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EXISTENCE OF SOLITARY TRAVELING WAVES IN FERMI–PASTA–ULAM SYSTEM ON 2D–LATTICE

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The article deals with the Fermi–Pasta–Ulam system that describes an infinite system of particles on 2D–lattice. The main result concerns the existence of solitary traveling wave solutions. By means of critical point theory, we obtain sufficient conditions for the existence of such solutions.

1. Introduction. In the present paper we study the Fermi–Pasta–Ulam system that describes the dynamics of an infinite system of nonlinearly coupled particles on two dimensional lattice. Let $q_{n,m}(t)$ be a generalized coordinate of the (n,m)-th particle at time t. It is assumed that each particle interacts nonlinearly with its four nearest neighbors. The equation of motion of the system considered is of the form

$$\ddot{q}_{n,m} = U'(q_{n+1,m} - q_{n,m}) - U'(q_{n,m} - q_{n-1,m}) +$$
$$+ U'(q_{n,m+1} - q_{n,m}) - U'(q_{n,m} - q_{n,m-1}), (n,m) \in \mathbb{Z}^2, \tag{1}$$

where U is the potential of interaction. Equations (1) form an infinite system of ordinary differential equations.

Systems of such type are of interest in view of numerous applications in physics [1], [17], [18]. A comprehensive presentation of existing results on traveling waves for 1D Fermi-Pasta-Ulam lattices is given in [23]. The existence of periodic traveling waves in Fermi–Pasta–Ulam system on 2D–lattice is studied in [3].

On the other hand, some results on chains of oscillators are known in the literature. In particular, in [21] certain results of such type are obtained by means of bifurcation theory, while in [9] and [13] the existence of periodic and solitary traveling waves is studied by means of critical point theory. In papers [4], [15], [19], [20] traveling waves for infinite systems of linearly coupled oscillators on 2D–lattice are studied, while [8] and [24] deal with periodic in time solutions for such systems. Paper [22] is devoted to periodic and homoclinic traveling waves for infinite one-dimensional chain of nonlinearly coupled nonlinear particles. In [6] it is obtained a result on the existence of subsonic periodic traveling waves for the system of nonlinearly coupled nonlinear oscillators on 2D–lattice, while in [7] supersonic periodic traveling waves for such systems are studied.

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Papers [11], [14], [16] are devoted to the well-posedness of initial value problem for infinite systems of linearly coupled nonlinear oscillators on 2D–lattice.

In paper [2] it is obtained a result on the existence of heteroclinic traveling waves for the discrete sine–Gordon equation with linear interaction on 2D–lattice. Paper [10] is devoted to the existence of periodic traveling waves for the discrete sine–Gordon equation with nonlinear interaction on 2D–lattice, while in [12] it is obtained a result on existence of heteroclinic traveling waves for such equation.

In the present paper we obtain, by means of critical point theory, a result on the existence of solitary traveling waves for the Fermi–Pasta–Ulam system on 2D–lattice. This paper extends some of results obtained in [23].

2. Statement of a problem. A traveling wave solution of Eq. (1) is a function of the form

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct)\,,$$

where the profile function $u(s)$ of the wave, or simply profile, satisfies the equation

$$c^2u''(s) = U'(u(s+\cos\varphi) - u(s)) - U'(u(s) - u(s-\cos\varphi)) +$$
$$+U'(u(s+\sin\varphi) - u(s)) - U'(u(s) - u(s-\sin\varphi)). \tag{2}$$

The constant $c \neq 0$ is called the speed of the wave. Without loss of generality, we assume that $c > 0$ because otherwise we can replace φ by $\varphi + \pi$.

We consider two types of solutions:

— periodic traveling waves;

— solitary traveling waves.

In the first case profile satisfies periodic condition (see [23])

$$u'(s+2k) = u'(s), \quad s \in \mathbb{R}, \tag{3}$$

and in the second case profile satisfies boundary condition

$$\lim_{s\to\pm\infty} u'(s) = u'(\pm\infty) = 0. \tag{4}$$

In what follows, a solution of Eq. (2) is understood as a function $u(s)$ from the space $C^2(\mathbb{R})$ satisfying Eq. (2) for all $s \in \mathbb{R}$.

