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A. I. Bandura

COMPOSITION, PRODUCT AND SUM OF ANALYTIC FUNCTIONS OF BOUNDED L-INDEX IN DIRECTION IN THE UNIT BALL

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In this paper, we investigate a composition of entire function of one variable and analytic function in the unit ball. There are obtained conditions which provide equivalence of boundedness of L-index in a direction for such a composition and boundedness of l-index of initial function of one variable, where the continuous function $L\colon \mathbb{B}^n \to \mathbb{R}_+$ is constructed by the continuous function $l\colon \mathbb{C} \to \mathbb{R}_+$. We present sufficient conditions for boundedness of L-index in the direction for sum and for product of functions analytic in the unit ball.

The class of analytic functions in the unit ball having bounded L-index in direction is very wide because it contains all analytic functions with bounded multiplicities of zeros on every complex line $\{z^0 + t\mathbf{b}\colon t \in \mathbb{C}\}$. It is a statement of proved existence theorem. In the one-dimensional case these results are new for functions analytic in the unit disc.

1. Introduction. Let $\mathbb{B}^n = \{z \in \mathbb{C}^n\colon |z| < 1\}$. The paper is a continuation of [2, 6, 7]. There was generalized a concept of L-index boundedness in a direction for a class of analytic functions in the unit ball (see the definition below), including many criteria of L-index boundedness in the direction, where $L\colon \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function.

In this paper, we will apply some obtained results from [6] to deduce sufficient conditions of L-index boundedness in direction for some composite analytic functions in the unit ball and sum of these functions. Also we prove that analytic functions in the unit ball has bounded L-index in any direction for compactly embedded domain in the unit ball. Among other results we show that for any analytic function $F\colon \mathbb{B}^n \to \mathbb{C}$ with bounded multiplicities of zeros on every complex line $\{z^0 + t\mathbf{b}\colon t \in \mathbb{C}\}$ and any direction $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ there exists a positive continuous function $L\colon \mathbb{B}^n \to \mathbb{R}_+$ such that F is of bounded L-index in the direction $\mathbf{b}$. Mostly obtained results are also new for functions analytic in the unit disc.

Note that investigation of properties of analytic functions having bounded L-index in direction is very important in view of analytic theory of differential equations. These functions have regular behavior, uniform distribution of zeros in some sense and its growth estimates [15, 24]. It is known many various conditions providing index boundedness for every analytic solutions of some ordinary and partial differential equations and its system [14, 15, 24, 27].

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Let $\mathbb{D}=\{t\in\mathbb{C}\colon |t|<1\}$, $L\colon \mathbb{B}^n\to\mathbb{R}_+$ be a continuous function, $\mathbf{b}=(b_1,\dots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be a fixed direction, where $\mathbf{0}=(0,\dots,0)$, $\mathbf{1}=(1,\dots,1)$. For $z\in\mathbb{B}^n$ we denote $D_z=\{t\in\mathbb{C}\colon |t|\le\frac{1-|z|}{|\mathbf{b}|}\}$,

$$\lambda_{\mathbf{b}}(\eta)=\sup_{z\in\mathbb{B}^n}\sup_{t_1,t_2\in D_z}\left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})}\colon |t_1-t_2|\le\frac{\eta}{\min\{L(z+t_1\mathbf{b}),L(z+t_2\mathbf{b})\}}\right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{B}^n)$ stands for the class of positive continuous functions $L\colon\mathbb{B}^n\to\mathbb{R}_+$ satisfying

$$(\forall\eta\in[0,\beta])\colon\ \lambda_{\mathbf{b}}(\eta)<+\infty \tag{1}$$

and

$$L(z)>\frac{\beta|\mathbf{b}|}{1-|z|}, \tag{2}$$

where $\beta>1$ is some constant. If $n=1$ then $Q(\mathbb{D})\equiv Q_1(\mathbb{B}^1)$ and $\lambda(\eta)\equiv\lambda_1(\eta)$.

Similarly, $Q_{\mathbf{b}}^n$ stands for the class of positive continuous functions $L\colon\mathbb{C}^n\to\mathbb{R}_+$ satisfying (1) with

$$\lambda_{\mathbf{b}}(\eta)=\sup_{z\in\mathbb{C}^n}\sup_{t_1,t_2\in\mathbb{C}}\left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})}\colon |t_1-t_2|\le\frac{\eta}{\min\{L(z+t_1\mathbf{b}),L(z+t_2\mathbf{b})\}}\right\}.$$

2. Composition of entire functions of bounded L-index in direction. Analytic function $F\colon\mathbb{B}^n\to\mathbb{C}$ is called a function of *bounded L-index [5–7, 10] in a direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, if there exists $m_0\in\mathbb{Z}_+$ such that for every $m\in\mathbb{Z}_+$ and for each $z\in\mathbb{B}^n$

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m!L^m(z)}\le\max_{0\le k\le m_0}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)}, \tag{3}$$

where

$$\partial_{\mathbf{b}}^0F(z)=F(z),\partial_{\mathbf{b}}F(z)=\sum_{j=1}^n\frac{\partial F(z)}{\partial z_j}b_j,\quad \partial_{\mathbf{b}}^kF(z)=\partial_{\mathbf{b}}\Big(\partial_{\mathbf{b}}^{k-1}F(z)\Big),\quad k\ge 2.$$

There are also papers on analytic functions in the unit ball of bounded **L**-index in joint variables [4,9]. A connection between these classes is established in [10,11]. The least integer $m_0=m_0(\mathbf{b})$ satisfying (3) is called the *L-index in the direction* $\mathbf{b}$ *of the analytic function* F and is denoted by $N_{\mathbf{b}}(F,L)=m_0$. If $n=1$, $\mathbf{b}=1$, $L=l$, $F=f$, then $N(f,l)\equiv N_1(f,l)$ is called the l-index of the function f. In the case $n=1$ and $\mathbf{b}=1$ we obtain the definition of an analytic function in the unit disc of bounded l-index ([25]). Similarly, entire function $F\colon\mathbb{C}^n\to\mathbb{C}$ is called a function of *bounded L-index in a direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, if it satisfies (3) for all $z\in\mathbb{C}^n$. If $n=1$ and $L=l$ we obtain the definition of bounded l-index for entire functions of one variable [20], and if, in addition, $l\equiv 1$ we have the definition of an entire function of bounded index [21]. Theory of entire functions of bounded L-index in direction is developed in [15].

There are many papers on various classes of functions of bounded index (see bibliography [15, 24]). Nevertheless index boundedness of composite entire and analytic functions were considered only in [16,18,19,22,24]. In paper [22], there investigated l-index boudnedness of composition $f(P(z))$, where f is an entire function and P is a polynomial. In [18] there were presented conditions which provide l-index boundedness of the function $f(w(z))$, where f is a function analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon \ |z| < 1\}$, $w(z) = \frac{z-z_0}{1-z\overline{z}_0}e^{i\alpha}$, $z_0 \in \mathbb{D}$, $\alpha \in \mathbb{R}$. The most general result of such type was obtained in [19] for composite analytic function in arbitrary domains in complex plane. M. M. Sheremeta [24, p. 99] also proved that an entire function $f(z)$ has bounded index if and only if the analytic function $f(\frac{1}{z})$ in $\mathbb{C} \setminus \{0\}$ has bounded l-index with $l(z) = \frac{1}{|z|^2}$.

Note that the multidimensional case [13, 16] was considered for the composition of two entire functions, where one of them is an entire function of several variables. The most general result is the following

Theorem 1 ([16]). *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, *$f$ be an entire function in* $\mathbb{C}$, *Φ be an entire function in* $\mathbb{C}^n$ *such that* $\partial_{\mathbf{b}}\Phi(z) \neq 0$ *and* $|\partial^j_{\mathbf{b}}\Phi(z)| \leq K|\partial_{\mathbf{b}}\Phi(z)|^j$, $K \equiv \mathrm{const} > 0$, *for all* $z \in \mathbb{C}^n$ *and for all* $j \leq p$, *where* $p = N(f, l)$ *or* $p = N_{\mathbf{b}}(F, L)$, *respectively.*

Suppose that $l \in Q$, $l(w) \geq 1$, $w \in \mathbb{C}$ *and* $L \in Q^n_{\mathbf{b}}$, $L(z) = \big|\partial_{\mathbf{b}}\Phi(z)\big|l(\Phi(z))$. *The entire function f has bounded l-index if and only if the entire function* $F(z) = f(\Phi(z))$ *has bounded L-index in the direction* $\mathbf{b}$.

Similar result ([19]) is also known for functions analytic in an arbitrary domain in the complex plane.

Our main theorem is the following

Theorem 2. *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $f\colon \mathbb{C}^m \to \mathbb{C}$ *be an entire function,* $\Phi\colon \mathbb{B}^n \to \mathbb{C}$ *be an analytic function, such that* $\partial_{\mathbf{b}}\Phi(z) \neq 0$ *and*

$$|\partial^j_{\mathbf{b}}\Phi(z)| \leq K|\partial_{\mathbf{b}}\Phi(z)|^j, \quad K \equiv \mathrm{const} > 0, \tag{4}$$

for all $z \in \mathbb{B}^n$ *and for all* $j \leq p$, *where* $p = N_{\mathbf{1}}(f, l)$ *or* $p = N_{\mathbf{b}}(F, L)$, *respective.*

Suppose that $l \in Q^m_{\mathbf{1}}$, $l(w) \geq 1$ $(w \in \mathbb{C}^m)$, $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $L(z) = \big|\partial_{\mathbf{b}}\Phi(z)\big|l(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$. *The entire function f has bounded l-index in the direction* $\mathbf{1}$ *if and only if the analytic function* $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ *has bounded L-index in the direction* $\mathbf{b}$.

To prove main theorem we need auxiliary propositions. They are analogs of Hayman's Theorem for entire functions and analytic functions in the unit ball. It was firstly proved by W. Hayman ([17]) for entire functions of one variable having bounded index.

