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TWO-POINT NONLOCAL PROBLEM FOR A WEAK NONLINEAR DIFFERENTIAL-OPERATOR EQUATION

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We study the solvability of a two-point nonlocal boundary-value problem for an operator-differential equation with weakly nonlinear right-hand side. The proof of theorems is carried out within the Nash-Moser iterative scheme. In this scheme, the important point is a construction of estimates to norms of inverse linearized operators arising at each step of this scheme as well as the related problem of small denominators. The inverse operator for linearized operator is obtained by the method developed in the work of M. Berti, P. Bolle (Duke Math. J., **134** (2), 359–419 (2006)). The problem of small denominators is solved by using the metric approach.

1. Introduction. Problems with nonlocal conditions for partial differential equations represent an important part of the modern theory of differential equations [11, 12, 15]. Such problems are mainly ill posed in the Hadamard sence, and their solvability is connected with the problem of small denominators that arise in the construction of solutions [1, 5, 6, 15].

A specific feature of the present work is the study of a nonlocal boundary-value problem for an operator-differential equation with nonlinear right-hand side. The proof of the solvability of problem will be executed by the Nash-Moser iterative scheme [13, 14].

Using the Nash-Moser iterative scheme Berti and Bolle [2] proved the existence of small amplitude, $2\pi/\omega$–periodic in time solution of completely resonant nonlinears with Dirichlet boundary conditions, for any ω belonging to a Cantor–like set of positive measure. The same problem for the wave equations with nonlinearities of class C^k has been investigated in [3] (required a modified Nash-Moser iterative scheme with interpolation estimates for the inverse linearized operators). In [4] it has been proved the existence of Cantor families of periodic solutions for nonlinear wave equations in higher spatial dimensions with periodic boundary conditions. The proofs are based on a differentiable Nash-Moser iterative scheme.

The nonlocal boundary value problems for a differential equation with nonlinearity and with operator $B = (B_1, \ldots, B_p)$, where $B_j \equiv z_j \frac{\partial}{\partial z_j}$, $j = 1, \ldots, p$, were considered in the papers [7–10]. By using of the Nash–Mozer iteration scheme, conditions of the solvability of the problems in the Sobolev spaces were established for the functions of several complex variables, in the Hörmander–Hilbert spaces and in the scale of spaces of functions, which are Dirichlet-Taylor series with fixed spectrum.

In this work, the crucial point is a construction of estimates of norms of inverse linearized operators in each iteration. The estimation is related to the problem of small denominators, which is solved within metric approach.

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2. Notations and statement of the problem. Let $\mathbf{X}$ be a separable Hilbert space; $A_1, \dots, A_p$ be linear operators that have a common spectral representation, that is, there is a complete orthonormal system of elements $x_k \in \mathbf{X}$, $k \in \mathbb{N}$ such that the equalities

$$A_i x_k = \alpha_{ik} x_k, \quad i = 1, \dots, p, \quad k \in \mathbb{N},$$

hold for some complex numbers α_{ik}.

Write $\alpha_k = (\alpha_{1k}, \dots, \alpha_{pk})$, $\|\alpha_k\|^2 = |\alpha_{1k}|^2 + \dots + |\alpha_{pk}|^2$, and let

$$\|\alpha_k\| > Ck^{\beta_0}, \quad C > 0, \quad \beta_0 > 0.$$

We introduce the function $\zeta_1(x) = \sum_{k \in \mathbb{N}} \left(1 + \|\alpha_k\|^2\right)^{-x}$ defined for $x > 1/(2\beta_0)$, and also the function $\zeta_2(x) = \sum_{m \in \mathbb{Z}} \left(1 + |m|^2\right)^{-x}$ defined for $x > 1/2$.