3. Periodic waves. The following results are obtained in [3].

Let E_k be the Hilbert space defined by

$$E_k = \{u \in H^1_{loc}(\mathbb{R}) : u'(s+2k) = u'(s), u(0) = 0\}$$

with the scalar product

$$(u,v)_k = \int_{-k}^{k} u'(s)v'(s)ds$$

and corresponding norm $\|u\|_k = (u,u)^{\frac{1}{2}}$. E_k is 1-codimensional subspace of the Hilbert space

$$\widetilde{E}_k = \{u \in H^1_{loc}(\mathbb{R}) : u'(s+2k) = u'(s)\}$$

with

$$\int_{-k}^{k} u'(s)v'(s)ds + u(0)v(0)$$

as the scalar product.

On $\widetilde{E}_k$ we define operators $\widetilde{E}_k \to \widetilde{E}_k$:

$$(Au)(s) := u(s+\cos\varphi) - u(s) = \int_s^{s+\cos\varphi} u'(\tau)d\tau,$$

$$(Bu)(s) := u(s+\sin\varphi) - u(s) = \int_s^{s+\sin\varphi} u'(\tau)d\tau.$$

We introduce the functional

$$J_k(u) := \int_{-k}^{k} \Big\{\frac{c^2}{2}|u'(s)|^2 - U(Au(s)) - U(Bu(s))\Big\}ds$$

defined on the space U_k.

We assume that

(i) *function* $U(r)$ *is* C^1 *on* $\mathbb{R}$ *and* $U(0) = U'(0) = 0$.

Then any critical point of J_k is C^2–solution of (2) satisfying (3) (see [3], Lemma 3).

Now we impose the following conditions:

(i') $U(r) = \frac{c_0^2}{2} + V(r)$, *where* $c_0 \geq 0, V \in C^1(\mathbb{R})$, $V(0) = V'(0) = 0$ *and* $V'(r) = o(|r|)$ *as* $r \to 0$,

and either

(ii^+) *there exist* $r_0 > 0$ *and* $\mu > 2$ *such that* $V(r_0) > 0$ *and for* $r \geq 0$

$$0 \leq \mu V(r) \leq rV'(r),$$

or

(ii^-) *there exist* $r_0 < 0$ *and* $\mu > 2$ *such that* $V(r_0) > 0$ *and for* $r \leq 0$

$$0 \leq \mu V(r) \leq rV'(r).$$

The following theorem ([3], Theorem 1) is obtained with the aid of the mountain pass theorem.

Theorem 1. *Assume* (i') *and* $k \geq 1$. *Then*

(a) *under assumption* (ii^+) *for every* $c > c_0$ *equation (2) has a nontrivial nondecreasing solution* $u_k \in E_k$;

(b) *under assumption* (ii^-) *for every* $c > c_0$ *equation (2) has a nontrivial nonincreasing solution* $u_k \in E_k$.

Moreover, in both cases there exist $\delta > 0$ *and* $M > 0$, *independent of* k, *such that the corresponding critical value* $J_k(u_k)$ *satisfies*

$$0 < \delta \leq J_k(u_k) \leq M.$$

We note that from the point of view of physics, increasing waves are *expansion waves*, while decreasing waves are *compression waves*. The next theorem ([3], Theorem 4) concerns the existence of not necessary monotone waves.

Theorem 2. *Assume*

(i″) $U(r) = \frac{a}{2}r^2 + V(r)$, *where* $V \in C^1(\mathbb{R})$, $V(0) = V'(0) = 0$ *and* $V'(r) = o(|r|)$ *as* $r \to 0$;

(ii′) *there exist* $r_0 \in \mathbb{R}$ *and* $\mu > 2$ *such that* $V(r_0) > 0$ *and*

$$\mu V(r) \le rV'(r), r \in \mathbb{R}.$$

Let $c^2 > \max\{a, 0\}$. *Then for every* $k \ge 1$ *equation (2) has a nontrivial solution* u_k *satisfying (3). Moreover, there exist* $\delta > 0$ *and* $M > 0$, *independent of* k, *such that the corresponding critical value* $J_k(u_k)$ *satisfies*

$$0 < \delta \le J_k(u_k) \le M.$$

4. Solitary waves. In a sense, the case of solitary waves is a limit case of the periodic waves. Therefore, solitary waves will be constructed by considering critical points of the functional J_k and then passing to the limit as $k \to \infty$.

