УДК 517.98

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SYMMETRIC POLYNOMIALS ON THE CARTESIAN POWER OF L_p ON THE SEMI-AXIS

T. V. Vasylyshyn. *Symmetric polynomials on the Cartesian power of L_p on the semi-axis*, Mat. Stud. **50** (2018), 93–104.

The paper deals with polynomials in the complex Banach space $(L_p[0,+\infty))^n$, which are the nth Cartesian power of the complex Banach space of Lebesgue measurable integrable in a power p complex-valued functions on $[0,+\infty)$, where $1 \le p < +\infty$. It is proved that if p is an integer, then every continuous symmetric polynomial on $(L_p[0,+\infty))^n$ can be uniquely represented as an algebraic combination of some "elementary" p-homogeneous symmetric polynomials. It is also proved that if p is not an integer, then every continuous symmetric polynomial on $(L_p[0,+\infty))^n$ is constant. Results of the paper can be used for investigations of algebras of symmetric continuous polynomials and of symmetric analytic functions on $(L_p[0,+\infty))^n$.

1. Introduction. Symmetric polynomials and symmetric analytic functions on Banach spaces with symmetric structures were studied by a number of authors [2–5, 7–13] (see also a survey [1]). Firstly symmetric polynomials on the real Banach space of Lebesgue measurable integrable in a power p functions on $[0,1]$, where $1 \le p < \infty$, were studied by Nemirovski and Semenov in [7]. Symmetric polynomial and symmetric analytic functions on the complex Banach space of Lebesgue measurable essentially bounded complex-valued functions on $[0,1]$ were studied in [2, 8, 10, 11]. Symmetric and finitely-symmetric polynomials on the complex Banach space of Lebesgue measurable essentially bounded complex-valued functions on the semi-axis were studied in [3]. Symmetric polynomials on Cartesian powers of some Banach spaces (such polynomials are also called "block-symmetric") were studied in [5, 9, 12, 13]. In particular, in [12] it is constructed some "elementary" symmetric polynomials on the complex Banach space $(L_p[0,1])^n$, which is the nth Cartesian power of the complex Banach space of Lebesgue measurable integrable in a power p complex-valued functions on $[0,1]$, where $1 \le p < \infty$. It is proved that every symmetric continuous polynomial on $(L_p[0,1])^n$ can be uniquely represented as an algebraic combination of these "elementary" polynomials.

In [4] it is proved that every symmetric continuous polynomial on the real Banach space $L_p[0,+\infty)$ of Lebesgue measurable integrable in a power p functions on $[0,+\infty)$, where p is a positive integer number, is an algebraic combination (see definition below) of the polynomial $L_p[0,+\infty) \ni x \mapsto \int_{[0,+\infty)} |x(t)|^p\, dt \in \mathbb{R}$. If $1 \le p < \infty$ and p is not integer, then every symmetric continuous polynomial on $L_p[0,+\infty)$ is a constant.

In this work we consider symmetric continuous polynomials on the complex Banach space $(L_p[0,+\infty))^n$, which is the nth Cartesian power of the complex Banach space of Lebesgue

2010 *Mathematics Subject Classification:* 46E15, 46E25, 46G20, 46G25.
Keywords: polynomial; symmetric polynomial; algebraic combination.
doi:10.15330/ms.50.1.93-104

measurable integrable in a power p complex-valued functions on $[0, +\infty)$, where $1 \leq p < \infty$. We show that if p is integer, then every symmetric continuous polynomial on $(L_p[0, +\infty))^n$ can be uniquely represented as an algebraic combination of p-homogeneous polynomials on $(L_p[0, +\infty))^n$, which act as integrals of products of powers of components of an element of the space $(L_p[0, +\infty))^n$. Also we show that if p is not integer, then every continuous symmetric polynomial on $(L_p[0, +\infty))^n$ is a constant.

2. Preliminaries. We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all nonnegative integers.

Let X be a complex Banach space. A mapping $A\colon X^m \to \mathbb{C}$, where $m \in \mathbb{N}$, is called an *m-linear mapping* if it is linear with respect to every of its m arguments separately. A mapping $P\colon X \to \mathbb{C}$ is called an *m-homogeneous polynomial* if there exists an m-linear mapping $A_P\colon X^m \to \mathbb{C}$ such that its restriction to the diagonal is equal to P, that is, $P(x) = A_P(\underbrace{x, \ldots, x}_{m})$ for every $x \in X$. The mapping A_P is called the m-linear mapping associated with P.

A mapping $P = P_0 + P_1 + \ldots + P_N$, where $P_0 \in \mathbb{C}$ and P_j is a j-homogeneous polynomial for every $j \in \{1, \ldots, N\}$, is called a *polynomial* of degree at most N.

A mapping $f\colon X \to \mathbb{C}$ is called an *algebraic combination* of mappings $f_1, \ldots, f_k\colon X \to \mathbb{C}$ if there exists a polynomial $Q\colon \mathbb{C}^k \to \mathbb{C}$ such that

$$f(x) = Q(f_1(x), \ldots, f_k(x))$$

for every $x \in X$. Mappings $f_1, \ldots, f_k\colon X \to \mathbb{C}$ are called *algebraically independent* if $Q(f_1(x), \ldots, f_k(x)) = 0$ for every $x \in X$ if and only if the polynomial Q is identically equal to zero. If mappings $f_1, \ldots, f_k$ are algebraically independent and polynomials $Q_1, Q_2\colon \mathbb{C}^k \to \mathbb{C}$ are such that

$$Q_1(f_1(x), \ldots, f_k(x)) = Q_2(f_1(x), \ldots, f_k(x))$$

for every $x \in X$, then the polynomial Q_1 is identically equal to the polynomial Q_2. Thus, every algebraic combination of algebraically independent mappings is unique.

Let $p \in [1, +\infty)$ and $n \in \mathbb{N}$. Let Ω be the Lebesgue measurable subset of $\mathbb{R}$ with positive measure. Let $L_p(\Omega)$ be the complex Banach space of functions $y\colon \Omega \to \mathbb{C}$ for which the pth power of the absolute value is Lebesgue integrable, with the norm

$$\|y\|_{p,\Omega} = \left(\int_\Omega |y(t)|^p \, dt \right)^{1/p}.$$

Let $(L_p(\Omega))^n$ be the nth Cartesian power of $L_p(\Omega)$ with norm

$$\|y\|_{p,n,\Omega} = \left(\sum_{s=1}^n \int_\Omega |y_s(t)|^p \, dt \right)^{1/p},$$

where $y = (y_1, \ldots, y_n) \in (L_p(\Omega))^n$.