Theorem 3 ([12]). *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *and* $L \in Q^n_{\mathbf{b}}$. *An entire function $F(z)$ has bounded L-index in the direction* $\mathbf{b}$ *if and only if there exist numbers* $p \in \mathbb{Z}_+$, $R > 0$ *and* $C > 0$ *such that for every* $z \in \mathbb{C}^n$, $|z| \geq R$,

$$\frac{|\partial^{p+1}_{\mathbf{b}}F(z)|}{L^{p+1}(z)} \leq C \max\left\{\frac{|\partial^k_{\mathbf{b}}F(z)|}{L^k(z)}\colon \ 0 \leq k \leq p\right\}. \tag{5}$$

Theorem 4 ([5,6]). *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *and* $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. *An analytic function* $F\colon \mathbb{B}^n \to \mathbb{C}$ *is of bounded L-index in the direction* $\mathbf{b}$ *if and only if there exist* $p \in \mathbb{Z}_+$ *and* $C > 0$ *such that for every* $z \in \mathbb{B}^n$ *inequality* (5) *holds.*

Proof of Theorem 2. Denote $\nabla f = \partial_{\mathbf{1}} f = \sum_{j=1}^m \frac{\partial f}{\partial z_j}$, $\nabla^k f \equiv \partial^k_{\mathbf{1}} f$ for $k \geq 2$. Our proof is similar to the proof of the corresponding theorem in [16]. Firstly, we prove that

$$\partial^k_{\mathbf{b}}F(z) = \nabla^k f(\Phi(z), \ldots, \Phi(z))\,(\partial_{\mathbf{b}}\Phi(z))^k + \sum_{j=1}^{k-1} \nabla^j f(\Phi(z), \ldots, \Phi(z))Q_{j,k}(z), \tag{6}$$

where

$$Q_{j,k}(z)=\sum_{\substack{n_1+2n_2+\ldots+kn_k=k\\0\le n_1\le j-1}} c_{j,k,n_1,\ldots,n_k}\left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2}\ldots\left(\partial_{\mathbf{b}}^k\Phi(z)\right)^{n_k},$$

and $c_{j,k,n_1,\ldots,n_k}$ are non-negative integer numbers. We also will show that

$$\nabla^k f(\Phi(z),\ldots,\Phi(z))=\frac{\partial_{\mathbf{b}}^k F(z)}{\left(\partial_{\mathbf{b}}\Phi(z)\right)^k}+\frac{1}{\left(\partial_{\mathbf{b}}\Phi(z)\right)^{2k}}\sum_{j=1}^{k-1}\partial_{\mathbf{b}}^j F(z)\left(\partial_{\mathbf{b}}\Phi(z)\right)^j Q_{j,k}^*(z), \tag{7}$$

where

$$Q_{j,k}^*(z)=\sum_{m_1+2m_2+\ldots+km_k=2(k-j)} b_{j,k,m_1,\ldots,m_k}(\partial_{\mathbf{b}}\Phi(z))^{m_1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2}\ldots\left(\partial_{\mathbf{b}}^k\Phi(z)\right)^{m_k},$$

and $b_{j,k,m_1,\ldots,m_k}$ are some integer coefficients.

The validity of formulas (6) and (7) will be checked by the method of mathematical induction. Of course, for $k=1$ equalities (6) and (7) hold. Assume that they are valid for $k=s$. Let us to prove them for $k=s+1$. Evaluate directional derivative in (6)

$$\begin{aligned}\partial_{\mathbf{b}}^{s+1}F(z)=&\nabla^{s+1}f(\Phi(z),\ldots,\Phi(z))\left(\partial_{\mathbf{b}}\Phi(z)\right)^{s+1}+s\nabla^s f(\Phi(z),\ldots,\Phi(z))\left(\partial_{\mathbf{b}}\Phi(z)\right)^{s-1}\partial_{\mathbf{b}}^2\Phi(z)+\\&+\sum_{j=1}^{s-1}\left(\nabla^{j+1}f(\Phi(z),\ldots,\Phi(z))\partial_{\mathbf{b}}\Phi(z)Q_{j,s}(z)+\nabla^j f(\Phi(z),\ldots,\Phi(z))\partial_{\mathbf{b}}Q_{j,s}(z)\right)=\\&=\nabla^{s+1}f(\Phi(z),\ldots,\Phi(z))\left(\partial_{\mathbf{b}}\Phi(z)\right)^{s+1}+\\&+\nabla^s f(\Phi(z),\ldots,\Phi(z))\left(s\left(\partial_{\mathbf{b}}\Phi(z)\right)^{s-1}\partial_{\mathbf{b}}^2\Phi(z)+\partial_{\mathbf{b}}\Phi(z)Q_{s-1,s}(z)\right)+\\&+\sum_{j=2}^{s-1}\nabla^j f(\Phi(z),\ldots,\Phi(z))\left(\partial_{\mathbf{b}}\Phi(z)Q_{j-1,s}(z)+\partial_{\mathbf{b}}Q_{j,s}(z)\right)+\nabla f(\Phi(z),\ldots,\Phi(z))\partial_{\mathbf{b}}Q_{1,s}(z).\end{aligned}$$

Since

$$\begin{aligned}&s\left(\partial_{\mathbf{b}}\Phi(z)\right)^{s-1}\partial_{\mathbf{b}}^2\Phi(z)+\sum_{\substack{n_1+2n_2+\ldots+sn_s=s\\0\le n_1\le s-2}} c_{s-1,s,n_1,\ldots,n_s}\left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1+1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)\ldots\left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s}=\\&=\sum_{\substack{m_1+2m_2+\ldots+sm_s=s+1\\0\le m_1\le s-1}}\tilde{c}_{s,s+1,m_1,\ldots,m_s}\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2}\ldots\left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{m_s}=Q_{s,s+1}(z),\\&\partial_{\mathbf{b}}Q_{1,s}(z)=\sum_{2n_2+\ldots+sn_s=s} c_{1,s,0,n_2,\ldots,n_s}\Big(n_2\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2-1}\left(\partial_{\mathbf{b}}^3\Phi(z)\right)^{n_3+1}\ldots\left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s}+\\&+\ldots+n_s\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2}\left(\partial_{\mathbf{b}}^3\Phi(z)\right)^{n_3}\ldots\left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s-1}\partial_{\mathbf{b}}^{s+1}\Phi(z)\Big)=\\&=\sum_{2m_2+\ldots+(s+1)m_{s+1}=s+1}\tilde{c}_{1,s+1,0,m_2,\ldots,m_{s+1}}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2}\ldots\left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{m_s}\times\\&\times\left(\partial_{\mathbf{b}}^{s+1}\Phi(z)\right)^{m_{s+1}}=Q_{1,s+1}(z),\\&\partial_{\mathbf{b}}\Phi(z)Q_{j-1,s}(z)+\partial_{\mathbf{b}}Q_{j,s}(z)=\\&=\sum_{\substack{n_1+2n_2+\ldots+sn_s=s\\0\le n_1\le j-2}} c_{j-1,s,n_1,\ldots,n_s}\left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1+1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2}\ldots\left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s}+\end{aligned}$$

$$+\sum_{\substack{n_1+2n_2+\ldots+kn_s=s\\0\le n_1\le j-1}} c_{j,s,n_1,n_2,\ldots,n_s}\Big(n_1\left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1-1}\left(\partial^2_{\mathbf{b}}\Phi(z)\right)^{n_2+1}\ldots\left(\partial^s_{\mathbf{b}}\Phi(z)\right)^{n_s}+$$

$$+\ldots+n_s\left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1}\left(\partial^2_{\mathbf{b}}\Phi(z)\right)^{n_2}\ldots\left(\partial^s_{\mathbf{b}}\Phi(z)\right)^{n_s-1}\partial^{s+1}_{\mathbf{b}}\Phi(z)\Big)$$

$$=\sum_{\substack{m_1+2m_2+\ldots+(s+1)m_{s+1}=s+1\\0\le m_1\le j-1}}\tilde{c}_{j,s+1,m_1,\ldots,m_{s+1}}(\partial_{\mathbf{b}}\Phi(z))^{n_1}\ldots(\partial^s_{\mathbf{b}}\Phi(z))^{n_s}\left(\partial^{s+1}_{\mathbf{b}}\Phi(z)\right)^{n_{s+1}}=Q_{j,s+1}(z),$$

we obtain (6) with $s+1$ instead of k.

Using mathematical induction as in (6) it can be proved that (7) holds. After differentiation in the direction $\mathbf{b}$ equation (7) gives

$$\nabla^{s+1}f(\Phi(z),\ldots,\Phi(z))=\frac{\partial^{s+1}_{\mathbf{b}}F(z)}{\left(\partial_{\mathbf{b}}\Phi(z)\right)^{s+1}}-s\partial^2_{\mathbf{b}}\Phi(z)\partial^s_{\mathbf{b}}F(z)\left(\partial_{\mathbf{b}}\Phi(z)\right)^{-s-2}+$$

$$+\sum_{j=1}^{s-1}\Big\{\partial^{j+1}_{\mathbf{b}}F(z)\left(\partial_{\mathbf{b}}\Phi(z)\right)^{j-2s-1}Q^*_{j,s}(z)+$$

$$+\partial^j_{\mathbf{b}}F(z)\left(\partial_{\mathbf{b}}\Phi(z)\right)^{j-2s-2}\left((j-2s)\partial^2_{\mathbf{b}}\Phi(z)Q^*_{j,s}(z)+\partial_{\mathbf{b}}\Phi(z)\partial_{\mathbf{b}}Q^*_{j,s}(z)\right)\Big\}=$$

$$=\frac{\partial^{s+1}_{\mathbf{b}}F(z)}{\left(\partial_{\mathbf{b}}\Phi(z)\right)^{s+1}}+\partial^s_{\mathbf{b}}F(z)\left(\partial_{\mathbf{b}}\Phi(z)\right)^{-s-2}\Big(-s\partial^2_{\mathbf{b}}\Phi(z)+Q^*_{s-1,s}(z)\Big)+$$

$$+\sum_{j=2}^{s-1}\Big\{\partial^j_{\mathbf{b}}F(z)\left(\partial_{\mathbf{b}}\Phi(z)\right)^{j-2s-2}\left(\partial_{\mathbf{b}}\Phi(z)\partial_{\mathbf{b}}Q^*_{j,s}(z)+(j-2s)\partial^2_{\mathbf{b}}\Phi(z)Q^*_{j,s}(z)+Q^*_{j-1,s}(z)\right)\Big\}+$$

$$+\partial_{\mathbf{b}}F(z)\left(\partial_{\mathbf{b}}\Phi(z)\right)^{-2s-1}\left((1-2s)\partial^2_{\mathbf{b}}\Phi(z)Q^*_{1,s}(z)+\partial_{\mathbf{b}}\Phi(z)\partial_{\mathbf{b}}Q^*_{1,s}(z)\right).$$