4.1. Variational setting. Let E be the Hilbert space defined by

$$E = \{u \in H^1_{loc}(\mathbb{R}) : u' \in Ł^2(\mathbb{R}), u(0) = 0\}$$

with the scalar product

$$(u, v) = \int_{-\infty}^{+\infty} u'(s)v'(s)ds$$

and corresponding norm $\|u\| = (u, u)^{\frac{1}{2}}$. Note that the condition $u' \in Ł^2(\mathbb{R})$ in the definition of E corresponds to the boundary condition (4) and the condition $u(0) = 0$ is meaningful because every element of $H^1_{loc}(\mathbb{R})$ is a continuous function. By $\|\cdot\|_*$ we denote the dual norm on E^*, the dual space to E.

Actually, E is 1-codimensional subspace of the Hilbert space

$$\widetilde{E} = \{u \in H^1_{loc}(\mathbb{R}) : u' \in Ł^2(\mathbb{R})\}$$

with

$$\int_{-\infty}^{+\infty} u'(s)v'(s)ds + u(0)v(0)$$

as the scalar product.

On $\widetilde{E}$ we define operators $\widetilde{E} \to \widetilde{E}$:

$$(Au)(s) := u(s + \cos\varphi) - u(s) = \int_s^{s+\cos\varphi} u'(\tau)d\tau,$$

$$(Bu)(s) := u(s + \sin\varphi) - u(s) = \int_s^{s+\sin\varphi} u'(\tau)d\tau.$$

Lemma 1. *The operators* A *and* B *are linear bounded operators from* E *to* $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *satisfying*

$$\|Au\|_{L^\infty(\mathbb{R})} \le |\cos\varphi| \cdot \|u\|, \|Au\|_{L^2(\mathbb{R})} \le |\cos\varphi| \cdot \|u\|,$$
$$\|Bu\|_{L^\infty(\mathbb{R})} \le |\sin\varphi| \cdot \|u\|, \|Bu\|_{L^2(\mathbb{R})} \le |\sin\varphi| \cdot \|u\|,$$

and

$$\lim_{t\to\pm\infty} (Au)(t) = (Au)(\pm\infty) = 0,$$
$$\lim_{t\to\pm\infty} (Bu)(t) = (Bu)(\pm\infty) = 0.$$

Proof. Denote by

$$\widehat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi t} u(t)dt$$

the Fourier transform of a function u. Then

$$(\widehat{Au})(\xi) = (e^{i\xi \cos\varphi} - 1)\widehat{u}(\xi),$$

$$|(\widehat{Au})(\xi)|^2 = 2(1 - \cos(\xi\cos\varphi))|\widehat{u}(\xi)|^2 = 4\sin^2 \frac{\xi\cos\varphi}{2} |\widehat{u}(\xi)|^2 \le \xi^2 \cos^2\varphi |\widehat{u}(\xi)|^2.$$

Hence, by the Parseval identity, we have

$$\|Au\|_{L^2(\mathbb{R})} \le |\cos\varphi| \cdot \|u\|.$$

Other inequalities are proved similarly.

Since, by the Cauchy-Bunyakovsky-Schwartz inequality,

$$|(Au)(s)| \le \int_s^{s+\cos\varphi} |u'(t)|dt \le \Big(\int_s^{s+\cos\varphi} |u'(t)|^2 dt\Big)^{1/2} \Big(\int_s^{s+\cos\varphi} dt\Big)^{1/2} =$$

$$= |\cos\varphi| \Big(\int_s^{s+\cos\varphi} |u'(t)|^2 dt\Big)^{1/2},$$

$$|(Bu)(s)| \le \int_s^{s+\sin\varphi} |u'(t)|dt \le \Big(\int_s^{s+\sin\varphi} |u'(t)|^2 dt)^{1/2} \Big(\int_s^{s+\sin\varphi} dt\Big)^{1/2} =$$

$$= |\sin\varphi| \Big(\int_s^{s+\sin\varphi} |u'(t)|^2 dt\Big)^{1/2},$$

and $u' \in L^2(\mathbb{R})$, then $(Au)(\pm\infty) = 0, (Bu)(\pm\infty) = 0$. □

Note that the operator

$$(Pu)(s) := \int_0^s |u'(t)|dt$$

acts continuously from E into itself.

We assume that

(i''') *function* $U(r)$ *is* C^1 *on* $\mathbb{R}, U(0) = U'(0) = 0$ *and for some* $R > 0$

$$\sup_{|u|\le R} \left| \frac{U'(r)}{r} \right| < +\infty.$$

We remark that the assumption (i''') is slightly stronger than (i) and is slightly weaker than (i'').