Let Ξ_Ω be the set of all bijections $\sigma\colon \Omega \to \Omega$ such that both σ and σ^{-1} are measurable and preserve the Lebesgue measure. A function $f\colon (L_p(\Omega))^n \to \mathbb{C}$ is called *symmetric* if

$$f((y_1 \circ \sigma, \ldots, y_n \circ \sigma)) = f((y_1, \ldots, y_n))$$

for every $(y_1, \ldots, y_n) \in (L_p(\Omega))^n$ and for every $\sigma \in \Xi_\Omega$.

Let $L_p[a,b] := L_p([a,b])$ for every interval $[a,b]$, and $L_p[0,+\infty) := L_p([0,+\infty))$. For every multi-index $k = (k_1,\ldots,k_n) \in \mathbb{Z}_+^n$ such that $1 \le |k| \le \lfloor p \rfloor$, where $|k| = k_1 + \ldots + k_n$ and $\lfloor p \rfloor$ is the floor of p, and for every interval $[a,b]$, let us define a mapping $R_{k,[a,b]} \colon (L_p[a,b])^n \to \mathbb{C}$ by

$$R_{k,[a,b]}(y) = \int_{[a,b]} \prod_{\substack{s=1\\k_s>0}}^{n} (y_s(t))^{k_s}\,dt, \tag{1}$$

where $y = (y_1,\ldots,y_n) \in (L_p[a,b])^n$. If $p \in \mathbb{N}$, for every multi-index $k = (k_1,\ldots,k_n) \in \mathbb{Z}_+^n$ such that $|k| = p$, let us define a mapping $R_{k,[0,+\infty)} \colon (L_p[0,+\infty))^n \to \mathbb{C}$ by

$$R_{k,[0,+\infty)}(y) = \int_{[0,+\infty)} \prod_{\substack{s=1\\k_s>0}}^{n} (y_s(t))^{k_s}\,dt, \tag{2}$$

where $y = (y_1,\ldots,y_n) \in (L_p[0,+\infty))^n$. For example, if $p = 3$, $n = 2$, and $k = (2,1)$, then

$$R_{k,[0,+\infty)}(y) = \int_{[0,+\infty)} (y_1(t))^2 y_2(t)\,dt,$$

where $y = (y_1,y_2) \in (L_3[0,+\infty))^2$.

Note that $R_{k,[a,b]}$ and $R_{k,[0,+\infty)}$ are symmetric $|k|$-homogeneous polynomials. The continuity of these polynomials can be proved analogically to [12, Theorem 2.1]. We will use the following result, proved in [12].

Theorem 1 ([12], Theorem 2.10)**.** *Let $N \in \mathbb{N}$. Every N-homogeneous symmetric continuous polynomial $P \colon (L_p[0,1])^n \to \mathbb{C}$ can be uniquely represented as an algebraic combination of polynomials $R_{k,[0,1]}$, where $k \in \mathbb{Z}_+^n$ are such that $1 \le |k| \le \min\{\lfloor p \rfloor, N\}$.*

Let us denote by $\ell_p(\mathbb{C}^n)$ the complex Banach space of all sequences $x = (x_1,x_2,\ldots)$, where $x_j = \big(x_j^{(1)},\ldots,x_j^{(n)}\big) \in \mathbb{C}^n$ for $j \in \mathbb{N}$, such that the series $\sum_{j=1}^{+\infty}\sum_{s=1}^{n} \big|x_j^{(s)}\big|^p$ is convergent, endowed with norm

$$\|x\|_{\ell_p(\mathbb{C}^n)} = \left(\sum_{j=1}^{+\infty}\sum_{s=1}^{n} \big|x_j^{(s)}\big|^p\right)^{1/p}.$$

A function $f \colon \ell_p(\mathbb{C}^n) \to \mathbb{C}$ is called *symmetric* if $f(x \circ \sigma) = f(x)$ for every $x \in \ell_p(\mathbb{C}^n)$ and for every bijection $\sigma \colon \mathbb{N} \to \mathbb{N}$, where $x \circ \sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$. For $m \in \mathbb{N}$, let $c_{00}^{(m)}(\mathbb{C}^n)$ be the subspace of $\ell_p(\mathbb{C}^n)$, which consists of all sequences of the form $x = (x_1,\ldots,x_m,\bar{0},\ldots)$, where $x_1,\ldots,x_m \in \mathbb{C}^n$ and $\bar{0} = (0,\ldots,0) \in \mathbb{C}^n$. For every $k \in \mathbb{Z}_+^n$ such that $|k| \ge 1$, let $H_k^{(m)} \colon c_{00}^{(m)}(\mathbb{C}^n) \to \mathbb{C}$ be defined by

$$H_k^{(m)}(x) = \sum_{j=1}^{m} \prod_{\substack{s=1\\k_s>0}}^{n} \big(x_j^{(s)}\big)^{k_s}. \tag{3}$$

Note that $H_k^{(m)}$ is a continuous $|k|$-homogeneous polynomial.

Proposition 1 ([5], Corollary 7)**.** *Let $M = \big\{k^{(1)},\ldots,k^{(s)}\big\} \subset \mathbb{Z}_+^n$ be such that $\big|k^{(j)}\big| \ge 1$ for every $j \in \{1,\ldots,s\}$. Then there exists $m \in \mathbb{N}$ such that for every $m' \ge m$ polynomials $H_{k^{(1)}}^{(m')},\ldots,H_{k^{(s)}}^{(m')}$ are algebraically independent on $c_{00}^{(m')}(\mathbb{C}^n)$.*

For every $k \in \mathbb{Z}_+^n$ such that $|k| \geq \lceil p \rceil$, where $\lceil p \rceil$ is the ceiling of p, let us define a polynomial $H_k\colon \ell_p(\mathbb{C}^n) \to \mathbb{C}$ by

$$H_k(x) = \sum_{j=1}^{+\infty} \prod_{\substack{s=1\\k_s>0}}^{n} \big(x_j^{(s)}\big)^{k_s}.$$

Theorem 2 ([5], Theorem 14)**.** *Let $N \in \mathbb{N}$. Let $P\colon \ell_p(\mathbb{C}^n) \to \mathbb{C}$ be a symmetric continuous N-homogeneous polynomial. If $1 \leq N < \lceil p \rceil$, then $P \equiv 0$. Otherwise, P can be uniquely represented as an algebraic combination of polynomials H_k, where $\lceil p \rceil \leq |k| \leq N$.*