Since

$$-s\partial^2_{\mathbf{b}}\Phi(z)+Q^*_{s-1,s}(z)=(-s+b_{s-1,s,m_1,\ldots,m_s})\partial^2_{\mathbf{b}}\Phi(z)=$$

$$=\sum_{\substack{m_1+2m_2+\ldots+sm_s+\\+(s+1)m_{s+1}=2}}\tilde{b}_{s,s+1,m_1,\ldots,m_{s+1}}\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1}\ldots\left(\partial^s_{\mathbf{b}}F(z)\right)^{m_s}\times$$

$$\times\left(\partial^{m_s+1}_{\mathbf{b}}F(z)\right)^{m_s+1}=Q^*_{s,s+1}(z),$$

$$(1-2s)\partial^2_{\mathbf{b}}\Phi(z)Q^*_{1,s}(z)+\partial_{\mathbf{b}}\Phi(z)\partial_{\mathbf{b}}Q^*_{1,s}(z)=(1-2s)\times$$

$$\times\sum_{\substack{m_1+2m_2+\ldots+sm_s=\\=2s-2}}b_{1,s,m_1,\ldots,m_s}\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1}\left(\partial^2_{\mathbf{b}}\Phi(z)\right)^{m_2+1}\ldots\left(\partial^s_{\mathbf{b}}F(z)\right)^{m_s}+$$

$$+\sum_{\substack{m_1+2m_2+\ldots+sm_s=\\=2s-2}}b_{1,s,m_1,\ldots,m_s}\Big\{m_1\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1}\left(\partial^2_{\mathbf{b}}\Phi(z)\right)^{m_2+1}\ldots\left(\partial^s_{\mathbf{b}}F(z)\right)^{m_s}+$$

$$+m_2\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1+1}\left(\partial^2_{\mathbf{b}}\Phi(z)\right)^{m_2-1}\left(\partial^3_{\mathbf{b}}\Phi(z)\right)^{m_3+1}\ldots\left(\partial^s_{\mathbf{b}}F(z)\right)^{m_s}+\ldots+$$

$$+m_s\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1+1}\ldots\left(\partial^s_{\mathbf{b}}F(z)\right)^{m_s-1}\partial^{s+1}_{\mathbf{b}}\Phi(z)\Big\}=$$

$$=\sum_{\substack{m_1+2m_2+\ldots+sm_s+\\+(s+1)m_{s+1}=2s}}\tilde{b}_{1,s+1,m_1,\ldots,m_{s+1}}\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1}\ldots\left(\partial^s_{\mathbf{b}}F(z)\right)^{m_s}\left(\partial^{s+1}_{\mathbf{b}}\Phi(z)\right)^{m_{s+1}}=Q^*_{1,s+1}(z),$$

and

$$\partial_{\mathbf{b}}\Phi(z)\partial_{\mathbf{b}}Q^*_{j,s}(z)+(j-2s)\partial^2_{\mathbf{b}}\Phi(z)Q^*_{j,s}(z)+Q^*_{j-1,s}(z)=$$

$$= \sum_{\substack{m_1+2m_2+\ldots+sm_s=\\=2(s-j)}} b_{j,s,m_1,\ldots,m_s} \Big\{ m_1 \left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2+1} \ldots \left(\partial_{\mathbf{b}}^s F(z)\right)^{m_s} +$$

$$+ \ldots + m_s \left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1+1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2} \ldots \left(\partial_{\mathbf{b}}^s F(z)\right)^{m_s-1} \partial_{\mathbf{b}}^{s+1}\Phi(z) \Big\} +$$

$$+ (j-2s) \sum_{\substack{m_1+2m_2+\ldots+sm_s=\\=2(s-j)}} b_{j,s,m_1,\ldots,m_s} \left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2+1} \ldots \left(\partial_{\mathbf{b}}^s F(z)\right)^{m_s} +$$

$$+ \sum_{\substack{m_1+2m_2+\ldots+sm_s=\\=2(s-j)+2}} b_{j-1,s,m_1,\ldots,m_s} \left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1} \ldots \left(\partial_{\mathbf{b}}^s F(z)\right)^{m_s} =$$

$$= \sum_{\substack{m_1+2m_2+\ldots+sm_s+\\+(s+1)m_{s+1}=2(s+1-j)}} \tilde{b}_{j,s+1,m_1,\ldots,m_{s+1}} \left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1} \ldots \left(\partial_{\mathbf{b}}^{s+1}\Phi(z)\right)^{m_{s+1}} = Q^*_{j,s+1}(z),$$

we conclude that (6) is valid with $s+1$ instead of k.

Let f be an entire function of bounded l-index. By Theorem 3 inequality (5) holds for $n = m$, $F = f$, $L = l$, $\mathbf{b} = \mathbf{1}$. Taking into account (4) and (6), for $k = p+1$ we obtain

$$\frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L^{p+1}(z)} \le \frac{|\nabla^{p+1} f(\Phi(z),\ldots,\Phi(z))|}{L^{p+1}(z)} |\partial_{\mathbf{b}}\Phi(z)|^{p+1} + \sum_{j=1}^{p} \frac{|\nabla^j f(\Phi(z),\ldots,\Phi(z))||Q_{j,p+1}(z)|}{L^{p+1}(z)} \le$$

$$\le \max\left\{ \frac{|\nabla^k f(\Phi(z),\ldots,\Phi(z))|}{l^k(\Phi(z))} \colon\ 0 \le k \le p \right\} \left(C + \sum_{j=1}^{p} \frac{|Q_{j,p+1}(z)|}{l^{p+1-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{p+1}} \right) \le$$

$$\le \max\left\{ \frac{|\nabla^k f(\Phi(z),\ldots,\Phi(z))|}{l^k(\Phi(z))} \colon\ 0 \le k \le p \right\} \left(C + \sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\0\le n_1\le j-1}} c_{j,p+1,n_1,\ldots,n_{p+1}} \times \right.$$

$$\left. \times \frac{|\left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2} \ldots \left(\partial_{\mathbf{b}}^{p+1}\Phi(z)\right)^{n_{p+1}}|}{l^{p+1-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{p+1}} \right) \le \max\left\{ \frac{|\nabla^k f(\Phi(z),\ldots,\Phi(z))|}{l^k(\Phi(z))} : 0 \le k \le p \right\} \times$$

$$\times \left(C + \sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\0\le n_1\le j-1}} \frac{c_{j,p+1,n_1,\ldots,n_{p+1}} K^{p+1}}{l^{p+1-j}(\Phi(z))} \right) \le C_1 \max_{0\le k\le p} \frac{|\nabla^k f(\Phi(z),\ldots,\Phi(z))|}{l^k(\Phi(z))}.$$

Using (7), we find the upper estimate for the fraction $\frac{|\nabla^k f(\Phi(z),\ldots,\Phi(z))|}{l^k(\Phi(z))}$:

$$\frac{|\nabla^k f(\Phi(z),\ldots,\Phi(z))|}{l^k(\Phi(z))} \le \frac{|\partial_{\mathbf{b}}^k F(z)|}{l^k(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^k} + \sum_{j=1}^{k-1} \frac{|\partial_{\mathbf{b}}^j F(z)||Q^*_{j,k}(z)|}{l^k(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2k-j}} \le$$

$$\le \max\left\{ \frac{1}{L^j(z)} \left|\partial_{\mathbf{b}}^j F(z)\right| : 1 \le j \le k \right\} \left(1 + \sum_{j=1}^{k-1} \frac{|Q^*_{j,k}(z)|}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}} \right) \le$$

$$\le \max\left\{ \frac{1}{L^j(z)} \left|\partial_{\mathbf{b}}^j F(z)\right| : 1 \le j \le k \right\} \left(1 + \sum_{j=1}^{k-1} \sum_{m_1+2m_2+\ldots+km_k=2(k-j)} |b_{j,k,m_1,\ldots,m_k}| \times \right.$$

$$\left. \times \frac{|\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2} \ldots \left(\partial_{\mathbf{b}}^k\Phi(z)\right)^{m_k}|}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}} \right) \le \max\left\{ \frac{|\partial_{\mathbf{b}}^j F(z)|}{L^j(z)} : 1 \le j \le k \right\} \times$$

$$\times\left(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}\frac{|b_{j,k,m_1,\ldots,m_k}|K^k}{l^{k-j}(\Phi(z))}\right)\leq C_2\max_{1\leq j\leq k}\frac{|\partial_{\mathbf{b}}^jF(z)|}{L^j(z)}.$$

Hence, it follows that

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)}\leq C_1C_2\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}:\ 0\leq k\leq p\right\}.$$

Therefore, by Theorem 4 the last inequality means that the function F has bounded L-index in the direction $\mathbf{b}$.