On E we consider the functional

$$J(u) := \int_{-\infty}^{+\infty} \Big\{ \frac{c^2}{2} |u'(s)|^2 - U(Au(s)) - U(Bu(s)) \Big\} ds.$$

Lemma 2. *Under assumption* (i''') *the functional* J *is* C^1 *on* E *and*

$$\langle J'(u), h\rangle = \int_{-\infty}^{+\infty} \{c^2 u'(s)h'(s) - U'(Au(s))Ah(s) - U'(Bu(s))Bh(s)\} ds \tag{5}$$

for $u, h \in E$.

Proof. The functional J can be expressed in the form

$$J(u) = \frac{c^2}{2}(u,u) - \Phi(u),$$

where

$$\Phi(u) := \int_{-\infty}^{+\infty} \{U(Au(s)) + U(Bu(s))\} ds.$$

Thus, we have to consider only the functional Φ because for the quadratic part the statement is obvious.

Assumption (i''') implies that for every $R > 0$

$$\sup_{|u| \le R} \left| \frac{U(r)}{r^2} \right| < +\infty.$$

Then, by Lemma 1, for every $u \in E$ the functions Au and Bu are continuous and there exist constants $C_1, C_2 > 0$ such that

$$|U(Au(t)| \le C_1 |Au(t)|^2, |U(Bu(t)| \le C_2 |Bu(t)|^2.$$

This implies that $\Phi < \infty$. A direct calculation shows that the Gateaux derivative of Φ exists and is given by

$$\langle \Phi'(u), h \rangle = \int_{-\infty}^{+\infty} \{U'(Au(s))Ah(s) + U'(Bu(s))Bh(s)\} ds.$$

Now we prove that Φ' is continuous. Let $\|h\| \le 1$ and $u_n \to u$ in E. Then $Au_n \to Au$ in E, also in $L^2(\mathbb{R})$ and in $L^\infty(\mathbb{R})$. Moreover, by Lemma 1, we have

$$|\langle \Phi'(u_n) - \Phi'(u), h \rangle| \le$$

$$\le \|Ah\|_{L^2(\mathbb{R})} \cdot \|U'(Au_n) - U'(Au)\|_{L^2(\mathbb{R})} + \|Bh\|_{L^2(\mathbb{R})} \cdot \|U'(Bu_n) - U'(Bu)\|_{L^2(\mathbb{R})} \le$$
$$\le \|U'(Au_n) - U'(Au)\|_{L^2(\mathbb{R})} + \|U'(Bu_n) - U'(Bu)\|_{L^2(\mathbb{R})}.$$

Due to assumption (i'''), there exists $C > 0$ such that

$$|U'(r)| \le C|r|, |r| \le R,$$

where $R = \max\{\|Au\|_{L^\infty(\mathbb{R})}, \|Au_n\|_{L^\infty(\mathbb{R})}, \|Bu\|_{L^\infty(\mathbb{R})}, \|Bu_n\|_{L^\infty(\mathbb{R})}\}$. Then

$$\|U'(Au_n) - U'(Au)\|^2_{L^2(\mathbb{R})} \le \int_{-a}^{a} |U'(Au_n) - U'(Au)|^2 dt + \int_{|t| \ge a} \left[|U'(Au_n)|^2 + |U'(Au)|^2 \right] dt \le$$

$$\le \int_{-a}^{a} |U'(Au_n) - U'(Au)|^2 dt + C \int_{|t| \ge a} \left[|Au_n|^2 + |Au|^2 \right] dt.$$

Since $Au_n \to Au$ in $L^2(\mathbb{R})$, for every $\varepsilon > 0$ there exists $a > 0$ independent of n such that the second integral above is less than $\varepsilon > 0$. Besides $Au_n \to Au$ uniformly on $[-a, a]$, then the first integral above is less than $\varepsilon > 0$ for n is large enough. Thus, for such n we have

$$\|U'(Au_n) - U'(Au)\|^2_{L^2(\mathbb{R})} \le \varepsilon + C\varepsilon.$$

Similarly, if n is large enough then

$$\|U'(Bu_n) - U'(Bu)\|^2_{L^2(\mathbb{R})} \le \varepsilon + C\varepsilon.$$

And this implies that

$$|\langle \Phi'(u_n) - \Phi'(u), h\rangle| \to 0$$

as $n \to \infty$. □

Lemma 3. *Under assumption (i''') any critical point of the functional J is a C^2–solution of Eq. (2) satisfying (4).*