3. The main result. For every $m \in \mathbb{N}$, let us define a mapping $v_{1,m}\colon (L_p[0,1])^n \to (L_p[0,m])^n$ in the following way. In the case $m = 1$, let $v_{1,m}(x) = x$ for every $x \in (L_p[0,1])^n$. In the case $m \geq 2$, let

$$v_{1,m}(x)(t) = \begin{cases} x(t), & \text{if } t \in [0,1],\\ 0, & \text{if } t \in (1,m] \end{cases}$$

for every $x \in (L_p[0,1])^n$ and $t \in [0,m]$. In a similar way, for every $m \in \mathbb{N}$, let us define a mapping $v_{m,\infty}\colon (L_p[0,m])^n \to (L_p[0,+\infty))^n$ by

$$v_{m,\infty}(x)(t) = \begin{cases} x(t), & \text{if } t \in [0,m],\\ 0, & \text{if } t \in (m,+\infty), \end{cases} \tag{4}$$

where $x \in (L_p[0,m])^n$ and $t \in [0,+\infty)$. Note that $v_{1,m}$ and $v_{m,\infty}$ are isometric linear mappings.

Lemma 1. *Let $m, N \in \mathbb{N}$ and $P\colon (L_p[0,+\infty))^n \to \mathbb{C}$ be a continuous symmetric N-homogeneous polynomial. Then the mapping $P \circ v_{m,\infty}$ is a continuous symmetric N-homogeneous polynomial on $(L_p[0,m])^n$.*

Proof. We have the following diagram:

$$(L_p[0,m])^n \xrightarrow{\ v_{m,\infty}\ } (L_p[0,+\infty))^n \xrightarrow{\ P\ } \mathbb{C}.$$

Since $v_{m,\infty}$ is an isometric linear mapping and P is a continuous N-homogeneous polynomial, it follows that $P \circ v_{m,\infty}$ is a continuous N-homogeneous polynomial. Let us show that $P \circ v_{m,\infty}$ is symmetric. Let $x \in (L_p[0,m])^n$ and $\sigma \in \Xi_{[0,m]}$. Let us define $\tilde{\sigma}\colon [0,+\infty) \to [0,+\infty)$ by

$$\tilde{\sigma}(t) = \begin{cases} \sigma(t), & \text{if } t \in [0,m],\\ t, & \text{if } t \in (m,+\infty). \end{cases}$$

Since $\sigma \in \Xi_{[0,m]}$, it follows that $\tilde{\sigma} \in \Xi_{[0,+\infty)}$. Let us show that $v_{m,\infty}(x \circ \sigma) = v_{m,\infty}(x) \circ \tilde{\sigma}$. Since $\tilde{\sigma}(t) \in [0,m]$ if and only if $t \in [0,m]$, and $\tilde{\sigma}(t) \in (m,+\infty)$ if and only if $t \in (m,+\infty)$, it follows that

$$(v_{m,\infty}(x) \circ \tilde{\sigma})(t) = v_{m,\infty}(x)(\tilde{\sigma}(t)) = \begin{cases} x(\tilde{\sigma}(t)), & \text{if } \tilde{\sigma}(t) \in [0,m],\\ 0, & \text{if } \tilde{\sigma}(t) \in (m,+\infty) \end{cases} =$$

$$= \begin{cases} x(\tilde{\sigma}(t)), & \text{if } t \in [0,m],\\ 0, & \text{if } t \in (m,+\infty) \end{cases} = \begin{cases} x(\sigma(t)), & \text{if } t \in [0,m],\\ 0, & \text{if } t \in (m,+\infty) \end{cases} = v_{m,\infty}(x \circ \sigma)(t).$$

Thus, $v_{m,\infty}(x \circ \sigma) = v_{m,\infty}(x) \circ \tilde{\sigma}$. Therefore, $(P \circ v_{m,\infty})(x \circ \sigma) = P(v_{m,\infty}(x) \circ \tilde{\sigma})$. Since P is symmetric and $\tilde{\sigma} \in \Xi_{[0,+\infty)}$, it follows that $P(v_{m,\infty}(x) \circ \tilde{\sigma}) = P(v_{m,\infty}(x))$. Therefore, $(P \circ v_{m,\infty})(x \circ \sigma) = (P \circ v_{m,\infty})(x)$. Thus, $P \circ v_{m,\infty}$ is symmetric. This completes the proof. □

For $m \in \mathbb{N}$, let us define a mapping $I_m \colon (L_p[0,1])^n \to (L_p[0,m])^n$ by

$$I_m(x)(t) = x(t/m),$$

where $x \in (L_p[0,1])^n$ and $t \in [0,m]$. Clearly, I_m is a linear bijection. Let us show that I_m is continuous. For $x = (x_1, \ldots, x_n) \in (L_p[0,1])^n$ we have

$$\begin{aligned} \|I_m(x)\|_{p,n,[0,m]} &= \left(\int_{[0,m]} |x_1(t/m)|^p \, dt + \ldots + \int_{[0,m]} |x_n(t/m)|^p \, dt \right)^{1/p} \\ &= \left(m \int_{[0,1]} |x_1(\tau)|^p \, d\tau + \ldots + m \int_{[0,1]} |x_n(\tau)|^p \, d\tau \right)^{1/p} = m^{1/p} \|x\|_{p,n,[0,1]}. \end{aligned}$$

Hence, I_m is continuous. Therefore, I_m is an isomorphism.