Conversely, suppose that the function F is of bounded L-index in the direction $\mathbf{b}$. Then it satisfies (5). In view of (4) and (7), we deduce

$$\frac{|\nabla^{p+1}f(\Phi(z),\ldots,\Phi(z))|}{l^{p+1}(\Phi(z))}\leq\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{l^{p+1}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}+\sum_{j=1}^{p}\frac{|\partial_{\mathbf{b}}^jF(z)||Q_{j,p+1}^*(z)|}{l^{p+1}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2p+2-j}}\leq$$

$$\leq\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}:0\leq k\leq p\right\}\left(C+\sum_{j=1}^{p}\frac{|Q_{j,p+1}^*(z)|}{l^{p+1-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(p+1-j)}}\right)\leq$$

$$\leq\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}:0\leq k\leq p\right\}\left(C+\sum_{j=1}^{p}\sum_{\substack{m_1+\ldots+(p+1)m_{p+1}=\\=2(p+1-j)}}|b_{j,p+1,m_1,\ldots,m_{p+1}}|\times\right.$$

$$\left.\times\frac{|\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2}\ldots\left(\partial_{\mathbf{b}}^{p+1}\Phi(z)\right)^{m_{p+1}}|}{l^{p+1-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(p+1-j)}}\right)\leq\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}:0\leq k\leq p\right\}\times$$

$$\times\left(C+\sum_{j=1}^{p}\sum_{\substack{m_1+\ldots+(p+1)m_{p+1}=\\=2(p+1-j)}}\frac{|b_{j,p+1,m_1,\ldots,m_{p+1}}|K^{2p+2-2j}}{l^{p+1-j}(\Phi(z))}\right)\leq C_3\max_{0\leq k\leq p}\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}.$$

Applying (6), we estimate

$$\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}\leq\frac{|\nabla^kf(\Phi(z),\ldots,\Phi(z))||\varphi'(z)|^k}{L^k(z)}+\sum_{j=1}^{k-1}\frac{|\nabla^jf(\Phi(z),\ldots,\Phi(z))||Q_{j,k}(z)|}{L^k(z)}\leq$$

$$\leq\max\left\{\frac{|\nabla^jf(\Phi(z),\ldots,\Phi(z))|}{l^j(\Phi(z))}:1\leq j\leq k\right\}\left(1+\sum_{j=1}^{k-1}\frac{|Q_{j,k}(z)|}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^k}\right)\leq$$

$$\leq C_4\max\left\{\frac{|\nabla^jf(\Phi(z),\ldots,\Phi(z))|}{l^j(\Phi(z))}:1\leq j\leq k\right\}.$$

It implies that

$$\frac{|\nabla^{p+1}f(\Phi(z),\ldots,\Phi(z))|}{l^{p+1}(\Phi(z))}\leq C_3C_4\max\left\{\frac{|\nabla^jf(\Phi(z),\ldots,\Phi(z))|}{l^j(\Phi(z))}:0\leq j\leq p\right\}.$$

Thus, by Theorem 3 ($n=m$, $F=f$, $L=l$, $\mathbf{b}=\mathbf{1}$) the function f has bounded l-index. □

Note that the condition $\partial_{\mathbf{b}}\Phi(z)\neq 0$ in Theorem 1 is generated by our method of the proof. In fact, we can remove it and prove more general proposition with some greater function L.

Theorem 5. *Let $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, $f\colon\mathbb{C}^m\to\mathbb{C}$ be an entire function, $\Phi\colon\mathbb{B}^n\to\mathbb{C}$ be an analytic function, $p=N_{\mathbf{1}}(f,l)$ or $p=N_{\mathbf{b}}(F,L)$ respective.*

Suppose that $l\in Q^m_{\mathbf{1}}$, $l(w)\geq 1$, $w\in\mathbb{C}^m$ and $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$ with

$$L(z)=\max_{1\leq j\leq p}\left\{1,\left|\partial^j_{\mathbf{b}}\Phi(z)\right|\right\} l(\underbrace{\Phi(z),\ldots,\Phi(z)}_{m\ times}).$$

The entire function f has bounded l-index in the direction $\mathbf{1}$ if and only if the analytic function $F(z)=f(\underbrace{\Phi(z),\ldots,\Phi(z)}_{m\ times})$ has bounded L-index in the direction $\mathbf{b}$.

The proof of this theorem is similar to that of Theorem 2 and also use analogs of Hayman's Theorem for entire functions of bounded L-index in direction (Theorems 3, 4).

Remark 1. One should observe that Theorems 2 and 5 are also new results in one-dimensional case, i.e. in the case of analytic functions in the unit disc. Moreover, if we replace the condition "Φ be an analytic function in the unit ball" by the condition "Φ be an entire function of several variables" in these theorems then we also deduce new results for composite entire functions. In comparison, there is removed the condition $\partial_{\mathbf{b}}\Phi(z)\neq 0$ and is considered more general composition than in [16].

Note that for $n=1$ the assumption in Theorem 2 are weaker than in [19] because we require validity of (4) for $j\leq p$ instead all values $j\in\mathbb{N}$.

3. Product theorem. To prove a theorem on product of analytic functions of bounded L-index in direction we need auxiliary propositions.

Lemma 1 ([5,6])**.** *Let $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\frac{1}{\beta}<\theta_1\leq\theta_2<+\infty$, $\theta_1L(z)\leq L^*(z)\leq\theta_2L(z)$. Analytic function $F(z)$ in $\mathbb{B}^n$ has bounded L^*-index in the direction $\mathbf{b}$ if and only if the function F has bounded L-index in the direction $\mathbf{b}$.*

Let $g_{z^0}(t):=F(z^0+t\mathbf{b})$. If for given $z^0\in\mathbb{B}^n$ $g_{z^0}(t)\neq 0$ for all $t\in D_{z^0}$, then $G^{\mathbf{b}}_r(F,z^0):=\varnothing$; if for given $z^0\in\mathbb{B}^n$ $g_{z^0}(t)\equiv 0$, then $G^{\mathbf{b}}_r(F,z^0):=\{z^0+t\mathbf{b}\colon t\in D_{z^0}\}$. And if for some $z^0\in\mathbb{B}^n$ $g_{z^0}(t)\not\equiv 0$ and a^0_k are zeros of the functions $g_{z^0}(t)$, i.e., $F(z^0+a^0_k\mathbf{b})=0$, then

$$G^{\mathbf{b}}_r(F,z^0):=\bigcup_k\left\{z^0+t\mathbf{b}\colon |t-a^0_k|\leq\frac{r}{L(z^0+a^0_k\mathbf{b})}\right\},\quad r>0.$$

Let

$$G^{\mathbf{b}}_r(F)=\bigcup_{z^0\in\mathbb{B}^n}G^{\mathbf{b}}_r(F,z^0). \tag{8}$$

By $n\big(r,z^0,1/F\big)=\sum_{|a^0_k|\leq r}1$ we denote counting functions of zeros a^0_k.

Theorem 6 ([5,6])**.** *Let F be an analytic function in $\mathbb{B}^n$, $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$ and $\mathbb{B}^n\setminus G^{\mathbf{b}}_{\beta}(F)\neq\varnothing$. The function $F(z)$ has bounded L-index in the direction $\mathbf{b}$ if and only if*

1) for every $r\in(0,\beta]$ there exists $P=P(r)>0$ such that for any $z\in\mathbb{B}^n\setminus G^{\mathbf{b}}_r(F)$

$$\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right|\leq PL(z); \tag{9}$$

2) for each $r \in (0, \beta]$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for all $z^0 \in \mathbb{B}^n$ with $F(z^0 + t\mathbf{b}) \not\equiv 0$ one has

$$n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{F}\right) \le \widetilde{n}(r). \tag{10}$$

Using Theorem 4 we prove the following

Theorem 7. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. An analytic function $F\colon \mathbb{B}^n \to \mathbb{C}$ has bounded L-index in the direction $\mathbf{b}$ if and only if there exist numbers $C \in (0, +\infty)$ and $N \in \mathbb{N}$ such that for all $z \in \mathbb{B}^n$*

$$\sum_{k=0}^{N} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} \ge C \sum_{k=N+1}^{\infty} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)}. \tag{11}$$

Proof. Proof of this theorem is similar to the proof of its analogs for entire functions of bounded L-index in direction [8] and for entire functions of bounded l-index [23].

Let $\frac{1}{\beta} < \theta < 1$. If the function F is of bounded L-index in the direction $\mathbf{b}$, then by Lemma 1 F is also of bounded L^*-index in the direction $\mathbf{b}$, where $L^*(z) = \theta L(z)$. Denote $N^* = N_{\mathbf{b}}(F, L_*)$ and $N = N_{\mathbf{b}}(F, L)$. Thus,

$$\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} \colon 0 \le k \le N^*\right\} = \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L_*^k(z)} \theta^k \colon 0 \le k \le N^*\right\} \ge$$
$$\ge \theta^{N^*} \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L_*^k(z)} \colon 0 \le k \le N^*\right\} \ge \theta^{N^*} \frac{|\partial_{\mathbf{b}}^j F(z)|}{j! L_*^j(z)} = \theta^{N^*-j} \frac{|\partial_{\mathbf{b}}^j F(z)|}{j! L^j(z)}$$

for all $j \ge 0$ and

$$\sum_{j=N^*+1}^{\infty} \frac{|\partial_{\mathbf{b}}^j F(z)|}{j! L^j(z)} \le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} \colon 0 \le k \le N^*\right\} \sum_{j=N^*+1}^{\infty} \theta^{j-N^*} =$$
$$= \frac{\theta}{1-\theta} \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} \colon 0 \le k \le N^*\right\} \le \frac{\theta}{1-\theta} \sum_{k=0}^{N^*} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)},$$

i.e. we obtain (11) with $N = N^*$ and $C = \frac{1-\theta}{\theta}$.

Now we prove the sufficiency. From (11) we obtain

$$\frac{|\partial_{\mathbf{b}}^{N+1} F(z)|}{(N+1)! L^{N+1}(z)} \le \sum_{k=N+1}^{\infty} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} \le \frac{1}{C} \sum_{k=0}^{N} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} \le \frac{N+1}{C} \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} \colon 0 \le k \le N\right\}.$$

Applying Theorem 4, we obtain a desired conclusion. □

We then consider an application of Theorem 6.

