УДК 517.95

O. Berezsky, M. Zarichnyi

GROMOV-FRÉCHET DISTANCE BETWEEN CURVES

O. Berezsky, M. Zarichnyi. *Gromov-Fréchet distance between curves*, Mat. Stud. **50** (2018), 88–92.

The Gromov-Fréchet distance is obtained from the Fréchet distance between metric curves similarly as the Gromov-Hausdorff distance is obtained from the Hausdorff distance.

We prove that the Gromov-Fréchet space is separable and non-complete.

1. Introduction. The Hausdorff metric is one of the most important tools for measuring dissimilarities between sets in metric spaces. In order to measure how far are metric spaces from being isometric, the notion of Gromov-Hausdorff metric was introduced ([7]; see also [6]). This metric is widely used in different areas of mathematics and related disciplines, in particular, in computer graphics and computational geometry, Riemannian geometry, and cosmology.

In the case when the sets under consideration are curves in a metric space, one can define the so-called Fréchet distance between them ([8]; see the definition below). The Fréchet metric is an object of study in numerous publications (see, e.g., [1, 4, 10, 11, 13, 14, 15, 16, 17]). The area of applications of the Fréchet metric includes computational geometry, bioinformatics, pattern recognition etc.

It is natural to find an analog to the Gromov-Hausdorff metric for the metric curves, when one replaces the Hausdorff metric by the Fréchet metric. In this way one obtains the Gromov-Fréchet distance between metric curves. We will consider some properties of the obtained distance. In particular, we show that the obtained space of (isometric classes of) metric curves is separable and non-complete.

We will also discuss some modifications of the obtained distance and formulate some open questions.

The authors are indebted to the referee for his/her valuable comments.

2. Preliminaries.

2.1. Frechét distance. We will denote the unit segment $[0,1]$ by $\mathbb{I}$, and by $H_+(\mathbb{I})$ the group of increasing homeomorphisms of $\mathbb{I}$. In the sequel we will need the notion of Fréchet metric between oriented curves in a metric space (X,d) (see, e.g., [8]). Given oriented parameterized curves $\gamma_i\colon \mathbb{I}\to X$, $i=1,2$, in a metric space, define

$$d_F(\gamma_1,\gamma_2)=\inf_{\alpha\in H_+(\mathbb{I})} d(\gamma_1\circ\alpha(t),\gamma_2(t)).$$

2010 *Mathematics Subject Classification:* 37E25, 54E35.
Keywords: Gromov-Fréchet distance; Gromov-Fréchet space; Gromov-Hausdorff distance.
doi:10.15330/ms.50.1.88-92

The function d_F is known to be a metric on the set of (equivalence classes of) parameterized curves in a metric space (the Fréchet metric).

2.2. Gromov-Hausdorff distance. Given a metric space (X, d), we denote by $\exp X$ the family of all nonempty compact subsets in X (the hyperspace of X). The Hausdorff metric d_H on $\exp X$ is defined by the formula:

$$d_H(A, B) = \max\{\max_{a\in A}\min_{a\in B} d(a, b), \max_{b\in B}\min_{a\in A} d(a, b)\}.$$

The Gromov-Hausdorff distance d_{GF} is defined as follows. For any two compact metric spaces (X_i, d_i), let

$$\begin{aligned} d_{GF}(X_1, X_2) = \inf\{&d_H(j_1(X_1), j_2(X_2)) \mid j_i \colon X_i \to Z \\ &\text{are isometric embeddings into a metric space } Z,\ i = 1, 2\}. \end{aligned}$$

2.3. Metric on identification spaces. Let (X_i, d_i), $i = 1, 2$, be disjoint metric spaces and $A_i \subset X_i$, $i = 1, 2$, be closed subsets for which there exists an isometry $h\colon A_1 \to A_2$. Denote by X the set obtained from the union of X_1 and X_2 by identifying x with $h(x)$, $x \in A_1$. We will assume that X_1, X_2 are subsets of X and endow X with the maximal metric d that extends d_i on X_i, $i = 1, 2$. Explicitly,

$$d(x, y) = \inf\{d_1(x, a) + d_2(a, y) \mid a \in A_1 = A_2\},\ x \in X_1,\ y \in X_2$$

(the other cases are similar).