Lemma 2. *Let $m, N \in \mathbb{N}$ and $P \colon (L_p[0,+\infty))^n \to \mathbb{C}$ be a continuous symmetric N-homogeneous polynomial. Then the mapping $P \circ v_{m,\infty} \circ I_m$ is a continuous symmetric N-homogeneous polynomial on $(L_p[0,1])^n$.*

Proof. We have the following diagram:

$$(L_p[0,1])^n \xrightarrow{I_m} (L_p[0,m])^n \xrightarrow{v_{m,\infty}} (L_p[0,+\infty))^n \xrightarrow{P} \mathbb{C}.$$

By Lemma 1, $P \circ v_{m,\infty}$ is a continuous symmetric N-homogeneous polynomial on $(L_p[0,m])^n$. Since I_m is an isomorphism and $P \circ v_{m,\infty}$ is a continuous N-homogeneous polynomial, it follows that $P \circ v_{m,\infty} \circ I_m$ is a continuous N-homogeneous polynomial. Let us show that $P \circ v_{m,\infty} \circ I_m$ is symmetric. Let $x \in (L_p[0,1])^n$ and $\sigma \in \Xi_{[0,1]}$. Let us define $\hat{\sigma} \colon [0,m] \to [0,m]$ by

$$\hat{\sigma}(t) = m\sigma(t/m).$$

Clearly, $\hat{\sigma} \in \Xi_{[0,m]}$. Let us show that $I_m(x \circ \sigma) = I_m(x) \circ \hat{\sigma}$. Note that for every $t \in [0,m]$

$$(I_m(x) \circ \hat{\sigma})(t) = I_m(x)(\hat{\sigma}(t)) = x(\hat{\sigma}(t)/m) = x(\sigma(t/m)) = (x \circ \sigma)(t/m) = I_m(x \circ \sigma)(t).$$

Thus, $I_m(x \circ \sigma) = I_m(x) \circ \hat{\sigma}$. Therefore,

$$(P \circ v_{m,\infty} \circ I_m)(x \circ \sigma) = (P \circ v_{m,\infty})(I_m(x \circ \sigma)) = (P \circ v_{m,\infty})(I_m(x) \circ \hat{\sigma}).$$

Since $P \circ v_{m,\infty}$ is symmetric and $\hat{\sigma} \in \Xi_{[0,m]}$, it follows that

$$(P \circ v_{m,\infty})(I_m(x) \circ \hat{\sigma}) = (P \circ v_{m,\infty})(I_m(x)).$$

Therefore,

$$(P \circ v_{m,\infty} \circ I_m)(x \circ \sigma) = (P \circ v_{m,\infty} \circ I_m)(x).$$

Thus, $P \circ v_{m,\infty} \circ I_m$ is symmetric. This completes the proof. □

For every nonempty finite set $M \subset \mathbb{Z}_+^n$ and for every mapping $l\colon M \to \mathbb{Z}_+$, let

$$\varkappa(l, M) = \sum_{k \in M} |k| l(k).$$

Lemma 3. *Let $P\colon (L_p[0, +\infty))^n \to \mathbb{C}$ be a continuous symmetric N-homogeneous polynomial, where $N \in \mathbb{N}$. Then, for every $m \in \mathbb{N}$, the polynomial $P \circ v_{m,\infty}$ can be uniquely represented in the form*

$$P \circ v_{m,\infty} = \sum_{\substack{l\colon M_1 \to \mathbb{Z}_+ \\ \varkappa(l, M_1) = N}} \alpha_l \prod_{\substack{k \in M_1 \\ l(k) > 0}} R_{k,[0,m]}^{l(k)},$$

where

$$M_1 = \big\{ k \in \mathbb{Z}_+^n :\ 1 \le |k| \le \min\{\lfloor p \rfloor, N\} \big\} \tag{5}$$

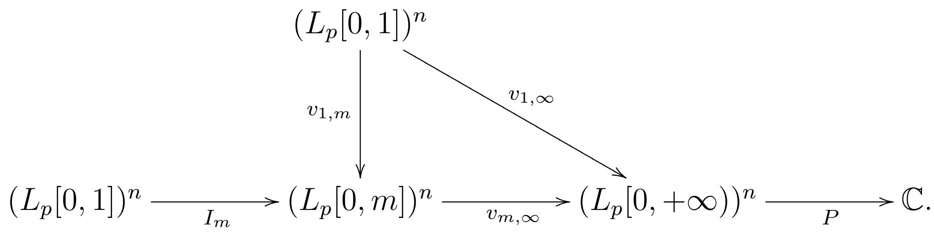
and the coefficients $\alpha_l \in \mathbb{C}$ does not depend on m.

Proof. By Lemma 1, the mapping $P \circ v_{1,\infty}$ is a continuous symmetric N-homogeneous polynomial on $(L_p[0, 1])^n$. Therefore, by Theorem 1, the polynomial $P \circ v_{1,\infty}$ can be uniquely represented as an algebraic combination of polynomials $R_{k,[0,1]}$, where $k \in \mathbb{Z}_+^n$ are such that $1 \le |k| \le \min\{\lfloor p \rfloor, N\}$. In other words, the polynomial $P \circ v_{1,\infty}$ can be uniquely represented in the form

$$P \circ v_{1,\infty} = \sum_{\substack{l\colon M_1 \to \mathbb{Z}_+ \\ \varkappa(l, M_1) = N}} \alpha_l \prod_{\substack{k \in M_1 \\ l(k) > 0}} R_{k,[0,1]}^{l(k)}, \tag{6}$$

where $\alpha_l \in \mathbb{C}$ and M_1 is defined by (5).

Let $m \in \mathbb{N}$. Consider the following diagram:



Evidently, $v_{m,\infty} \circ v_{1,m} = v_{1,\infty}$. By Lemma 2, the mapping $P \circ v_{m,\infty} \circ I_m$ is a continuous symmetric N-homogeneous polynomial on $(L_p[0, 1])^n$. Therefore, by Theorem 1, the polynomial $P \circ v_{m,\infty} \circ I_m$ can be uniquely represented as an algebraic combination of polynomials $R_{k,[0,1]}$, where $k \in \mathbb{Z}_+^n$ are such that $1 \le |k| \le \min\{\lfloor p \rfloor, N\}$. In other words, the polynomial $P \circ v_{m,\infty} \circ I_m$ can be uniquely represented in the form

$$P \circ v_{m,\infty} \circ I_m = \sum_{\substack{l\colon M_1 \to \mathbb{Z}_+ \\ \varkappa(l, M_1) = N}} \beta_l^{(m)} \prod_{\substack{k \in M_1 \\ l(k) > 0}} R_{k,[0,1]}^{l(k)}, \tag{7}$$

where $\beta_l^{(m)} \in \mathbb{C}$ and M_1 is defined by (5). Since I_m is an isomorphism, by (7),

$$P \circ v_{m,\infty} = \sum_{\substack{l\colon M_1 \to \mathbb{Z}_+ \\ \varkappa(l, M_1) = N}} \beta_l^{(m)} \prod_{\substack{k \in M_1 \\ l(k) > 0}} \big(R_{k,[0,1]} \circ I_m^{-1} \big)^{l(k)}. \tag{8}$$