Theorem 8. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $F\colon \mathbb{B}^n \to \mathbb{C}$ be an analytic function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $\Phi\colon \mathbb{B}^n \to \mathbb{C}$ be an analytic function in the unit ball and $\Psi(z) = F(z)\Phi(z)$. The function $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$ if and only if the function $\Phi(z)$ is of bounded L-index in the direction $\mathbf{b}$.*

Proof. The similar result was obtained for entire functions of bounded L-index in direction in [8]. Our proof is similar to the proof for entire functions in [8] but now we use Theorem 6, deduced for functions analytic in the unit ball. Since an analytic function $F(z)$ has bounded L-index in the direction $\mathbf{b}$, by Theorem 6 for every $r \in (0,\beta)$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for all $z^0 \in \mathbb{B}^n$, satisfying $F(z^0 + t\mathbf{b}) \not\equiv 0$, the estimate $n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{F}\right) \le \widetilde{n}(r)$ holds. Hence,

$$n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Phi}\right) \le n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Psi}\right) \le n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Phi}\right) + \widetilde{n}(r).$$

Thus, condition 2 of Theorem 6 either holds or does not hold for functions $\Psi(z)$ and $\Phi(z)$ simultaneously. If $\Phi(z)$ has bounded L-index in the direction $\mathbf{b}$, then for every $r \in (0,\beta)$ there exist numbers $P_F(r) > 0$ and $P_\Phi(r) > 0$ such that $\left|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}\right| \le P_f(r)L(z)$, $\left|\frac{\partial_{\mathbf{b}} \Phi(z)}{\Phi(z)}\right| \le P_\Phi(r)L(z)$ for each $z \in (\mathbb{B}^n \setminus G_r^{\mathbf{b}}(F)) \cap (\mathbb{B}^n \setminus G_r^{\mathbf{b}}(\Phi))$. Since

$$\mathbb{B}^n \setminus G_r^{\mathbf{b}}(\Psi) \subset (\mathbb{B}^n \setminus G_r^{\mathbf{b}}(F)) \cap (\mathbb{B}^n \setminus G_r^{\mathbf{b}}(\Phi)), \quad \left|\frac{\partial_{\mathbf{b}}\Psi(z)}{\Psi(z)}\right| \le \left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| + \left|\frac{\partial_{\mathbf{b}}\Phi(z)}{\Phi(z)}\right|,$$

for all $z \in \mathbb{B}^n \setminus G_r^{\mathbf{b}}(\Psi)$ we have $\left|\frac{\partial_{\mathbf{b}}\Psi(z)}{\Psi(z)}\right| \le (P_F(r) + P_\Phi(r))L(z)$, i.e. by Theorem 6 the function $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$.

On the contrary, let $\Psi(z)$ be of bounded L-index in the direction $\mathbf{b}$, $r > 0$. At first we show that for every $z^0 \in \mathbb{B}^n \setminus G_r^{\mathbf{b}}(F)$ $(r > 0)$ and for every $\widetilde{d^k} = z^0 + d_k^0\mathbf{b}$, where d_k^0 are zeros of function $\Phi(z^0 + t\mathbf{b})$, we have

$$|z^0 - \widetilde{d^k}| > \frac{r|\mathbf{b}|}{2L(z^0)\lambda_{\mathbf{b}}(r)}. \tag{12}$$

On the other hand, let there exist $z^0 \in \mathbb{B}^n \setminus G_r^{\mathbf{b}}(\Phi)$ and $\widetilde{d^k} = z^0 + d_k^0\mathbf{b}$ such that $|z^0 - \widetilde{d^k}| \le \frac{r|\mathbf{b}|}{2L(z^0)\lambda_{\mathbf{b}}(r)}$. Then by the definition of $\lambda_{\mathbf{b}}$ we have the next estimate $L(\widetilde{d^k}) \le \lambda_{\mathbf{b}}(r)L(z^0)$. Hence $|z^0 - \widetilde{d^k}| = |\mathbf{b}| \cdot |d_k^0| \le \frac{r|\mathbf{b}|}{2L(\widetilde{d^k})}$, i.e. $|d_k^0| \le \frac{r}{2L(\widetilde{d^k})}$, but it contradicts $z^0 \in \mathbb{B}^n \setminus G_r^{\mathbf{b}}(\Phi)$.

We consider

$$\overline{K}_0 = \left\{ z^0 + t\mathbf{b} \colon |t| \le \frac{r}{2L(z^0)\lambda_{\mathbf{b}}(r)} \right\}.$$

It does not contain zeros of $\Phi(z^0 + t\mathbf{b})$, which may contain zeros $\widetilde{c^k} = z^0 + c_k^0\mathbf{b}$ of the function $\Psi(z^0 + t\mathbf{b})$. Since $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$, the set $\overline{K}_0$ by Theorem 6 contains at most $\widetilde{n}_1 = \widetilde{n}_1(\frac{r}{2\lambda_{\mathbf{b}}(r)})$ zeros c_k^0 of the function $\Psi(z^0 + t\mathbf{b})$. For all $c_k^0 \in \overline{K}_0$, using the definition of $Q_{\mathbf{b}}(\mathbb{B}^n)$, we obtain the following inequality

$$L(z^0 + c_k^0\mathbf{b}) \ge \frac{1}{\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)} L(z^0).$$

Thus, every set $m_k^0 = \{z^0 + t\mathbf{b} \colon |t - c_k^0| \le \frac{r_1}{L(z^0 + c_k^0\mathbf{b})}\}$ with $r_1 = \frac{r}{4(\widetilde{n}_1 + 1)\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)\lambda_{\mathbf{b}}(r)}$ is contained in the set

$$s_k^0 = \left\{ z^0 + t\mathbf{b} \colon |t - c_k^0| \le \frac{r_1\lambda_{\mathbf{b}}(\frac{r}{\lambda_{\mathbf{b}}(r)})}{L(z^0)} \right\}.$$

The total sum of diameters of these sets does not exceed

$$\frac{2\widetilde{n}_1 r_1 \lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)}{L(z^0)} = \frac{r}{2\lambda_{\mathbf{b}}(r)L(z^0)} \cdot \frac{\widetilde{n}_1}{(\widetilde{n}_1+1)} < \frac{r}{2\lambda_{\mathbf{b}}(r)L(z^0)}.$$

Therefore, there exists $r^* \in \left(0, \frac{r}{2\lambda_{\mathbf{b}}(r)}\right)$ such that if $|t| = \frac{r^*}{L(z^0)}$, then $z^0 + t\mathbf{b} \notin G_{r_1}^{\mathbf{b}}(\Psi)$, and therefore $z^0 + t\mathbf{b} \notin G_{r_1}^{\mathbf{b}}(F)$. By Theorem 6 for all these points $z^0 + t\mathbf{b}$ we obtain

$$\left|\frac{\partial_{\mathbf{b}}\Phi(z^0+t\mathbf{b})}{\Phi(z^0+t\mathbf{b})}\right| \le \left|\frac{\partial_{\mathbf{b}}\Psi(z^0+t\mathbf{b})}{\Psi(z^0+t\mathbf{b})}\right| + \left|\frac{\partial_{\mathbf{b}}F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})}\right| \le (P_\Psi^* + P_F^*)L(z^0+t\mathbf{b}), \tag{13}$$

where P_Ψ^* and P_F^* depend only on r_1, i.e. only on r. Since the function $\frac{\partial_{\mathbf{b}}\Phi(z)}{\Phi(z)}$ is analytic in $\overline{K}_0$, applying the maximum modulus principle to the function $\frac{\partial_{\mathbf{b}}\Phi(z^0+t\mathbf{b})}{\Phi(z^0+t\mathbf{b})}$ as a function of variable t, we obtain that the modulus of this function at the point $t = 0$ does not exceed the maximum modulus of this function on the circle $\{t \in \mathbb{C}\colon |t| = \frac{r^*}{L(z^0)}\}$. It means that obtained inequality (13) holds for z^0.

Thus, for arbitrary $r \in (0, \beta)$ and $z^0 \in \mathbb{B}^n \setminus G_r^{\mathbf{b}}(F)$ we have proved the first condition of Theorem 6. Above we have already shown that the second condition of Theorem 6 is true. Hence, by the mentioned theorem the function $\Phi(z)$ has bounded L-index in the direction $\mathbf{b}$. □

4. Boundedness of L-index in direction for sum of analytic functions. There are known sufficient conditions of index boundedness for sum of two entire functions of one variables [26]. These results were generalized for entire functions of bounded L-index in direction [1] and for entire functions of bounded index in joint variables [3]. But similar conditions for analytic functions in the unit ball (or in the unit disk) are not known. Therefore, in this subsection we consider the following **question**: *what are sufficient conditions for L-index boundedness in direction for the sum of two functions analytic in the unit ball?*

We need the following theorem.

Theorem 9 ([5,6]). *Let $\beta > 1$, $L \in Q_{\mathbf{b},\beta}(\mathbb{B}^n)$. An analytic function $F(z)$ in $\mathbb{B}^n$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if for any r_1 and for any r_2, $0 < r_1 < r_2 \le \beta$, there exists $P_1 = P_1(r_1, r_2) \ge 1$ such that for each $z^0 \in \mathbb{B}^n$*

$$\max\left\{|F(z^0+t\mathbf{b})|\colon |t| = \frac{r_2}{L(z^0)}\right\} \le P_1 \max\left\{|F(z^0+t\mathbf{b})|\colon\ |t| = \frac{r_1}{L(z_0)}\right\}. \tag{14}$$

Let us consider intersection of the hyperplane $\langle z, \mathbf{b}\rangle = 0$ with the unit ball. The intersection we denote by $A = \{z \in \mathbb{B}^n\colon\ \langle z, \mathbf{b}\rangle = 0\}$, where $\langle z, \mathbf{b}\rangle := \sum_{j=1}^n z_j b_j$. Obviously $\bigcup_{z^0 \in A}\{z^0 + t\mathbf{b}\colon |t| \le \frac{1-|z_0|}{|\mathbf{b}|}\} = \mathbb{B}^n$.