2.4. Kantorovich metric. Let X be a metric space. By $P(X)$ we define the set of all probability measures of compact supports on X. The Kantorovich metric d_K on the set $P(X)$ is defined as follows:

$$d_K(\mu, \nu) = \sup\left\{\Big|\int_X \varphi d\mu - \int_X \varphi d\nu\Big| \mid \varphi\colon X \to \mathbb{R} \text{ is 1-Lipschitz}\right\}$$

(see, e.g., [18]).

3. Gromov-Frechét distance. A metric curve is a metric space which is homeomorphic to $\mathbb{I}$. We will consider the pointed metric curves in which the base point is an endpoint. Any isometry of metric curves is supposed to preserve the base points. A parameterized curve in a metric space X is a map $\gamma\colon \mathbb{I} \to X$. We denote by $\mathrm{supp}(\gamma)$ the *support* of γ, i.e. the set $\gamma(\mathbb{I})$. The support of γ is a metric curve and we suppose that $\gamma(0)$ is the base point in it.

Two parameterized curves $\gamma_i\colon \mathbb{I} \to X_i$, $i = 1, 2$, in metric spaces (X_i, d_i) are said to be isometric if there exists an isometry $h\colon \mathrm{supp}(\gamma_1) \to \mathrm{supp}(\gamma_2)$ and a homeomorphism $\alpha \in H_+(\mathbb{I})$ such that $h\circ\gamma_1 = \gamma_2\circ\alpha$.

Let $\gamma_i\colon \mathbb{I} \to X_i$, $i = 1, 2$, be parameterized curves in metric spaces (X_i, d_i).

Denote by $\mathfrak{A}$ the set of all triples of the form (Z, j_1, j_2), where Z is a metric space and $j_i\colon \mathrm{supp}(\gamma_i) \to Z$, $i = 1, 2$, are isometric embeddings. Remark that $\mathfrak{A} \neq \varnothing$.

Define

$$d_{GF}(\gamma_1, \gamma_2) = \inf\{d_F(j_1\circ\gamma_1, j_2\circ\gamma_2) \mid (Z, j_1, j_2) \in \mathfrak{A}\}.$$

The following is an easy consequence of the definition.

Proposition 1. $d_{GF}(\gamma_1, \gamma_2) \geq d_{GH}(\mathrm{supp}(\gamma_1), \mathrm{supp}(\gamma_2))$.

Like in the case of the Gromov-Hausdorff distance, we deal with the equivalence classes of curves up to base point preserving isometry. For the sake of simplicity, we sometimes prefer dealing with representatives of such classes rather than the classes themselves.

Theorem 1. *The function d_{GF} is a metric on the set of isometric classes of curves.*

Proof. Clearly $d_{GF}(\gamma_1,\gamma_2) \geq 0$ for every γ_1,γ_2. Now, assume that $d_{GF}(\gamma_1,\gamma_2) = 0$. By Proposition 1, then also $d_{GH}(\gamma_1,\gamma_2) = 0$. From the properties of the Gromov-Hausdorff metric, it easily follows that γ_1 and γ_2 are isometric.

From the definition, it easily follows that the function d_{GF} is symmetric.