For $x = (x_1, \ldots, x_n) \in (L_p[0, m])^n$ and $k \in \mathbb{Z}_+$ such that $k \in M_1$,

$$\big(R_{k,[0,1]} \circ I_m^{-1}\big)(x) = \int_{[0,1]} \prod_{\substack{s=1\\k_s>0}}^{n} (x_s(mt))^{k_s}\, dt = \frac{1}{m} \int_{[0,m]} \prod_{\substack{s=1\\k_s>0}}^{n} (x_s(\tau))^{k_s}\, d\tau = \frac{1}{m} R_{k,[0,m]}(x).$$

Thus, $R_{k,[0,1]} \circ I_m^{-1} = \frac{1}{m} R_{k,[0,m]}$. Therefore, by (8),

$$P \circ v_{m,\infty} = \sum_{\substack{l\colon M_1\to\mathbb{Z}_+\\ \varkappa(l,M_1)=N}} \beta_l^{(m)} \prod_{\substack{k\in M_1\\ l(k)>0}} \left(\frac{1}{m} R_{k,[0,m]}\right)^{l(k)} = \sum_{\substack{l\colon M_1\to\mathbb{Z}_+\\ \varkappa(l,M_1)=N}} \frac{\beta_l^{(m)}}{m^{\sum_{k\in M_1} l(k)}} \prod_{\substack{k\in M_1\\ l(k)>0}} R_{k,[0,m]}^{l(k)}. \quad (9)$$

By (9),

$$P \circ v_{m,\infty} \circ v_{1,m} = \sum_{\substack{l\colon M_1\to\mathbb{Z}_+\\ \varkappa(l,M_1)=N}} \frac{\beta_l^{(m)}}{m^{\sum_{k\in M_1} l(k)}} \prod_{\substack{k\in M_1\\ l(k)>0}} \big(R_{k,[0,m]} \circ v_{1,m}\big)^{l(k)}.$$

Taking into account that $v_{m,\infty} \circ v_{1,m} = v_{1,\infty}$ and $R_{k,[0,m]} \circ v_{1,m} = R_{k,[0,1]}$, we have

$$P \circ v_{1,\infty} = \sum_{\substack{l\colon M_1\to\mathbb{Z}_+\\ \varkappa(l,M_1)=N}} \frac{\beta_l^{(m)}}{m^{\sum_{k\in M_1} l(k)}} \prod_{\substack{k\in M_1\\ l(k)>0}} R_{k,[0,1]}^{l(k)}.$$

By the uniqueness of the representation (6), $\alpha_l = \dfrac{\beta_l^{(m)}}{m^{\sum_{k\in M_1} l(k)}}$ for every $l\colon M_1 \to \mathbb{Z}_+$ such that $\varkappa(l, M_1) = N$. Therefore, by (9),

$$P \circ v_{m,\infty} = \sum_{\substack{l\colon M_1\to\mathbb{Z}_+\\ \varkappa(l,M_1)=N}} \alpha_l \prod_{\substack{k\in M_1\\ l(k)>0}} R_{k,[0,m]}^{l(k)}.$$

This completes the proof. □

Theorem 3. *Let $N \in \mathbb{N}$. Let $P\colon (L_p[0, +\infty))^n \to \mathbb{C}$ be a symmetric continuous N-homogeneous polynomial. If $p \notin \mathbb{N}$ or $N < p$, then $P \equiv 0$. If $p \in \mathbb{N}$ and $N \geq p$, then P can be uniquely represented as an algebraic combination of polynomials $R_{k,[0,+\infty)}$, where $k \in \mathbb{Z}_+^n$ are such that $|k| = p$.*

Proof. By Lemma 3, for every $m \in \mathbb{N}$, the polynomial $P \circ v_{m,\infty}$ can be uniquely represented in the form

$$P \circ v_{m,\infty} = \sum_{\substack{l\colon M_1\to\mathbb{Z}_+\\ \varkappa(l,M_1)=N}} \alpha_l \prod_{\substack{k\in M_1\\ l(k)>0}} R_{k,[0,m]}^{l(k)}, \quad (10)$$

where M_1 is defined by (5) and the coefficients $\alpha_l \in \mathbb{C}$ does not depend on m.

Let us prove following facts:

(i) If $1 \leq N < \lceil p \rceil$, then $\alpha_l = 0$ for every $l\colon M_1 \to \mathbb{Z}_+$ such that $\varkappa(l, M_1) = N$.

(ii) If $p \notin \mathbb{N}$, then $\alpha_l = 0$ for every $l\colon M_1 \to \mathbb{Z}_+$ such that $\varkappa(l, M_1) = N$.

(iii) If $p \in \mathbb{N}$ and $p > 1$, then $\alpha_l = 0$ for every $l\colon M_1 \to \mathbb{Z}_+$ such that $\varkappa(l, M_1) = N$ and such that there exists $k \in \mathbb{Z}_+^n$, $1 \leq |k| < p$, such that $l(k) \neq 0$.

Let

$$M = \{k \in \mathbb{Z}_+^n : 1 \le |k| \le \max\{\lceil p \rceil, N\}\}.$$

By Proposition 1, there exists $m_0 \in \mathbb{N}$ such that polynomials $\{H_k^{(m_0)} : k \in M\}$, defined by (3), are algebraically independent on $c_{00}^{(m_0)}(\mathbb{C}^n)$. Consider the following diagram:

$$\begin{array}{ccccc} c_{00}^{(m_0)}(\mathbb{C}^n) & \xrightarrow{J'} & (L_p[0,m_0])^n & & \\ \Big\downarrow u & & \Big\downarrow v_{m_0,\infty} & & \\ \ell_p(\mathbb{C}^n) & \xrightarrow[J]{} & (L_p[0,+\infty))^n & \xrightarrow[P]{} & \mathbb{C}, \end{array} \tag{11}$$

where mappings u, J' and J are defined in the following way. The mapping u is the embedding of $c_{00}^{(m_0)}(\mathbb{C}^n)$ into $\ell_p(\mathbb{C}^n)$. The mapping J' is defined by

$$J'(a) = \Big(\sum_{j=1}^{m_0} a_j^{(1)} 1_{[j-1,j]}, \ldots, \sum_{j=1}^{m_0} a_j^{(n)} 1_{[j-1,j]}\Big),$$

where

$$a = \big((a_1^{(1)}, \ldots, a_1^{(n)}), \ldots, (a_{m_0}^{(1)}, \ldots, a_{m_0}^{(n)}), (0, \ldots, 0), \ldots\big) \in c_{00}^{(m_0)}(\mathbb{C}^n)$$

and

$$1_{[j-1,j]}(t) = \begin{cases} 1, & \text{if } t \in [j-1, j], \\ 0, & \text{if } t \in [0,+\infty) \setminus [j-1,j] \end{cases}$$

for every $j \in \mathbb{N}$. In a similar way, the mapping J is defined by

$$J(a) = \Big(\sum_{j=1}^{+\infty} a_j^{(1)} 1_{[j-1,j]}, \ldots, \sum_{j=1}^{+\infty} a_j^{(n)} 1_{[j-1,j]}\Big),$$

where $a = \big((a_1^{(1)}, \ldots, a_1^{(n)}), (a_2^{(1)}, \ldots, a_2^{(n)}), \ldots\big) \in \ell_p(\mathbb{C}^n)$.