Let $z^0 \in A$ be a given point. If $F(z^0 + t\mathbf{b}) \not\equiv 0$ as a function of variable $t \in \mathbb{C}$, then there exists $t_0 \in D_{z^0}$ such that $F(z^0 + t_0\mathbf{b}) \ne 0$. We denote

$$B(z^0, t) = \left\{t_0 \in D_{z^0}\colon |t_0 - t| < \min\left\{\frac{\beta}{2L(z^0+t\mathbf{b})}, \frac{1-|z^0+\mathbf{b}t|}{2|\mathbf{b}|}\right\}, F(z^0+t_0\mathbf{b}) \ne 0\right\},$$

$$B(z^0) = \bigcup_{|t| \le (1-|z^0|)/|\mathbf{b}|} B(z^0, t).$$

Theorem 10. *Let $L\colon \mathbb{B}^n \to \mathbb{R}_+$ be a positive continuous function satisfying (2) with $\beta \geq 3$, and $F\colon \mathbb{B}^n \to \mathbb{R}_+$, $G\colon \mathbb{B}^n \to \mathbb{R}_+$ be analytic functions in the unit ball which obey the following conditions:*

1) *$G(z)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ with $N_b(G, L) = N < +\infty$;*
2) *there exists $\alpha \in (0, 1)$ such that for all $z \in \mathbb{B}^n$ and $p \geq N + 1$ ($p \in \mathbb{N}$)*

$$\frac{|\partial_{\mathbf{b}}^p G(z)|}{p! L^p(z)} \leq \alpha \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z)|}{k! L^k(z)} : 0 \leq k \leq N\right\}; \tag{15}$$

3) *for every $z = z^0 + t\mathbf{b} \in \mathbb{B}^n$ with $z^0 \in A$ and some $t_0 \in B(z^0, t)$ with $r = |t - t_0| L(z^0 + t\mathbf{b})$ the inequality*

$$\max\left\{|F(z^0 + t'\mathbf{b})|\colon\ |t' - t_0| = \frac{2r}{L(z^0 + t\mathbf{b})}\right\} \leq \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0 + t\mathbf{b})|}{k! L^k(z^0 + t\mathbf{b})}\colon\ 0 \leq k \leq N\right\}; \tag{16}$$

is valid;

4) *either $(\exists c > 0)(\forall z^0 \in A)(\forall t \in \mathbb{D}_{z^0})$ ($\exists t_0 \in B(z^0, t)$ obeying (16) and if $|t - t_0| L(z^0 + t\mathbf{b}) \leq 1$) then*

$$\max\left\{|F(z^0 + t'\mathbf{b})|\colon\ |t' - t_0| = \frac{2}{L(z^0 + t\mathbf{b})}\right\}\Big/|F(z^0 + t_0\mathbf{b})| \leq c < +\infty,$$

or for $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$ $(\exists c > 0)(\forall z^0 \in A)$ $(\exists t_0 \in B(z^0))$ such that (16) is true and

$$\max\left\{|F(z^0 + t'\mathbf{b})|\colon\ |t' - t_0| = \frac{2\lambda_{\mathbf{b}}(1)}{L(z^0 + t_0\mathbf{b})}\right\}\Big/|F(z^0 + t_0\mathbf{b})| \leq c < +\infty, \tag{17}$$

where $\beta \geq 2\lambda_{\mathbf{b}}(1)$.

Then for every $\varepsilon \in \mathbb{C}$, $|\varepsilon| \leq \frac{1-\alpha}{2c}$, the function

$$H(z) = G(z) + \varepsilon F(z) \tag{18}$$

has bounded L-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(H, L) \leq N$.

Proof. We write Cauchy's formula for the analytic function $F(z^0 + t\mathbf{b})$ as function of one complex variable t

$$\frac{\partial_{\mathbf{b}}^p F(z^0 + t\mathbf{b})}{p!} = \frac{1}{2\pi i} \int\limits_{|t'-t| = \frac{r}{L(z^0 + t\mathbf{b})}} \frac{F(z^0 + t'\mathbf{b})}{(t' - t)^{p+1}} dt'. \tag{19}$$

For the chosen $r = |t - t_0| L(z^0 + t\mathbf{b})$ we deduce

$$\frac{r}{L(z^0 + t\mathbf{b})} = |t' - t| \geq |t' - t_0| - |t - t_0| = |t' - t_0| - \frac{r}{L(z^0 + t\mathbf{b})}.$$

Hence,

$$|t' - t_0| \leq \frac{2r}{L(z^0 + t\mathbf{b})}. \tag{20}$$

Equality (19) yields

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \le \frac{1}{2\pi L^p(z^0+t\mathbf{b})}\cdot\frac{L^{p+1}(z^0+t\mathbf{b})}{r^{p+1}}\times$$
$$\times\frac{2\pi r}{L(z^0+t\mathbf{b})}\cdot\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t|=\frac{r}{L(z^0+t\mathbf{b})}\right\}\le$$
$$\le\frac{1}{r^p}\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}. \tag{21}$$

If $r=|t-t_0|L(z^0+t\mathbf{b})>1$, then (21) yields

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}. \tag{22}$$

Let $r=|t-t_0|L(z^0+t\mathbf{b})\in(0;1]$. Setting $r=1$ in (19) and (20), we analogously deduce

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}=$$
$$=\frac{\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}}{\max\left\{|F(z^0+t'\mathbf{b})|\colon |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}}\times$$
$$\times\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\le$$
$$\le\frac{\max\left\{|F(z^0+t'\mathbf{b})|\colon |t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}\times$$
$$\times\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\le$$
$$\le c\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}, \tag{23}$$

where

$$c=\sup_{z^0\in A,|t|<(1-|z^0|)/|\mathbf{b}|}\frac{\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}\ge 1$$

and $t_0=t_0(z,t)\in B(z^0,t)$ is chosen in (16) and $|t_0-t|\le 1/L(z^0+t\mathbf{b})$. From $|t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}$ one has $|t'|\le|t_0|+\frac{2}{L(z^0+t\mathbf{b})}\le|t|+\frac{3}{L(z^0+t\mathbf{b})}$. Therefore, $\beta\ge 3$.

If $L\in Q$, then $\sup\left\{\frac{L(z^0+t_0\mathbf{b})}{L(z^0+t\mathbf{b})}\colon\ |t-t_0|\le\frac{1}{L(z^0+t\mathbf{b})}\right\}\le\lambda_{\mathbf{b}}(1)$. This means that $L(z^0+t\mathbf{b})\ge\frac{L(z^0+t_0\mathbf{b})}{\lambda_{\mathbf{b}}(1)}$. Using this inequality, we choose in (23)

$$c:=\sup_{z^0\in A}\frac{\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2\lambda_{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}\ge 1$$

with t_0 chosen in (16). Taking into account (22) and (23), one has

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le c\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\} \tag{24}$$

for all $n \in \mathbb{N} \cup \{\mathbf{0}\}$, $r \geq 0$, $z^0 \in A$, $t \in \mathbb{D}_{z^0}$.

We differentiate (18) p times, $p \geq N+1$, and then apply (15), (24) and (16)

$$\frac{|\partial_{\mathbf{b}}^p H(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \leq \frac{|\partial_{\mathbf{b}}^p G(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} + \frac{|\varepsilon||\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \leq$$
$$\leq \alpha \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})} : 0 \leq k \leq N\right\} +$$
$$+c|\varepsilon| \max\left\{|F(z^0+t'\mathbf{b})| \colon \ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\} \leq$$
$$\leq (\alpha + c|\varepsilon|) \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})} : 0 \leq k \leq N\right\}. \tag{25}$$

If $s \leq N$, then (24) is valid for $p = s$, but (15) does not hold. Thus, the differentiation of (18) leads to the following estimate

$$\frac{|\partial_{\mathbf{b}}^s H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} \geq \frac{|\partial_{\mathbf{b}}^S G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} - \frac{|\varepsilon||\partial_{\mathbf{b}}^s F(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} \geq$$
$$\geq \frac{|\partial_{\mathbf{b}}^s G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} - c|\varepsilon| \max\left\{|F(z^0+t'\mathbf{b})| \colon |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\}, \tag{26}$$

where $0 \leq s \leq N$. From (16) and (26) we deduce

$$\max_{0\leq s\leq N}\left\{\frac{|\partial_{\mathbf{b}}^s H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\} \geq (1-c|\varepsilon|) \max_{0\leq s\leq N}\left\{\frac{|\partial_{\mathbf{b}}^s G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\}. \tag{27}$$

If $c|\varepsilon| < 1$, then (25) and (27) imply

$$\frac{|\partial_{\mathbf{b}}^p H(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \leq \frac{\alpha + c|\varepsilon|}{1-c|\varepsilon|} \max_{0\leq s\leq N}\left\{\frac{|\partial_{\mathbf{b}}^s H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\} \tag{28}$$

for $p \geq N+1$. Assume that $\frac{\alpha+c|\varepsilon|}{1-c|\varepsilon|} \leq 1$. Hence, $|\varepsilon| \leq \frac{1-\alpha}{2c}$.

Let $N_{\mathbf{b}}(z^0+t\mathbf{b}, L, F)$ be the L-index in the direction $\mathbf{b}$ of the function F at the point $z^0+t\mathbf{b}$, i.e. $N_{\mathbf{b}}(z^0+t\mathbf{b}, L, F)$ is the smallest number m_0 for which inequality (3) holds with $z = z^0+t\mathbf{b}$.

For $|\varepsilon| \leq \frac{1-\alpha}{2c}$ validity of (28) means that for all $z^0 \in A$ and every $t \in D_{z^0}$ such that $F(z^0+t\mathbf{b}) \neq 0$ the L-index in the direction $\mathbf{b}$ at the point $z^0+t\mathbf{b}$ does not exceed N, i.e., $N_{\mathbf{b}}(z^0+t\mathbf{b}, F, L) \leq N$.