Proof. Let $g(s)$ is a function from $C_0^\infty(\mathbb{R})$ and $u \in E$ is a critical point of J. Then $h(s) = g(s) - g(0) \in E$ and

$$0 = \langle J'(u), h\rangle = \int_{-\infty}^{+\infty} \{c^2 u'(s)h'(s) - U'(Au(s))Ah(s) - U'(Bu(s))Bh(s)\}ds =$$

$$= \int_{-\infty}^{+\infty} \{c^2 u'(s)g'(s) - U'(u(s+\cos\varphi) - u(s))(g(s+\cos\varphi) - g(s)) -$$

$$-U'(u(s+\sin\varphi) - u(s))(g(s+\sin\varphi) - g(s))\}ds = \int_{-\infty}^{+\infty} \{c^2 u'(s)g'(s) -$$

$$-[U'(u(s) - u(s-\cos\varphi)) - U'(u(s+\cos\varphi) - u(s))]g(s) -$$

$$-[U'(u(s) - u(s-\sin\varphi)) - U'(u(s+\sin\varphi) - u(s))]g(s)\}ds =$$

$$= \int_{-\infty}^{+\infty} \{-c^2 u''(s)g(s) - [U'(u(s) - u(s-\cos\varphi)) - U'(u(s+\cos\varphi) - u(s))]g(s) -$$

$$-[U'(u(s) - u(s-\sin\varphi)) - U'(u(s+\sin\varphi) - u(s))]g(s)\}ds =$$

$$= \int_{-\infty}^{+\infty} \{-c^2 u''(s) + U'(u(s+\cos\varphi) - u(s)) - U'(u(s) - u(s-\cos\varphi)) +$$

$$+U'(u(s+\sin\varphi) - u(s)) - U'(u(s) - u(s-\sin\varphi))\}g(s)ds.$$

This implies that u is a weak solution of Eq. (2). By the embedding theorem, $u \in C_b(\mathbb{R})$. Since $u(s)$ and $U'(r)$ are continuous, the right-hand side of (2) is also continuous. Hence, $u''(s)$ is continuous, i.e. $u \in C^2(\mathbb{R})$ is a classical solution of Eq. (2) satisfying (4). □

4.2. Main results. The functional J satisfies a part of conditions of the mountain pass theorem. However, the Palais-Smale condition for this functional is not satisfied. Therefore, in this case, critical points of J will be constructed in a different way, namely, by passing to the limit as $k \to \infty$ in the critical points of J_k.

To get the main results we need the following lemmas.

Lemma 4.

(*a*) *Assume (i''), (ii') and $c^2 > \max(a, 0)$. Then there exists $\varepsilon > 0$ independent of k such that for any nontrivial critical points $u_k \in E_k$ of the functional J_k and $u \in E$ of the functional J*

$$\varepsilon \le (c^2 - a)\|u_k\|_k^2 \le \frac{2\mu}{\mu - 2} J_k(u_k), \quad \varepsilon \le (c^2 - a)\|u\|^2 \le \frac{2\mu}{\mu - 2} J(u).$$

(*b*) *Assume* (i'), (ii^+) *(resp.,* (ii^-)*) and* $c > c_0$*. Then the statement* (a) *holds for nontrivial critical points* $u_k \in PE_k$ *(resp.,* $u_k \in -PE_k$*) of* J_k *and* $u \in PE$ *(resp.,* $u \in -PE$*) of* J *with* $a = c_0^2$*.*

Proof. We consider the functional J_k (the case of J is similar). Let $u_k \in E_k$ be a nontrivial critical point of the functional J_k. Then, by condition (ii'), we have

$$J_k(u_k) = J_k(u_k) - \frac{1}{2}\langle J_k'(u_k), u_k\rangle =$$

$$= \int_{-k}^{k} \left[\left(\frac{1}{2}V'(Au_k(t))Au_k(t) - V(Au_k(t))\right) + \left(\frac{1}{2}V'(Bu_k(t))Bu_k(t) - V(Bu_k(t))\right)\right] dt \geq$$

$$\geq \frac{\mu-2}{2}\int_{-k}^{k} [V(Au_k(t)) + V(Bu_k(t))]\, dt.$$

Thus,

$$(c^2 - a)\|u_k\|_k^2 = 2J_k(u_k) + 2\int_{-k}^{k} [V(Au_k(t)) + V(Bu_k(t))]\, dt \leq \frac{2\mu}{\mu-2}J_k(u_k).$$