Let us prove the triangle inequality. Suppose that $\gamma_i\colon \mathbb{I} \to X_i$, $i = 1,2,3$, be parameterized curves. Given $\varepsilon > 0$, one can find metric spaces Y_{12}, Y_{23} and embeddings $j_i\colon \mathrm{supp}(\gamma_i) \to Y_{12}$, $i = 1,2$, and $k_i\colon \mathrm{supp}(\gamma_i) \to Y_{23}$, $i = 2,3$, such that

$$d_F(j_1{\circ}\gamma_1, j_2{\circ}\gamma_2) \leq d_{GF}(\gamma_1,\gamma_2) + \varepsilon,\ d_F(k_2{\circ}\gamma_2, k_3{\circ}\gamma_3) \leq d_{GF}(\gamma_2,\gamma_3) + \varepsilon.$$

Without loss of generality, one may assume that the maps j_i, $i = 1,2$, and k_i, $i = 2,3$, are inclusions, i.e., $\mathrm{supp}(\gamma_i) \subset Y_{12}$, $i = 1,2$, and $\mathrm{supp}(\gamma_i) \to Y_{23}$, $i = 2,3$. Thus, there exist increasing homeomorphisms $\alpha,\beta\colon \mathbb{I} \to \mathbb{I}$ such that

$$\sup\{d(\gamma_1(t), \gamma_2{\circ}\alpha(t)) \mid t \in \mathbb{I}\} \leq d_{GF}(\gamma_1,\gamma_2) + \varepsilon,$$
$$\sup\{d(\gamma_2(t), \gamma_3{\circ}\beta(t)) \mid t \in \mathbb{I}\} \leq d_{GF}(\gamma_2,\gamma_3) + \varepsilon.$$

Let Z be the identification space obtained from the spaces Y_{12} and Y_{23} by identification of the set $\mathrm{supp}(j_2{\circ}\gamma_2)$ and the set $\mathrm{supp}(k_2{\circ}\gamma_2)$ along the unique isometry map $k_2{\circ}j_2^{-1}$ preserving the base points. By abusing notation one may assume that $k_2{\circ}j_2^{-1}$ is the identity map.

Then

$$d(\gamma_1(t), \gamma_3{\circ}\alpha{\circ}\beta(t)) \leq d(\gamma_1(t), \gamma_2{\circ}\alpha(t)) + d(\gamma_2{\circ}\alpha(t), \gamma_3\alpha{\circ}\beta(t))$$

for all $t \in \mathbb{I}$ and therefore

$$\begin{aligned} d_{GF}(\gamma_1,\gamma_3) \leq d_F(\gamma_1,\gamma_3) &\leq \sup_{t\in\mathbb{I}} d(\gamma_1(t), \gamma_3{\circ}\alpha{\circ}\beta(t)) \\ &\leq \sup_{t\in\mathbb{I}} \big(d(\gamma_1(t), \gamma_2{\circ}\alpha(t)) + d(\gamma_2{\circ}\alpha(t), \gamma_3{\circ}\alpha{\circ}\beta(t))\big) \\ &\leq \sup_{t\in\mathbb{I}} d(\gamma_1(t), \gamma_2{\circ}\alpha(t)) + \sup_{t\in\mathbb{I}} d(\gamma_2{\circ}\alpha(t), \gamma_3{\circ}\alpha{\circ}\beta(t)) \\ &\leq d_{GF}(\gamma_1,\gamma_2) + \varepsilon + d_{GF}(\gamma_2,\gamma_3) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we are done. □

By $\mathfrak{C}$ we denote the metric space of the equivalence classes of curves endowed with the Gromov-Frechét metric.

Theorem 2. *The space $\mathfrak{C}$ is separable.*

Proof. Let $\gamma\colon \mathbb{I} \to X$ be a parameterized curve and let $\varepsilon > 0$. One can find a finite subset $\{t_1,\dots,t_k\} \subset \mathbb{I}$ such that $0 = t_1 < t_2 < \dots < t_{k-1} < t_k = 1$ and the diameter of the image $\gamma([t_i,t_{i+1}])$ is less than $\varepsilon/6$ for every $i \in \{1,\dots,k-1\}$. Since the set of finite metric spaces with rational distances is dense in the space of all compact metric spaces endowed with the Gromov-Hausdorff metric (see [5]), there exists a finite metric space $Y = \{y_1,\dots,y_k\}$ with rational distances such that $d_G(\gamma(\{t_1,\dots,t_k\}),Y) < \varepsilon/6$. Without loss of generality one may assume that $\gamma(\mathbb{I})$ and Y are subsets of a metric space M so that $d(\gamma(t_i),y_i) < (\varepsilon/6)$ for all $i\in \{1,\dots,k\}$. Note that