If for some $z^0 \in A$ $\quad F(z^0+t\mathbf{b}) \equiv 0$, then we have $H(z^0+t\mathbf{b}) \equiv G(z^0+t\mathbf{b})$ and $N_{\mathbf{b}}(z^0+t\mathbf{b}, F, L) = N_{\mathbf{b}}(z^0+t\mathbf{b}, G, L) \leq N$. Thus, $H(z)$ has bounded L-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H, L) \leq N$. It completes the proof of Theorem 10. □

Remark 2. Every analytic function $F \colon \mathbb{B}^n \to \mathbb{C}$ with $N_{\mathbf{b}}(F, L) = 0$ satisfies inequality (17) (see proof of the necessity in [6, Theorem 2]).

If $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, then condition 2) in Theorem 10 always holds. The following theorem is valid.

Theorem 11. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\alpha \in (1/\beta, 1)$ and F, G be analytic functions in $\mathbb{B}^n$ which satisfy condition:*

1) $G(z)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.

2) for every $z = z^0 + t\mathbf{b} \in \mathbb{B}^n$, where $z^0 \in A$, and some $t_0 \in B(z^0, t)$, and $r = |t - t_0| L(z^0 + t\mathbf{b})$

$$\max\left\{|F(z^0 + t'\mathbf{b})|\colon\ |t' - t_0| = \frac{2r}{L(z^0 + t\mathbf{b})}\right\} \le$$
$$\le \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0 + t\mathbf{b})|}{k! L^k(z^0 + t\mathbf{b})} : 0 \le k \le N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\}.$$

3) $c := \sup\limits_{z^0 \in A} \frac{\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0| = \frac{2\lambda_2^{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|} < \infty$ where t_0 is chosen in 2).

If $|\varepsilon| \le \frac{1-\alpha}{2c}$, then the function $H(z) = G(z) + \varepsilon F(z)$ has bounded L-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H, L) \le N_{\mathbf{b}}(G_\alpha, L_\alpha)$, where $G_\alpha(z) = G(z/\alpha)$, $L_\alpha(z) = L(z/\alpha)$.

Proof. Condition 2) in Theorem 10 always holds for $N = N_b(G_\alpha, L_\alpha)$ instead of $N = N_{\mathbf{b}}(G, L)$. Indeed by Theorem 9 inequality (14) is satisfied for the function G. Substituting $\frac{z^0}{\alpha}$, $\frac{t}{\alpha}$ and $\frac{t_0}{\alpha}$ instead z^0, t and t_0 in (14) we obtain

$$\max\left\{|G((z^0 + t\mathbf{b})/\alpha)|\colon |t - t_0| = \frac{r_2\alpha}{L((z^0 + t_0\mathbf{b})/\alpha)}\right\} \le$$
$$\le P_1 \max\left\{|G((z^0 + t\mathbf{b})/\alpha)|\colon |t - t_0| = \frac{r_1\alpha}{L((z_0 + t_0\mathbf{b})/\alpha)}\right\}. \tag{29}$$

By Theorem 9 inequality (29) means that $G_\alpha = G(z/\alpha)$ has bounded L_α-index in the direction $\mathbf{b}$ and vice versa. Then for $p \ge N_{\mathbf{b}}(G_\alpha, L_\alpha) + 1$ and $\alpha \in (1/\beta, 1)$

$$\frac{|\partial_{\mathbf{b}}^p G_\alpha(z)|}{p! L_\alpha^p(z)} = \frac{|\partial_{\mathbf{b}}^p G(z/\alpha)|}{p! \alpha^p L^p(z/\alpha)} \le \max\left\{\frac{|\partial_{\mathbf{b}}^s G_\alpha(z)|}{s! L_\alpha^s(z)}\colon\ 0 \le s \le N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\} =$$
$$= \max\left\{\frac{|\partial_{\mathbf{b}}^s G(z/\alpha)|}{s! \alpha^s L^s(z/\alpha)}\colon\ 0 \le s \le N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\}.$$

Multiplying by α^p, we deduce

$$\frac{|\partial_{\mathbf{b}}^p G(z/\alpha)|}{p! L^p(z/\alpha)} \le \max\left\{\frac{\alpha^{p-s}|\partial_{\mathbf{b}}^s G(z/\alpha)|}{s! L^s(z/\alpha)}\colon\ 0 \le s \le N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\} \le$$
$$\le \alpha \max\left\{\frac{|\partial_{\mathbf{b}}^s G(z/\alpha)|}{s! L^s(z/\alpha)}\colon\ 0 \le s \le N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\}. \tag{30}$$

Since z is arbitrary, inequality (30) yields (15). □

It is easy to see that $N_{\mathbf{b}}(G_\alpha, L_\alpha) \le N_{\mathbf{b}}(G, L)$ for $\alpha \in (0, 1)$. Thus, $N_{\mathbf{b}}(G_\alpha, L_\alpha)$ in Theorem 11 can be replaced by $N_{\mathbf{b}}(G, L)$.

Corollary 1. *Let $l \in Q(\mathbb{D})$, $\alpha \in (1/\beta, 1)$, $\beta > \lambda(1)$ and f, g be analytic functions in the unit disc $\mathbb{D}$, satisfying the conditions:*

1) $g(z)$ has bounded l-index;

2) *for every* $t\in\mathbb{C}$ *there exists* t_0 *such that* $f(t_0)\neq 0$, $|t_0-t|<\min\{\frac{\beta}{2l(t)};\frac{1-|t|}{2}\}$ *and for* $r=|t-t_0|l(t)$ *one has*

$$\max\left\{|f(t')|\colon\ |t'-t_0|=\frac{2r}{l(t)}\right\}\leq\max\left\{\frac{|g^{(k)}(t)|}{k!l^k(t)}\colon 0\leq k\leq N(g_\alpha,l_\alpha)\right\}.$$

3) *for all* t_0 *chosen in condition 2) one has* $\max\left\{|f(t')|\colon |t'-t_0|=\frac{2\lambda(1)}{l(t_0)}\right\}\Big/|f(t_0)|\leq c<+\infty$.

If $|\varepsilon|\leq\frac{1-\alpha}{2c}$, *then the function* $h(z)=g(z)+\varepsilon f(z)$ *is of bounded* l*-index with* $N(h,l)\leq N(g_\alpha,l_\alpha)$, *where* $g_\alpha(z)=g(z/\alpha)$, $l_\alpha(z)=l(z/\alpha)$.

Theorems 10 and 11 are new even for $n=1$, i.e. for analytic functions in the unit disc.

5. L**-index in direction in a domain compactly embedded in the unit ball.** Let D be an arbitrary bounded domain in $\mathbb{B}^n$ such that $\operatorname{dist}(D,\partial\mathbb{B}^n)>0$. If inequality (3) holds for all $z\in D$ instead of $\partial\mathbb{B}^n$, then the analytic function $F\colon\partial\mathbb{B}^n\to\mathbb{C}$ is called a *function of bounded* L*-index in the direction* $\mathbf{b}$ *in the domain* D. The least such integer m_0 is called the L*-index in the direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ *in domain* D and is denoted by $N_{\mathbf{b}}(F,L,D)=m_0$. The notation $\overline{D}$ stands for a closure of the domain D.

Lemma 2. *Let* D *be an arbitrary bounded domain in* $\mathbb{B}^n$ *such that* $d=\operatorname{dist}(D,\partial\mathbb{B}^n)=\inf_{z\in D}(1-|z|)>0$, $\beta>1$, $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ *be an arbitrary direction. If* $L\colon\mathbb{B}^n\to\mathbb{R}_+$ *is continuous function such that* $L(z)\geq\frac{\beta|b|}{d}$, *and* $F\colon\mathbb{B}^n\to\mathbb{C}$ *is analytic function such that* $(\forall z^0\in\overline{D})\colon\quad F(z^0+t\mathbf{b})\not\equiv 0$, *then* $N_{\mathbf{b}}(F,L,D)<\infty$.

Proof. For every fixed $z^0\in\overline{D}$ we expand the analytic function $F(z^0+t\mathbf{b})$ in a power series by powers of t in the disc $\left\{t\in\mathbb{C}\colon |t|\leq\frac{1}{L(z^0)}\right\}$

$$F(z^0+t\mathbf{b})=\sum_{m=0}^{\infty}\frac{\partial_{\mathbf{b}}^mF(z^0)}{m!}t^m. \tag{31}$$

The quantity $\frac{|\partial_{\mathbf{b}}^mF(z^0)|}{m!}$ is the modulus of a coefficient of the power series (31) at the point $t\in\mathbb{C}$ such that $|t|=\frac{1}{L(z^0)}$. Since $F(z)$ is function, for every $z_0\in\overline{D}$

$$\frac{|\partial_{\mathbf{b}}^mF(z^0)|}{m!L^m(z^0)}\to 0\quad(m\to\infty),$$

i.e., there exists $m_0=m(z^0,\mathbf{b})$ such that inequality (3) holds at the point $z=z^0$ for all $m\in\mathbb{Z}_+$.

