in L^0$ *and*

$$1 + \frac{x\gamma'(x)}{\gamma(x)} \ge h > 0, \quad x \ge x_0, \tag{12}$$

Suppose that condition (3) *holds and the exponents* λ_n *of entire Dirichlet series* (1) *satisfy condition* (7). *In order that series* (1) *belong to modified generalized convergence* $\alpha\beta$*-class, it is necessary and in the case, when* $\varkappa_n(F) \nearrow +\infty$, $n \to +\infty$, *it is sufficient that*

$$\sum_{n=1}^{+\infty} (\alpha(\lambda_n) - \alpha(\lambda_{n-1}))\gamma_1\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right) < +\infty, \quad \gamma_1(x) = \int_x^{+\infty} \frac{d \ln \sigma}{\gamma(\sigma)}. \tag{13}$$

Proof. At first, we remark that condition (3) implies

$$\int_{x_0}^{+\infty} \frac{d \ln x}{\gamma(x)} dx = \int_{x_0}^{+\infty} \frac{1}{\beta(x)} dx < +\infty.$$

Therefore, in view of Theorem 2 and Lemma 2 it is necessary to find conditions under which (9) holds. Since by Lemma 3 $\nu(\sigma, F^0) = \nu(\sigma, F)$ and for the majorant of Newton (10) $\varkappa_n(F^0) \nearrow +\infty$ as $n \to +\infty$, we have

$$\int_{\sigma_0}^{+\infty} \frac{\alpha(\lambda_{\nu(\sigma,F)})}{\beta(\sigma)} d\sigma = \int_{\sigma_0}^{+\infty} \frac{\alpha(\lambda_{\nu(\sigma,F^0)})}{\sigma\gamma(\sigma)} d\sigma = \sum_{n=1}^{+\infty} \int_{\varkappa_{n-1}(F^0)}^{\varkappa_n(F^0)} \frac{\alpha(\lambda_{\nu(\sigma,F^0)})}{\sigma\gamma(\sigma)} d\sigma + \text{const} =$$

$$= \sum_{n=1}^{+\infty} \alpha(\lambda_n) \int\limits_{\varkappa_{n-1}(F^0)}^{\varkappa_n(F^0)} \frac{d\ln \sigma}{\gamma(\sigma)} + \text{const} = \sum_{n=1}^{+\infty} \alpha(\lambda_n)(\gamma_1(\varkappa_{n-1}(F^0)) - \gamma_1(\varkappa_n(F^0))) + \text{const} =$$

$$= \sum_{n=1}^{+\infty} (\alpha(\lambda_n) - \alpha(\lambda_{n-1}))\gamma_1(\varkappa_{n-1}(F^0)) + \text{const}. \tag{14}$$

From conditions (12) and $\gamma \in L^0$ we obtain

$$(\gamma(x) + x\gamma'(x))\gamma_1(x) = (\gamma(x) + x\gamma'(x)) \int\limits_x^{+\infty} \frac{d\ln \sigma}{\gamma(\sigma)} \geq (\gamma(x) + x\gamma'(x)) \int\limits_x^{2x} \frac{d\ln \sigma}{\gamma(\sigma)} \geq$$

$$\geq \frac{\gamma(x) + x\gamma'(x)}{\gamma(2x)} \ln 2 \geq \frac{\gamma(x) + x\gamma'(x)}{c\gamma(x)} \ln 2 = \frac{\ln 2}{c}\left(1 + \frac{x\gamma'(x)}{\gamma(x)}\right) \geq \frac{h \ln 2}{c} > 0.$$

We choose $p > 1$ such that $\frac{p-1}{p} < \frac{h\ln 2}{c}$. Then

$$(\gamma^{1/p}(x))'' = \frac{1}{p}\gamma_1^{1/p-2}(x)\left\{\gamma_1(x)\gamma_1''(x) - \frac{p-1}{p}(\gamma_1'(x))^2\right\} =$$

$$= \frac{1}{p}\gamma_1^{1/p-2}(x)\left\{\gamma_1(x)\frac{\gamma(x) + x\gamma'(x)}{x^2\gamma^2(x)} - \frac{p-1}{p}\frac{1}{x^2\gamma^2(x)}\right\} =$$

$$= \frac{\gamma_1^{1/p-2}(x)}{x^2\gamma^2(x)}\left\{(\gamma(x) + x\gamma'(x))\gamma_1(x) - \frac{p-1}{p}\right\} \geq \frac{\gamma_1^{1/p-2}(x)}{x^2\gamma^2(x)}\left\{\frac{h\ln 2}{c} - \frac{p-1}{p}\right\} > 0,$$

i. e. the function $\gamma^{1/p}(x)$ is convex on $[x_0, +\infty)$.