To obtain the lower bound, we assume on the contrary that there exists a sequence of nontrivial critical points $u_{k_n} \in E_{k_n}$ such that $\|u_{k_n}\|_{k_n} \to 0$ as $n \to \infty$ (it is not necessary that $k_n \to \infty$). Then, by Lemma 1 from [3] (similar lemma to Lemma 1), $\|Au_{k_n}\|_{L^\infty(-k_n,k_n)} \to 0$, $\|Bu_{k_n}\|_{L^\infty(-k_n,k_n)} \to 0$, and by assumption ($i'$)

$$|V'(Au_{k_n})Au_{k_n} + V'(Bu_{k_n})Bu_{k_n}| \leq \varepsilon_n \left(|Au_{k_n}|^2 + |Bu_{k_n}|^2\right),$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Since $\langle J_k'(u_{k_n}), u_{k_n}\rangle = 0$, we have

$$c^2\|u_{k_n}\|_{k_n}^2 =$$

$$= \int_{-k_n}^{k_n} \left[a\left(|Au_{k_n}(t)|^2 + |Bu_{k_n}(t)|^2\right) + V'(Au_{k_n}(t))Au_{k_n}(t) + V'(Bu_{k_n}(t))Bu_{k_n}(t)\right] \leq$$

$$\leq (a + \varepsilon_n)\|u_{k_n}\|_{k_n}^2,$$

i.e. $c^2 - a - \varepsilon_n \leq 0$. But $c^2 > a$ and we got a contradiction that proves statement (a) of the lemma.

Statement (b) follows from (a) with $a = c_0^2$. It is enough to modify the potential $V(r)$ so that the new potential coincides with $V(r)$ for $r > 0$ (resp., $r < 0$) and vanishes for $r < 0$ (resp., $r > 0$). □

We note that this lemma is still valid for nonzero elements $u_k \in E_k$ (resp., $u \in E$) satisfying $\langle J_k(u_k), u_k\rangle = 0$ (resp., $\langle J(u), u\rangle = 0$).

Lemma 5. *Assume* (i'') *and* $c^2 > \max(a, 0)$*. Let* $u_k \in E_k$ *be a sequence such that* $\|u_k\|_k$ *is bounded and* $\|J_k'(u_k)\|_{k,*} \to 0$ *as* $k \to \infty$*. Then also* $\|u_k\|_k \to 0$ *as* $k \to \infty$*, or for any* $r > 0$ *there exist* $\theta > 0$*, a subsequence of* u_k *(still denoted by* u_k*) and* $\eta_k \in \mathbb{R}$ *such that*

$$\int_{\eta_k - r}^{\eta_k + r} \left[|Au_k(t)|^2 + |Bu_k(t)|^2\right] dt \geq \theta.$$

Proof. Let

$$\lim_{k\to\infty}\sup_{\eta\in\mathbb{R}}\int_{\eta-r}^{\eta+r}\left[|Au_k(t)|^2+|Bu_k(t)|^2\right]dt=0$$

and $g_k\in C_0^\infty(\mathbb{R})$ such that

$$0\le g_k(t)\le 1,$$
$$g_k(t)=1 \quad if \quad |t|\le k,$$
$$g_k(t)=0 \quad if \quad |t|\ge k+1,$$
$$|g_k'(t)|\le C,$$

where $C>0$ is independent of k. We set

$$f_k(t)=g_k(t)(Au_k(t)+Bu_k(t)).$$

Obviously, $f_k\in H^1(\mathbb{R})$. It is readily to verify that $\|f_k\|_{H^1(\mathbb{R})}$ is bounded and

$$\lim_{k\to\infty}\sup_{\eta\in\mathbb{R}}\int_{\eta-r}^{\eta+r}|f_k(t)|^2dt=0.$$

Then by Lemma B.2 ([23]) $\|f_k\|_{L^p(\mathbb{R})}\to 0$ for all $p>2$. Since

$$\|Au_k\|_{L^p(-k,k)}+\|Bu_k\|_{L^p(-k,k)}\le C_1\|f_k\|_{L^p(\mathbb{R})}$$

with some $C_1>0$, we have

$$\|Au_k\|_{L^p(-k,k)}+\|Bu_k\|_{L^p(-k,k)}\to 0$$

for all $p>2$.