Clearly, $v_{m_0,\infty} \circ J' = J \circ u$. Thus, diagram (11) is commutative. Note that u, J' and J are linear mappings. Also note that, for every $a \in \ell_p(\mathbb{C}^n)$,

$$\begin{aligned} \|J(a)\|_{p,n,[0,+\infty)} &= \Big(\sum_{s=1}^{n} \int_{[0,+\infty)} \sum_{j=1}^{+\infty} |a_j^{(s)}|^p 1_{[j-1,j]}(t)\, dt\Big)^{1/p} = \\ &= \Big(\sum_{s=1}^{n} \sum_{j=1}^{+\infty} |a_j^{(s)}|^p\Big)^{1/p} = \|a\|_{\ell_p(\mathbb{C}^n)}. \end{aligned}$$

Thus, J is isometric. In a similar way, it can be checked that J' is isometric.

Note that, for every $a \in c_{00}^{(m_0)}(\mathbb{C}^n)$ and $k \in \mathbb{Z}_+^n$ such that $|k| \ge 1$,

$$(R_{k,[0,m_0]} \circ J')(a) = \int_{[0,m_0]} \prod_{\substack{s=1 \\ k_s>0}}^{n} \Big(\sum_{j=1}^{m_0} a_j^{(s)} 1_{[j-1,j]}(t)\Big)^{k_s} dt =$$

$$= \int_{[0,m_0]} \sum_{j=1}^{m_0} \prod_{\substack{s=1 \\ k_s>0}}^{n} \big(a_j^{(s)}\big)^{k_s} 1_{[j-1,j]}(t)\, dt = \sum_{j=1}^{m_0} \prod_{\substack{s=1 \\ k_s>0}}^{n} \big(a_j^{(s)}\big)^{k_s} = H_k^{(m_0)}(a),$$

that is, $R_{k,[0,m_0]} \circ J' = H_k^{(m_0)}$. Therefore, by (10),

$$P \circ v_{m_0,\infty} \circ J' = \sum_{\substack{l\colon M_1 \to \mathbb{Z}_+ \\ \varkappa(l,M_1)=N}} \alpha_l \prod_{\substack{k\in M_1 \\ l(k)>0}} \big(H_k^{(m_0)}\big)^{l(k)}. \tag{12}$$

Since P is a continuous N-homogeneous polynomial and J is a continuous linear mapping, it follows that the mapping $P \circ J$ is a continuous N-homogeneous polynomial. It can be checked that $P \circ J$ is symmetric.

Consider the case $1 \le N < \lceil p \rceil$. By Theorem 2, $P \circ J \equiv 0$. Therefore, $P \circ J \circ u \equiv 0$. Since $J \circ u = v_{m_0,\infty} \circ J'$, it follows that $P \circ v_{m_0,\infty} \circ J' \equiv 0$. Thus, by (12),

$$\sum_{\substack{l\colon M_1 \to \mathbb{Z}_+ \\ \varkappa(l,M_1)=N}} \alpha_l \prod_{\substack{k\in M_1 \\ l(k)>0}} \big(H_k^{(m_0)}\big)^{l(k)} \equiv 0. \tag{13}$$

Since $M_1 \subset M$ and polynomials $\{H_k^{(m_0)} :\ k \in M\}$ are algebraically independent, it follows that polynomials $\{H_k^{(m_0)} :\ k \in M_1\}$ are algebraically independent. Therefore, by (13), $\alpha_l = 0$ for every $l\colon M_1 \to \mathbb{Z}_+^n$ such that $\varkappa(l, M_1) = N$. This completes the proof of (i).

Consider the case $N \ge \lceil p \rceil$ and $p > 1$. By Theorem 2, $P \circ J$ can be uniquely represented as an algebraic combination of polynomials H_k, where $\lceil p \rceil \le |k| \le N$, that is, $P \circ J$ can be uniquely represented in the form

$$P \circ J = \sum_{\substack{r\colon M_2 \to \mathbb{Z}_+ \\ \varkappa(r,M_2)=N}} \gamma_r \prod_{\substack{k\in M_2 \\ r(k)>0}} (H_k)^{r(k)}, \tag{14}$$

where $\gamma_r \in \mathbb{C}$ and

$$M_2 = \big\{k \in \mathbb{Z}_+^n :\ \lceil p \rceil \le |k| \le N\big\}.$$

By (14), taking into account $H_k \circ u = H_k^{(m_0)}$,

$$P \circ J \circ u = \sum_{\substack{r\colon M_2 \to \mathbb{Z}_+ \\ \varkappa(r,M_2)=N}} \gamma_r \prod_{\substack{k\in M_2 \\ r(k)>0}} \big(H_k^{(m_0)}\big)^{r(k)}. \tag{15}$$