For simplicity, we suppose that in series (10) $a_0^0 = 1$. Then for other coefficients we have

$$\ln a_n^0 = \ln a_n^0 - \ln a_{n-1}^0 + \cdots + \ln a_1^0 - \ln a_0^0 + \ln a_0^0 =$$
$$= -\varkappa_{n-1}(F^0)(\lambda_n - \lambda_{n-1}) - \cdots - \varkappa_0(F^0)(\lambda_1 - \lambda_0),$$

i. e.

$$\frac{1}{\lambda_n}\ln\frac{1}{a_n^0} = \frac{\varkappa_0(F^0)\lambda_1^* + \cdots + \varkappa_{n-1}(F^0)\lambda_n^*}{\lambda_1^* + \cdots + \lambda_n^*}, \quad \lambda_n^* = \lambda_n - \lambda_{n-1}. \tag{15}$$

Since the function α is concave, the sequence $\mu_n = \frac{\alpha(\lambda_n)-\alpha(\lambda_{n-1})}{\lambda_n-\lambda_{n-1}}$ is nonincreasing. Therefore, by Lemma 4 with

$$f(x) = \gamma_1(x),\ \lambda_n^* = \lambda_n - \lambda_{n-1},\ C_n = \frac{1}{\lambda_n}\ln\frac{1}{a_n^0}$$

and $c_n = \varkappa_{n-1}(F^0)$, we have

$$\sum_{n=1}^{+\infty}(\alpha(\lambda_n) - \alpha(\lambda_{n-1}))\gamma_1\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right) = \sum_{n=1}^{+\infty}\mu_n\lambda_n^* f(C_n) \leq$$

$$\leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{+\infty}\mu_n\lambda_n^* f(c_n) = \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{+\infty}(\alpha(\lambda_n) - \alpha(\lambda_{n-1}))\gamma_1\left(\varkappa_{n-1}(F^0)\right). \tag{16}$$

If (9) holds then (14) and (16) imply (13) with a_n^0 instead of $|a_n|$.

On the contrary, (15) implies $\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\le\varkappa_{n-1}(F^0)$. And if (13) holds with a_n^0 instead $|a_n|$ then in view of (14) we obtain (9) because the function γ_1 is decreasing.

Theorem 3 is proved for the majorant of Newton. Therefore, condition (13) with a_n^0 instead of $|a_n|$ is necessary and sufficient for the belonging of F^0 to the modified generalized convergence $\alpha\beta$-class.

Since $|a_n|\le a_n^0$, we obtain the necessity of condition (13). If the sequence $(\varkappa_n(F))$ does not decrease then $|a_n|=a_n^0$, $\varkappa_n(F^0)=\varkappa_n(F)$ and condition (13) is sufficient. The proof of Theorem 3 is complete. □

Remark 1. Since $\beta(x)=x\gamma(x)$, the conditions $\alpha\in L^0$ and $\ln n=O\left(\lambda_n\gamma^{-1}(\alpha(\lambda_n))\right)$ as $n\to+\infty$ in Theorem 3 are much weaker than the condition $\alpha(e^x)\in L^0$ and $\ln n=o\left(\lambda_n\beta^{-1}(\alpha(\lambda_n))\right)$ as $n\to+\infty$ in Theorem 1.

Remark 2. In a general case, the following condition

$$\ln n=O\big(\lambda_n\gamma^{-1}(\alpha(\lambda_n))\big)$$

as $n\to+\infty$ in Theorem 3 cannot be relaxed. Indeed, let for example $\alpha(x)\equiv x$ and $\gamma(x)\equiv x^{p-1}$, $p>1$, for $x\ge x_0$. Then $\lambda_n\gamma^{-1}(\alpha(\lambda_n))=\lambda_n^{p/(p-1)}$ and condition (5) has the form

$$\int\limits_{\sigma_0}^{+\infty}\frac{\ln M(\sigma,F)}{\sigma^{p+1}}d\sigma<+\infty.$$

It is proved [10] that the last condition is equivalent to the condition

$$\int\limits_{\sigma_0}^{+\infty}\frac{\ln \mu(\sigma,F)}{\sigma^{p+1}}d\sigma<+\infty$$

for every Dirichlet series with given sequence of exponents (λ_n) if and only if

$$\ln n=O\Big(\lambda_n^{p/(p-1)}\Big)\quad(n\to+\infty).$$

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