Let $\varepsilon_k:=\|J_k'(u_k)\|_{k,*}\to 0$ as $k\to\infty$. Then

$$\langle J_k'(u_k),u_k\rangle=$$

$$=\int_{-k}^{k}\left[c^2|u_k'(t)|^2-a\left(|Au_k(t)|^2+|Bu_k(t)|^2\right)-V'(Au_k(t))Au_k(t)-V'(Bu_k(t))Bu_k(t)\right]dt\le$$

$$\le\varepsilon_k\|u_k\|_k.$$

Due to Lemma 1 from [3], we have

$$\|Au_k\|_{L^\infty(-k,k)}+\|Bu_k\|_{L^\infty(-k,k)}\le C.$$

Fix any $p>2$. Then, by (i''), for every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|V'(r)r|\le\varepsilon r^2+C_\varepsilon|r|^p,\quad |r|\le C.$$

Thus, we have

$$c^2\|u_k\|_k^2\le$$

$$\le\int_{-k}^{k}\left[a\left(|Au_k(t)|^2+|Bu_k(t)|^2\right)+V'(Au_k(t))Au_k(t)+V'(Bu_k(t))Bu_k(t)\right]dt+\varepsilon_k\|u_k\|_k\le$$

$$\leq \int_{-k}^{k} \left[(a+\varepsilon)\left(|Au_k(t)|^2 + |Bu_k(t)|^2\right) + C_\varepsilon\left(|Au_k(t)|^p + |Bu_k(t)|^p\right)\right] dt + \varepsilon_k \|u_k\|_k =$$

$$= (a+\varepsilon)\left(\|Au_k(t)\|^2_{L^2(-k,k)} + \|Bu_k(t)\|^2_{L^2(-k,k)}\right) +$$

$$+ C_\varepsilon\left(\|Au_k(t)\|^p_{L^p(-k,k)} + \|Bu_k(t)\|^p_{L^p(-k,k)}\right) + \varepsilon_k \|u_k\|_k.$$

For $a < 0$ we can choose $\varepsilon > 0$ small enough so that $a + \varepsilon < 0$. Then

$$c^2\|u_k\|_k^2 \leq C_\varepsilon\left(\|Au_k(t)\|^p_{L^p(-k,k)} + \|Bu_k(t)\|^p_{L^p(-k,k)}\right) + \varepsilon_k\|u_k\|_k,$$

which implies that $\|u_k\|_k \to 0$ as $k \to \infty$.

For $a > 0$, by Lemma 1 from [3], we have

$$(c^2 - a - \varepsilon)\|u_k\|_k^2 \leq C_\varepsilon\left(\|Au_k(t)\|^p_{L^p(-k,k)} + \|Bu_k(t)\|^p_{L^p(-k,k)}\right) + \varepsilon_k\|u_k\|_k.$$

Then, since $c^2 > a$, we can choose $\varepsilon > 0$ small enough so that $c^2 - a - \varepsilon > 0$. And also $\|u_k\|_k \to 0$. □

Lemma 6.

(a) *Assume* (i''), (ii') *and* $c^2 > \max(a, 0)$. *Let* $u_k \in E_k$ *be a sequence of nontrivial critical points of the functional* J_k *such that the critical values* $J_k(u_k)$ *are uniformly bounded. Then there exist nontrivial critical point* $u \in E$ *of the functional* J *and a sequence* $\eta_k \in \mathbb{R}$ *such that a subsequence of* $u_k(t+\eta_k) - u_k(\eta_k)$ *converges to* u *uniformly on compact intervals together with first and second derivatives.*

(b) *Assume* (i'), (ii^+) *(resp.,* (ii^-)*) and* $c > c_0$. *Then the statement* (a) *holds for nontrivial critical points* $u_k \in PE_k$ *(resp.,* $u_k \in -PE_k$*) of* J_k *and* $u \in PE$ *(resp.,* $u \in -PE$*) of* J *with* $a = c_0^2$.

Proof. Due to Lemma 4, $\|u_k\|_k \nrightarrow 0$. Then, by Lemma 5, for any $r > 0$ there exist $\theta > 0$, a subsequence of u_k (still denoted by u_k) and $\eta_k \in \mathbb{R}$ such that

$$\int_{\eta_k - r}^{\eta_k + r} \left[|Au_k(t)|^2 + |Bu_k(t)|^2\right] dt \geq \theta. \tag{6}$$

We set

$$v_k := u_k(t+\eta_k) - u_k(\eta_k).$$

Then $\|v_k\|_k = \|u_k\|_k$, $J_k(v_k) = J_k(u_k)$ and $J_k'(v_k) = J_k'(u_k)$. Moreover, since $\|v_k\|_k$ is bounded, there exists a subsequence (still denoted by v_k) that converges weakly to $u \in H^1_{loc}(\mathbb{R})$ (i.e. weakly in $H^1(a,b)$ for any finite interval (a,b)).