$$d(y_i,y_{i+1}) < d(y_i,\gamma(t_i)) + d(\gamma(t_i),\gamma(t_{i+1})) + d(\gamma(t_{i+1}),y_{i+1}) < \frac{3\varepsilon}{6} = \frac{\varepsilon}{2}.$$

Recall that $P(Y)$ is the space of probability measures on the space Y endowed with the Kantorovich metric. For any $y \in Y$, by δ_y the Dirac measure concentrated at y is denoted. Let

$$Z = \bigcup_{i=1}^{k-1}\{\theta\delta_{y_i} + (1-\theta)\delta_{y_{i+1}} \mid \theta \in \mathbb{I}\} \subset P(X).$$

We regard Z as a parameterized curve (with $\delta_{\gamma(0)}$ as the base point). We are going to prove that $d_F(\gamma,Z) < \varepsilon$. To this end, consider an arbitrary homeomorphism $\alpha\colon \mathrm{supp}(\gamma) \to Z$ sending $\gamma(t_i)$ to δ_{y_i} for every $i = 1,\dots,k$. Then, for $t \in [t_i,t_{i+1}]$, we obtain

$$d(\gamma(t),\alpha\circ\gamma(t)) \le d(\gamma(t),\gamma(t_i)) + d(\gamma(t_i),y_i) + d_K(\delta_{y_i},\alpha(t)) \le \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{2} < \varepsilon$$

Then also $d_{GF}(\gamma,Z) < \varepsilon$.

Since the family of parameterized curves obtained by this procedure is countable, we conclude that the space $\mathfrak{C}$ is separable.

□

Remark 1. Instead of the Kantorovich metric one can use the Kuratowski embedding of discrete metric spaces into normed spaces, see [12].

The following example demonstrates that the space $\mathfrak{C}$ is not complete. Consider a sequence $(\gamma_i)_{i=1}^\infty$ of curves in $\mathbb{R}^2$ satisfying the conditions:

1. $(\mathrm{supp}(\gamma_i))_{i=1}^\infty$ converges to the unit square $\mathbb{I}^2$ in the Hausdorff metric;
2. $d_F(\gamma_i,\gamma_{i+1}) \le 2^{-i}$ for all i

(one can take as $(\gamma_i)_{i=1}^\infty$ a sequence of approximations to square-filling Hilbert curve; see, e.g., [2]). Then $(\gamma_i)_{i=1}^\infty$ is a Cauchy sequence in $\mathfrak{C}$ and let us suppose that γ is its limit. By Proposition 1, the sequence $(\mathrm{supp}(\gamma_i))_{i=1}^\infty$ converges to $\mathrm{supp}(\gamma)$ in the metric d_{GH}. However, since $d_{GH} \le d_H$, we see that also $(\mathrm{supp}(\gamma_i))_{i=1}^\infty$ converges to $\mathbb{I}^2$ in the metric d_{GH} and this provides a contradiction.

4. Remarks. One can define some modifications of the Gromov-Frechét distance.

1) One can consider also the non-monotonic version of the Frechét distance (see, e.g. [4]). Substituting this distance in the formula for d_{GF} one obtains the non-monotonic Gromov-Frechét distance.

2) There exists also a version of the Fréchet distance for closed curves (i.e., homeomorphic images of S^1). Similarly, one can define the Gromov-Frechét distance between the metric closed curves.