Denote $\tau(m) = -\ln \mu / T + i\, 2\pi m / T$, where $\ln \mu$ is the principal value of the logarithm. Let us consider the spaces $\mathbf{X}_{d,r}(\Omega)$, $d, r \in \mathbb{R}$, $\Omega \subseteq (\mathbb{N} \times \mathbb{Z})$, being the Hilbert spaces of functions

$$u(t, \Omega) = \sum_{(k,m) \in \Omega} u_{k,m}\, e^{\tau(m)t}\, x_k,$$

equipped with the scalar product

$$\big(u(\cdot, \Omega), v(\cdot, \Omega)\big)_{d,r,\Omega} = \sum_{(k,m) \in \Omega} (1 + \|\alpha_k\|^2)^d (1 + m^2)^r u_{k,m} \bar{v}_{k,m},$$

where

$$v(t, \Omega) = \sum_{(k,m) \in \Omega} v_{k,m}\, e^{\tau(m)t}\, x_k,$$

$\bar{v}_{k,m}$ is complex conjugate with $v_{k,m}$.

The scalar product induces the norm $\|\cdot\|_{d,r,\Omega}$. For simplicity, we denote $\mathbf{X}_{d,r}(\mathbb{N} \times \mathbb{Z}) = \mathbf{X}_{d,r}$, $u(t, \mathbb{N} \times \mathbb{Z}) = u(t)$, $\|\cdot\|_{d,r,\mathbb{N} \times \mathbb{Z}} = \|\cdot\|_{d,r}$.

Note if $u \in \mathbf{X}_{d,r}$, then $\|u\|_{d,r,\Omega} \le \|u\|_{d,r}$ for any d, r and Ω.

Obviously, if $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \varnothing$, then $\mathbf{X}_{d,r}(\Omega) = \mathbf{X}_{d,r}(\Omega_1) \oplus \mathbf{X}_{d,r}(\Omega_2)$, where $\oplus$ means the direct sum; so for any function $u(t, \Omega) \in \mathbf{X}_{d,r}(\Omega)$ $u(t, \Omega) = u(t, \Omega_1) + u(t, \Omega_2)$ holds, where $u(t, \Omega_1) \in \mathbf{X}_{d,r}(\Omega_1)$, $u(t, \Omega_2) \in \mathbf{X}_{d,r}(\Omega_2)$.

The operators $\frac{d}{dt} \colon \mathbf{X}_{d,r}(\Omega) \to \mathbf{X}_{d,r-1}(\Omega)$ and $A_j \colon \mathbf{X}_{d,r}(\Omega) \to \mathbf{X}_{d-1,r}(\Omega)$, $j = 1, \dots, p$, act as follows

$$\frac{du}{dt} = \sum_{(k,m) \in \Omega} \tau(m) u_{k,m}\, e^{\tau(m)t}\, x_k, \qquad A_j u = \sum_{(k,m) \in \Omega} \alpha_{jk} u_{k,m}\, e^{\tau(m)t}\, x_k$$

for any

$$u = u(t, \Omega) = \sum_{(k,m) \in \Omega} u_{k,m}\, e^{\tau(m)t}\, x_k \in \mathbf{X}_{d,r}$$

and $(d, r) \in \mathbb{R}^2$.

Let us consider the problem with nonlocal conditions for the operator-differential equation with constant coefficients and a nonlinear right-hand side

$$L(d_t, A)u \equiv \sum_{|\hat{s}| \le n} a_{\hat{s}} A_1^{s_1} \dots A_p^{s_p} d_t^{s_0} u(t) = \varepsilon f(u), \quad d_t = d/dt, \tag{1}$$

$$\mu d_t^m u\big|_{t=0} - d_t^m u\big|_{t=T} = 0, \qquad m = 0, 1, \ldots, n-1, \tag{2}$$

in the space $\mathbf{X}_{d,r}$, where $a_{\hat{s}}$, ε and μ are complex parameters ($a_{(n,0,\ldots,0)}{=}1$, $\mu \neq 0$), and $\hat{s} = (s_0, s)$, $s = (s_1, \ldots, s_p) \in \mathbb{Z}_+^p$, $|\hat{s}| = s_0 + s_1 + \ldots + s_p$.