We prove that $\sup\{m_0\colon z^0\in\overline{D}\}<+\infty$. On the contrary we assume that the set of all values m_0 is unbounded in z^0, i.e., $\sup\{m_0\colon z^0\in\overline{D}\}=+\infty$. Hence, for every $m\in\mathbb{Z}_+$ there exists $z^{(m)}\in\overline{D}$ and $p_m>m$

$$\frac{1}{p_m!L^{p_m}(z^{(m)})}\left|\frac{\partial^{p_m}F(z^{(m)})}{\partial\mathbf{b}^{p_m}}\right|>\max\left\{\frac{1}{k!L^k(z^{(m)})}\left|\frac{\partial^kF(z^{(m)})}{\partial\mathbf{b}^k}\right|\colon\ 0\leq k\leq m\right\}. \tag{32}$$

Since $\{z^{(m)}\}\subset\overline{D}$, there exists a subsequence $z'^{(m)}\to z'\in\overline{G}$ as $m\to+\infty$. By Cauchy's integral formula

$$\frac{\partial_{\mathbf{b}}^pF(z)}{p!}=\frac{1}{2\pi i}\int_{|t|=r}\frac{F(z+t\mathbf{b})}{t^{p+1}}dt$$

for any $p \in \mathbb{N}$, $z \in D$. Rewrite (32) as following

$$\max\left\{\frac{1}{k!L^k(z^{(m)})}\left|\frac{\partial^k F(z^{(m)})}{\partial \mathbf{b}^k}\right|:\ 0 \le k \le m\right\} <$$
$$< \frac{1}{L^{p_m}(z^{(m)})}\int\limits_{|t|=r/L(z^{(m)})}\frac{|F(z^{(m)}+t\mathbf{b})|}{|t|^{p_m+1}}|dt| \le \frac{1}{r^{p_m}}\max\{|F(z)|\colon z\in D_r\}, \tag{33}$$

where $D_r = \bigcup_{z^*\in\overline{D}}\{z \in \mathbb{C}^n\colon |z-z^*| \le \frac{|b|r}{L(z^*)}\}$. We can choose $r \in (1,\beta)$, because F is a function analytic in the unit ball. Evaluating the limit for every directional derivative of fixed order in (33) as $m \to \infty$ we obtain

$$(\forall k \in \mathbb{Z}_+)\colon\quad \frac{1}{k!L^k(z')}\left|\frac{\partial^k F(z')}{\partial \mathbf{b}^k}\right| \le \varlimsup_{m\to\infty}\frac{1}{r^{p_m}}\max\{|F(z)|\colon z\in D_r\} \le 0.$$

Thus, all derivatives in the direction $\mathbf{b}$ of the function F at the point z' equals 0 and $F(z')=0$. In view of (31) $F(z'+t\mathbf{b}) \equiv 0$. It is a contradiction. □

6. Existence theorem. We consider the function $F(z^0+t\mathbf{b})$ where $z^0 \in \mathbb{B}^n$ is fixed. If $F(z^0+t\mathbf{b}) \not\equiv 0$, then we denote by $p_{\mathbf{b}}(z^0+a_k^0\mathbf{b})$ the multiplicity of the zero a_k^0 of the function $F(z^0+t\mathbf{b})$. If $F(z^0+t\mathbf{b}) \equiv 0$ for some $z^0 \in \mathbb{B}^n$, then we put $p_{\mathbf{b}}(z^0+t\mathbf{b}) = -1$.

Theorem 12. *In order that for an analytic function $F\colon \mathbb{B}^n \to \mathbb{C}$ there exist a positive continuous function $L\colon \mathbb{B}^n \to \mathbb{R}_+$ such that $F(z)$ is a function of bounded L-index in the direction $\mathbf{b}$ it is necessary and sufficient that $\exists p \in \mathbb{Z}_+\ \forall z^0 \in \mathbb{B}^n\ \forall k\ p_{\mathbf{b}}(z^0+a_k^0\mathbf{b}) \le p$.*

Proof. Our proof is based on the proof for entire functions from [13] and for analytic functions in the unit ball of bounded **L**-index in joint variables from [10].
Necessity. To simplify the notation we consider everywhere in the proof $p_k^0 \equiv p_{\mathbf{b}}(z^0+a_0^k\mathbf{b})$. Necessity follows from the definition of analytic function of bounded L-index in direction. Indeed, assume on the contrary that $\forall p \in \mathbb{Z}_+\ \exists z^0\ \exists k\ p_k^0 > p$. This means that

$$\partial_{\mathbf{b}}^{p_k^0} F(z^0+a_k^0\mathbf{b}) \ne 0 \ \text{ and }\ \partial_{\mathbf{b}}^{j} F(z^0+a_k^0\mathbf{b}) = 0$$

for all $j \in \{1,\dots,p_k^0-1\}$. Therefore L-index in the direction b at the point $z^0+a_k^0\mathbf{b}$ is not less than $p_k^0 > p$

$$N_{\mathbf{b}}(F,L,z^0+a_k^0\mathbf{b}) > p.$$

If $p \to +\infty$, then we obtain that $N_{\mathbf{b}}(F,L,z^0+a_k^0\mathbf{b}) \to +\infty$. But this contradicts the boundedness of L-index in the direction of the function F.

Sufficiency. If for some $z^0 \in \mathbb{B}^n$, $F(z^0+t\mathbf{b}) \equiv 0$, then inequality (3) is obvious.

Let p be the smallest integer such that $\forall z^0 \in \mathbb{B}^n\ F(z^0+t\mathbf{b}) \not\equiv 0$, and $\forall k\ p_k(z^0) \le p$. For any point $z \in \mathbb{B}^n$ we define unambiguously the choice of $z^0 \in \mathbb{C}^n$ and $t_0 \in \mathbb{C}$ such that $z = z^0 + t_0\mathbf{b}$. We choose a point $z^0 \in \mathbb{B}^n$ on the hyperplane $\langle z,\mathbf{b}\rangle = 0$, i.e. the point z^0 is a projection of point z on the hyperplane. Therefore, there exists $t_0 \in D_{z^0}$ such that $z = z^0 + t\mathbf{b}$. Let $R \in (0, \frac{1-|z^0|}{|\mathbf{b}|})$. We define $r_0 = \frac{1}{2}\min\{1-R, R\}$. We put $K_R = \{t \in \mathbb{C}\colon R - r_0 \le |t| \le R + r_0\}$ for all $R \in (0, \frac{1-|z^0|}{|\mathbf{b}|})$ and

$$m_1(z^0,R) = \min_{a_k^0\in K_R}\max_{0\le s\le p}\left\{\frac{|\partial_{\mathbf{b}}^s F(z^0+a_k^0\mathbf{b})|}{s!}\right\},$$

where a_k^0 are zeros of the function $F(z^0+t\mathbf{b})$.

Since F is analytic, there exists $\varepsilon=\varepsilon(z^0,R)>0$ such that

$$\frac{|\partial_{\mathbf{b}}^{s_0}F(z^0+t\mathbf{b})|}{s_0!}\geq\frac{m_1(z^0,R)}{2}$$

for some $s_0=s(a_k^0)\in\{0,\dots,p\}$ and for all $t\in K_R\cap\{t\in\mathbb{C}\colon |t-a_k^0|<\varepsilon(R,z^0)\}$ and for all k. We denote $G_\varepsilon^0=\bigcup_{a_k^0\in K_R}\{t\in\mathbb{C}\colon\ |t-a_k^0|<\varepsilon\}$, $m_2(z^0,R)=\min\{|F(z^0+t\mathbf{b})|\colon |t|\leq R+r_0, t\notin G_\varepsilon^0\}$,

$$Q(R,z^0)=\min\left\{\frac{m_1(R,z^0)}{2},m_2(R,z^0),1\right\}.$$

We take $R=|t_0|$. Then at least one of the numbers $|F(z^0+t_0\mathbf{b})|$, $|\partial_{\mathbf{b}}F(z^0+t_0\mathbf{b})|$, $\dots$, $\frac{1}{p!}|\partial_{\mathbf{b}}^pF(z^0+t_0\mathbf{b})|$ is not less than $Q(R,z^0)$ (respectively, $\frac{1}{s_0!}\left|\partial_{\mathbf{b}}^{s_0}F(z^0+t_0)\mathbf{b})\right|$ for $t_0\in G_\varepsilon^0$ and $|F(z^0+t_0\mathbf{b})|$ for $t\notin G_\varepsilon$). Hence

$$\max\left\{\frac{1}{j!}\left|\partial_{\mathbf{b}}^jF(z^0+t_0\mathbf{b})\right|:0\leq j\leq p\right\}\geq Q(R,z^0). \tag{34}$$

On the other hand, for $|t_0|=R$ and $j\geq p+1$ Cauchy's inequality is valid

$$\frac{1}{j!}\left|\partial_{\mathbf{b}}^jF(z^0+t_0\mathbf{b})\right|=\left|\frac{1}{2\pi i}\int\limits_{|\tau-t_0|=r_0}\frac{F(z^0+\tau\mathbf{b})}{(\tau-t_0)^{j+1}}d\tau\right|\leq\frac{1}{r_0^j}\max\{|F(z^0+\tau\mathbf{b})|\colon |\tau|\leq R+r_0\}. \tag{35}$$

We choose a positive continuous function $L(z)$ such that

$$L(z^0+t_0\mathbf{b})\geq\max\left\{\frac{\max\{1,\max\{|F(z^0+t\mathbf{b})|\colon |\tau|\leq R+r_0\}\}}{Q(R,z^0)r_0^2},\frac{\beta}{1-|z^0+t_0\mathbf{b}|}\right\}>1.$$

From (34) and (35) with $|t_0|=R$ and $j\geq 2\cdot p$ we obtain

$$\frac{\frac{1}{j!L^j(z^0+t_0\mathbf{b})}\cdot\left|\partial_{\mathbf{b}}^jF(z^0+t_0\mathbf{b})\right|}{\max\left\{\frac{1}{k!L^k(z^0+t_0\mathbf{b})}\left|\partial_{\mathbf{b}}^kF(z^0+t_0\mathbf{b})\right|:0\leq k\leq p\right\}}\leq\frac{L^{-j}(z^0+t_0\mathbf{b})}{r_0^jQ(R,z^0)L^{-p}(z^0+t_0\mathbf{b})}\times$$
$$\times\left(\frac{\max\{1,\max\{|F(z^0+t_0\mathbf{b})|\colon |\tau|\leq R+r_0\}\}}{Q(R,z^0)r^2}\right)^{j/2}\leq L^{p-j/2}(z^0+t_0\mathbf{b})\leq 1.$$

Since $z=z^0+t\mathbf{b}$, we have

$$\frac{\left|\partial_{\mathbf{b}}^jF(z)\right|}{j!L^j(z)}\leq\max\left\{\frac{\left|\partial_{\mathbf{b}}^kF(z)\right|}{k!L^k(z)}:0\leq k\leq p\right\}.$$

In view of arbitrariness of z, the function F has bounded L-index in the direction $\mathbf{b}$. □

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Department of Advanced Mathematics,
Ivano-Frankivsk National Technical University of Oil and Gas,
Ivano-Frankivsk, Ukraine
andriykopanytsia@gmail.com

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