3) The notion of Fréchet distance for parameterized surfaces is considered in [3]. Let $\alpha\colon \mathbb{I}^2 \to X$ be a parameterized surface in a metric space X. For any $s \in \mathbb{I}$, let the map $\alpha_s\colon \mathbb{I} \to X$ be defined as follows: $\alpha_s(t) = \alpha(t,s)$, $t \in \mathbb{I}$. We denote by $\bar\alpha$ the map $s \mapsto \alpha_s\colon \mathbb{I} \to \mathrm{Curve}(X)$, where $\mathrm{Curve}(X)$ stands for the metric space of curves in X endowed with the

Frechét metric. Now, the Fréchet distance between parameterized surfaces is defined by the formula: $D_F(\alpha, \beta) = d_F(\bar{\alpha}, \bar{\beta})$. On the base of this distance, one can define the corresponding Gromov-Fréchet distance.

4) Discrete Frechét distance is defined in [1, 14, 15]. To define the discrete Gromov-Frechét distance, we assume that the curves under consideration are polygonal in the sense that they are isometric to polygonal curves in normed spaces.

REFERENCES

1. P.K. Agarwal, R.B. Avraham, H. Kaplan, M. Sharir, *Computing the discrete Fréchet distance in subquadratic time*, SIAM J. Comput., **43** (2014), №2, 429–449.
2. J. Alber, R. Niedermeier, *On multidimensional curves with Hilbert property*, Theory of Computing Systems, **33** (2000), №4, 295–312.
3. H. Alt, M. Buchin, *Can we compute the similarity between surfaces?* Discrete Comput. Geom., **43** (2010), №1, 78–99.
4. Alt H., Godau M. *Computing the Fréchet distance between two polygonal curves*, Int. J. of Computational Geometry and Applications, **5** (1995), 75–91.
5. D. Burago, Yu. Burago, S. Ivanov, A course in metric geometry, AMS GSM 33, 2001.
6. D.A. Edwards, The structure of superspace, published in: Studies in Topology, Academic Press, 1975, 121–133.
7. M. Gromov, Groups of polynomial growth and expanding maps, Publications mathematiques I.H.E.S., 53, 1981.
8. M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Mathematico di Palermo, **22** (1906), 1–74.
9. G.M. Ewing, Calculus of variations with applications, Dover Publ., 1985.
10. G. Rote, *Computing the Fréchet distance between piecewise smooth curves*, Computational Geometry, **37** (2007), 162–174.
11. M.I. Schlesinger, E.V. Vodolazskiy, V.M. Yakovenko, *Fréchet similarity of closed polygonal curves*, International Journal of Computational Geometry & Applications, **26**, (2016), №1, 53–66.
12. C. Kuratowski, Quelques problèmes concernant les espaces métriques non-separables, Fundamenta Mathematicae, **25** (1935), 534–545.
13. A. Mosig, M. Clausen, *Approximately matching polygonal curves with respect to the Fréchet distance*, Computational Geometry, **30** (2005), 113–127.
14. T. Eiter, H. Mannila, Computing discrete Frechet distance, Technical Report CDTR 94/64, Christian Doppler Laboratory for Expert Systems, TU Vienna, Austria, 1994.
15. Hee-Kap Ahn, *Christian Knauer, Marc Scherfenberg, Lena Schlipf, Antoine Vigneron, Computing the discrete Fréchet distance with imprecise input*, International Journal of Computational Geometry & Applications, **22** (2012), №1, 27–44.
16. K. Buchin, M. Buchin, C. Wenk, *Computing the Fréchet distance between simple polygons*, Computational Geometry, **41** (2008), 2–20.
17. A.F. Cook IV, A. Driemel, J. Sherette, C. Wenk, *Computing the Frechet distance between folded polygons*, Computational Geometry, **50** (2015), 1–16.
18. C. Villani, Topics in optimal transportation, Amer. Math. Soc., Providence, Rhode Island, 2000.

Ternopil National Economic University, Ternopil, Ukraine
ob@tneu.edu.ua

Department of Mechanics and Mathematics
Lviv National University, Lviv, Ukraine
zarichnyi@yahoo.com

Received 27.08.2018