Since $p > 1$, it follows that the set

$$M_1' := M \setminus M_2 = \big\{k \in \mathbb{Z}_+^n :\ 1 \le |k| < p\big\}$$

is nonempty. For every $q\colon M \to \mathbb{Z}_+$ such that $\varkappa(q, M) = N$, let us define complex numbers A_q and Γ_q in the following way. If $M = M_1$, let $A_q = \alpha_q$. Otherwise, let

$$A_q = \begin{cases} \alpha_{q|_{M_1}}, & \text{if } q|_{M\setminus M_1} \equiv 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$\Gamma_q = \begin{cases} \gamma_{q|_{M_2}}, & \text{if } q|_{M\setminus M_2} \equiv 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then equalities (12) and (15) can be rewritten as

$$P\circ v_{m_0,\infty}\circ J'=\sum_{\substack{q\colon M\to\mathbb{Z}_+\\ \varkappa(q,M)=N}} A_q\prod_{\substack{k\in M\\ q(k)>0}}\left(H_k^{(m_0)}\right)^{q(k)} \tag{16}$$

and

$$P\circ J\circ u=\sum_{\substack{q\colon M\to\mathbb{Z}_+\\ \varkappa(q,M)=N}} \Gamma_q\prod_{\substack{k\in M\\ q(k)>0}}\left(H_k^{(m_0)}\right)^{q(k)}, \tag{17}$$

respectively. Since $P\circ v_{m_0,\infty}\circ J'=P\circ J\circ u$, by (16) and (17),

$$\sum_{\substack{q\colon M\to\mathbb{Z}_+\\ \varkappa(q,M)=N}} (A_q-\Gamma_q)\prod_{\substack{k\in M\\ q(k)>0}}\left(H_k^{(m_0)}\right)^{q(k)}\equiv 0.$$

Since polynomials $\{H_k^{(m_0)}:\ k\in M\}$ are algebraically independent, it follows that

$$A_q-\Gamma_q=0 \tag{18}$$

for every $q\colon M\to\mathbb{Z}_+$ such that $\varkappa(q,M)=N$.

Let us prove (ii). Let $p\notin\mathbb{N}$. If $N<\lceil p\rceil$, then (i) implies (ii). Consider the case $N\ge\lceil p\rceil$. In this case, $M=M_1\sqcup M_2$ and both sets M_1 and M_2 are nonempty. Let $l\colon M_1\to\mathbb{Z}_+$ be such that $\varkappa(l,M_1)=N$. Let $\hat{l}\colon M\to\mathbb{Z}_+$ be defined by

$$\hat{l}(k)=\begin{cases} l(k), & \text{if } k\in M_1,\\ 0, & \text{if } k\in M_2.\end{cases}$$

Then $\varkappa(\hat{l},M)=N$. Since $\hat{l}|_{M\setminus M_1}\equiv 0$, it follows that $A_{\hat{l}}=\alpha_l$. Since $\varkappa(l,M_1)\ne 0$, it follows that there exists $k_0\in M_1$ such that $l(k_0)\ne 0$. Since $\hat{l}(k_0)=l(k_0)\ne 0$ and $k_0\in M_1=M\setminus M_2$, it follows that $\hat{l}|_{M\setminus M_2}\not\equiv 0$. Consequently, $\Gamma_{\hat{l}}=0$. By (18), $A_{\hat{l}}=\Gamma_{\hat{l}}$, that is, $\alpha_l=0$. This completes the proof of (ii).

Let us prove (iii). Let $p\in\mathbb{N}$ and $p>1$. If $N<p$, then (i) implies (iii). Consider the case $N\ge p$. In this case,

$$M_1=\left\{k\in\mathbb{Z}_+^n:\ 1\le|k|\le p\right\},$$
$$M_2=\left\{k\in\mathbb{Z}_+^n:\ p\le|k|\le N\right\}$$

and

$$M=\left\{k\in\mathbb{Z}_+^n:\ 1\le|k|\le N\right\}.$$

Let $l\colon M_1\to\mathbb{Z}_+$ be such that $\varkappa(l,M_1)=N$ and there exists $k_0\in\mathbb{Z}_+^n$, $1\le|k_0|<p$, such that $l(k_0)\ne 0$. Since $1\le|k_0|<p$, it follows that $k_0\in M\setminus M_2$. Consider the case $M_1=M$. In this case, $A_l=\alpha_l$ and $\Gamma_l=0$, since $l|_{M\setminus M_2}\not\equiv 0$. Therefore, by (18), $\alpha_l=0$. Consider the case $M_1\ne M$. Let $\hat{l}\colon M\to\mathbb{Z}_+$ be defined by

$$\hat{l}(k)=\begin{cases} l(k), & \text{if } k\in M_1,\\ 0, & \text{if } k\in M\setminus M_1.\end{cases}$$

Then $\varkappa(\hat{l},M)=N$. Since $\hat{l}|_{M\setminus M_1}\equiv 0$, it follows that $A_{\hat{l}}=\alpha_l$. Since $\hat{l}(k_0)=l(k_0)\ne 0$ and $k_0\in M\setminus M_2$, it follows that $\hat{l}|_{M\setminus M_2}\not\equiv 0$ and, consequently, $\Gamma_{\hat{l}}=0$. Therefore, by (18), $\alpha_l=0$. This completes the proof of (iii).

By (10), taking into account (i)–(iii), for every $m \in N$, if $p \notin \mathbb{N}$ or $N < p$, then $P \circ v_{m,\infty} \equiv 0$, and if $p \in \mathbb{N}$ and $N \geq p$, then

$$P \circ v_{m,\infty} = \sum_{\substack{w\colon M_0 \to \mathbb{Z}_+ \\ \varkappa(w, M_0) = N}} \alpha_{\tilde{w}} \prod_{\substack{k \in M_0 \\ w(k) > 0}} R_{k,[0,m]}^{w(k)},$$

where

$$M_0 = \big\{k \in \mathbb{Z}_+^n : \ |k| = p\big\}$$

and $\tilde{w}\colon M_1 \to \mathbb{Z}_+$ is defined by

$$\tilde{w}(k) = \begin{cases} w(k), & \text{if } k \in M_0, \\ 0, & \text{otherwise} \end{cases}$$

for every $w\colon M_0 \to \mathbb{Z}_+$.

Let $D = \bigcup_{m=1}^{\infty} v_{m,\infty}\big((L_p[0,m])^n\big)$. Note that D is a linear subspace of $(L_p[0,+\infty))^n$. It can be checked that D is dense in $(L_p[0,+\infty))^n$.