We show that $u \in \tilde{E}$. Indeed, $v_k' \to u'$ weakly in L^2_{loc}. Hence, for every $a < b$

$$\int_a^b |u'(t)|^2 dt \leq \liminf_{k\to\infty} \int_a^b |v_k'(t)|^2 dt \leq \liminf_{k\to\infty} \|v_k'\|_k^2 \leq C.$$

Thus, passing to the limit as $a \to -\infty$ and $b \to +\infty$, we have

$$\int_{-\infty}^{+\infty} |u'(t)|^2 dt \leq C < +\infty,$$

i.e. $u \in \tilde{E}$.

Now we check that $u \neq 0$. By the compactness of Sobolev embedding, $Av_k \to Au$ and $Bv_k \to Bu$ strongly in $L^\infty_{loc}(\mathbb{R})$ (i.e. uniformly on finite intervals) and in $L^2_{loc}(\mathbb{R})$. Then, by (6), we obtain

$$\int_{-r}^{r} \left[|Au(t)|^2 + |Bu(t)|^2\right] dt \geq \theta > 0.$$

This implies that $u \neq 0$.

We note that $U'(Av_k) \to U'(Au)$ and $U'(Av_k) \to U'(Au)$ in $L^\infty_{loc}(\mathbb{R})$, because $Av_k \to Au$ and $Bv_k \to Bu$ in $L^\infty_{loc}(\mathbb{R})$.

Let $g \in C^\infty(\mathbb{R}), g(0) = 0$ and $g' \in C_0^\infty(\mathbb{R})$. Then, for k is large enough, $S := \operatorname{supp} Ag \cup \operatorname{supp} Bg \subset [-k, k]$. And for such k, let $g_k \in E_k$ be the primitive function of the $2k$-periodic extension of $g'|_{[-k,k]}$. Then

$$\langle J'(u), g\rangle = \int_{-\infty}^{+\infty} \left[c^2 u'(t)g'(t) - U'(Au(t))Ag(t) - U'(Bu(t))Bg(t)\right] dt =$$

$$= \int_{S} \left[c^2 u'(t)g'(t) - U'(Au(t))Ag(t) - U'(Bu(t))Bg(t)\right] dt =$$

$$= \lim_{k\to\infty} \int_{S} \left[c^2 v_k'(t)g'(t) - U'(Av_k(t))Ag(t) - U'(Bv_k(t))Bg(t)\right] dt =$$

$$= \lim_{k\to\infty} \int_{-k}^{k} \left[c^2 v_k'(t)g'(t) - U'(Av_k(t))Ag(t) - U'(Bv_k(t))Bg(t)\right] dt = 0.$$

Hence, u is a nontrivial solution of Eq. (2).

Finally, the right hand side of Eq. (2) for v_k converges in $L^\infty_{loc}(\mathbb{R})$ to the right hand side of that equation for u. Therefore, $v_k'' \to u''$, hence, $v_k' \to u'$ and $v_k \to u$ in $L^\infty_{loc}(\mathbb{R})$. In particular, $u(0) = 0$ and $u \in E$. And statement (a) is proved.

Statement (b) follows from (a) with $a = c_0^2$. Enough to modify the potential $V(r)$ so that the new potential coincides with $V(r)$ for $r > 0$ (resp., $r < 0$) and vanishes for $r < 0$ (resp., $r > 0$), and note that the limit of a sequence of monotone functions is also a monotone function. □

Note that this lemma is still valid if, instead of a sequence of critical points, we consider a sequence $u_k \in E_k$ such that $\|J_k'(u_k)\|_{k,*} \to 0$ and $J_k(u_k)$ is bounded.

Combining Lemma 6 with Theorems 1 and 2, we obtain the following results.

Theorem 3. *Assume (i'). Then*

(a) under assumption (ii^+) for every $c > c_0$ equation (2) has a nontrivial nondecreasing solution $u \in E$;

(b) under assumption (ii^-) for every $c > c_0$ equation (2) has a nontrivial nonincreasing solution $u \in E$.

Theorem 4. *Assume (i''), (ii') and $c^2 > \max\{a, 0\}$. Then equation (2) has a nontrivial solution u satisfying (4).*

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