Let $x \in D$. Then there exists $m \in \mathbb{N}$ such that $x \in v_{m,\infty}\big((L_p[0,m])^n\big)$. Consequently, there exists $y \in (L_p[0,m])^n$ such that $x = v_{m,\infty}(y)$. Then

$$(P \circ v_{m,\infty})(y) = \begin{cases} \displaystyle\sum_{\substack{w\colon M_0 \to \mathbb{Z}_+ \\ \varkappa(w, M_0) = N}} \alpha_{\tilde{w}} \prod_{\substack{k \in M_0 \\ w(k) > 0}} \big(R_{k,[0,m]}(y)\big)^{w(k)}, & \text{if } p \in \mathbb{N} \text{ and } N \geq p, \\ 0, & \text{if } p \notin \mathbb{N} \text{ or } N < p. \end{cases}$$

Since $(P \circ v_{m,\infty})(y) = P(v_{m,\infty})(y) = P(x)$ and $R_{k,[0,m]}(y) = R_{k,[0,+\infty)}(x)$, it follows that

$$P(x) = \begin{cases} \displaystyle\sum_{\substack{w\colon M_0 \to \mathbb{Z}_+ \\ \varkappa(w, M_0) = N}} \alpha_{\tilde{w}} \prod_{\substack{k \in M_0 \\ w(k) > 0}} \big(R_{k,[0,+\infty)}(x)\big)^{w(k)}, & \text{if } p \in \mathbb{N} \text{ and } N \geq p, \\ 0, & \text{if } p \notin \mathbb{N} \text{ or } N < p. \end{cases} \tag{19}$$

Since polynomials P and $R_{k,[0,+\infty)}$, where $k \in M_0$, are continuous on $(L_p[0,+\infty))^n$ and the equality (19) holds for every element x of the dense subspace D of $(L_p[0,+\infty))^n$, it follows that the equality (19) holds for every element x of $(L_p[0,+\infty))^n$. Since the coefficients α_l in the representation (10) are unique, it follows that the representation (19) is unique. Thus, we have proved that if $p \notin \mathbb{N}$ or $N < p$, then $P \equiv 0$, otherwise P can be uniquely represented as an algebraic combination of polynomials $R_{k,[0,+\infty)}$, where $k \in \mathbb{Z}_+^n$ are such that $|k| = p$. □

Corollary 1. *Let P be a symmetric continuous polynomial on $(L_p[0,+\infty))^n$. If $p \notin \mathbb{N}$, then P is a constant. If $p \in \mathbb{N}$, then P can be uniquely represented as an algebraic combination of polynomials $R_{k,[0,+\infty)}$, where $k \in \mathbb{Z}_+^n$ are such that $|k| = p$.*

Proof. Let $P = P_0 + P_1 + \ldots + P_N$, where $P_0 \in \mathbb{C}$ and P_j is a j-homogeneous polynomial for every $j \in \{1, \ldots, N\}$. By the Cauchy integral formula (see [6, Corollary 7.3, p. 47]), since $P_0 + P_1 + \ldots + P_N$ is the Taylor series of P at 0, it follows that every P_j is symmetric and continuous, where $j \in \{0, \ldots, N\}$.

If $p \notin \mathbb{N}$, then, by Theorem 3, $P_j \equiv 0$ for every $j \in \{1, \ldots, N\}$, therefore, $P = P_0$.

If $p \in \mathbb{N}$, then, by Theorem 3, every P_j, where $j \in \{1, \ldots, N\}$, can be uniquely represented as an algebraic combination of polynomials $R_{k,[0,+\infty)}$, where $k \in \mathbb{Z}_+^n$ are such that $|k| = p$

(every polynomial P_j such that $P_j \equiv 0$, can be considered as a trivial algebraic combination of polynomials $R_{k,[0,+\infty)}$). Thus, the polynomial P can be represented as an algebraic combination of polynomials $R_{k,[0,+\infty)}$, where $k \in \mathbb{Z}_+^n$ are such that $|k| = p$. This representation is unique, because every P_j is uniquely determined by the values of P by the Cauchy integral formula. □

REFERENCES

1. I. Chernega, *Symmetric polynomials and holomorphic functions on infinite dimensional spaces*, Journal of Vasyl Stefanyk Precarpathian National University, **2** (2015), №4, 23–49. doi:10.15330/jpnu.2.4.23-49.
2. P. Galindo, T. Vasylyshyn, A. Zagorodnyuk, *The algebra of symmetric analytic functions on L_∞*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, **147** (2017), №4, 743–761. doi:10.1017/S0308210516000287.
3. P. Galindo, T. Vasylyshyn, A. Zagorodnyuk, *Symmetric and finitely symmetric polynomials on the spaces ℓ_∞ and $L_\infty[0,+\infty)$*, Mathematische Nachrichten, **291** (2018), №11–12, 1712–1726. doi:10.1002/mana.201700314.
4. M. González, R. Gonzalo, J. A. Jaramillo, *Symmetric polynomials on rearrangement invariant function spaces*, J. London Math. Soc., **59** (1999), №2, 681–697. doi:10.1112/S0024610799007164.
5. V. Kravtsiv, T. Vasylyshyn, A. Zagorodnyuk, *On algebraic basis of the algebra of symmetric polynomials on $\ell_p(\mathbb{C}^n)$*, Journal of Function Spaces, **2017** (2017), Article ID 4947925, 8 pages. doi:10.1155/2017/4947925.
6. J. Mujica, Complex Analysis in Banach Spaces, North Holland, 1986.
7. A. S. Nemirovskii, S. M. Semenov, *On polynomial approximation of functions on Hilbert space*, Mat. USSR Sbornik, **21** (1973), №2, 255–277. doi:10.1070/SM1973v021n02ABEH002016
8. T. Vasylyshyn, *Symmetric continuous linear functionals on complex space L_∞*, Carpathian Math. Publ., **6** (2014), №1, 8–10. doi:10.15330/cmp.6.1.8-10.
9. T. Vasylyshyn, *Continuous block-symmetric polynomials of degree at most two on the space $(L_\infty)^2$*, Carpathian Math. Publ., **8** (2016), №1, 38–43. doi:10.15330/cmp.8.1.38-43.
10. T. Vasylyshyn, *Topology on the spectrum of the algebra of entire symmetric functions of bounded type on the complex L_∞*, Carpathian Math. Publ., **9** (2017), №1, 22–27. doi:10.15330/cmp.9.1.22-27.
11. T. Vasylyshyn, *Metric on the spectrum of the algebra of entire symmetric functions of bounded type on the complex L_∞*, Carpathian Math. Publ., **9** (2017), №2, 198–201. doi:10.15330/cmp.9.2.198-201.
12. T. Vasylyshyn, *Symmetric polynomials on $(L_p)^n$*, European Journal of Math., doi:10.1007/s40879-018-0268-3.
13. T. Vasylyshyn, A. Zagorodnyuk, *Continuous symmetric 3-homogeneous polynomials on spaces of Lebesgue measurable essentially bounded functions*, Methods of Functional Analysis and Topology (to appear).

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Received 05.06.2018