Let us consider the eigenvalue problem for the operator L in $\mathbf{X}_{d,r}$, generated by the differential expression $L(d_t, A)$ and the boundary conditions (2) with $\mu \neq 0$:

$$L(d_t, A)u = \lambda u, \qquad \mu d_t^m u\big|_{t=0} - d_t^m u\big|_{t=T} = 0, \qquad m = 0, 1, \ldots, n-1, \tag{3}$$

where λ is a spectral parameter.

Given vector $(k, m) \in \mathbb{N} \times \mathbb{Z}$ we denote by $R_{k,m}$ the set of vectors $(k^*, m^*) \in \mathbb{N} \times \mathbb{Z}$, for which the equality $L(\tau(m^*), \alpha_{k^*}) = L(\tau(m), \alpha_k)$ holds, where $\tau(m) = -\ln \mu/T + i\ 2\pi m/T$.

The eigenvalues of problem (3) are the numbers $\lambda_{k,m} = L(\tau(m), \alpha_k)$, $(k, m) \in \mathbb{N} \times \mathbb{Z}$, the eigenfunctions, corresponding to the eigenvalue $\lambda_{k,m}$, are the functions $x_{k^*} e^{\tau(m^*)t}$, $(k^*, m^*) \in R_{k,m}$ [15].

We look for a solution of problem (1), (2) by using the Nash–Moser iteration scheme in the form of the limit of a some sequence of smooth functions.

For any $N \in \mathbb{N}$ we represent the space $\mathbf{X}_{d,r}$ as the direct sum $\mathbf{X}_{d,r} = \mathbf{W}^{(N)} \oplus \mathbf{W}^{(N)\perp}$, where

$$\mathbf{W}^{(N)} = \Big\{u \in \mathbf{X}_{d,r} \colon u = \sum_{(k,m)\in\Omega^N} u_{k,m} e^{\tau(m)t} x_k\Big\},\ \Omega^N = \{(k, m)\colon 1 + \|\alpha_k\|^2 \leq N, 1 + m^2 \leq N\},$$

$$\mathbf{W}^{(N)\perp} = \Big\{u \in \mathbf{X}_{d,r} \colon u = \sum_{(k,m)\in(\mathbb{N}\times\mathbb{Z})\setminus\Omega^N} u_{k,m}\ e^{\tau(m)t}\ x_k\Big\}.$$

We denote by $\mathrm{P}_N \colon \mathbf{X}_{d,r} \to \mathbf{W}^{(N)}$ and $\mathrm{P}_N^{\perp} \colon \mathbf{X}_{d,r} \to \mathbf{W}^{(N)\perp}$, $N \in \mathbb{N}$, the projection operators in the space $\mathbf{X}_{d,r}$ onto $\mathbf{W}^{(N)}$ and $\mathbf{W}^{(N)\perp}$, respectively, i.e., $\mathbf{W}^{(N)} = \mathrm{P}_N \mathbf{X}_{d,r}$, $\mathbf{W}^{(N)\perp} = \mathrm{P}_N^{\perp} \mathbf{X}_{d,r}$. For any function $u \in \mathbf{X}_{d,r}$ they are defined by the following formulas:

$$\mathrm{P}_N\ u = \sum_{(k,m)\in\Omega^N} u_{k,m}\ e^{\tau(m)t}\ x_k, \qquad \mathrm{P}_N^{\perp}\ u = \sum_{(k,m)\in(\mathbb{N}\times\mathbb{Z})\setminus\Omega^N} u_{k,m}\ e^{\tau(m)t}\ x_k. \tag{4}$$

The definitions of the space $\mathbf{X}_{d,r}$ and the projector P_N imply that, for any $N \in \mathbb{N}$, $d \in \mathbb{R}$, $r \in \mathbb{R}$ the following inequalities hold:

$$\|\mathrm{P}_N u\|_{d+j_1,r+j_2} \leq N^{j_1+j_2} \|u\|_{d,r} \quad \text{for every} \quad u \in \mathbf{X}_{d,r}, \tag{5}$$

$$\|\mathrm{P}_N^{\perp} u\|_{d,r} \leq N^{-j_1-j_2} \|u\|_{d+j_1,r+j_2} \quad \text{for every} \quad u \in \mathbf{X}_{d+j_1,r+j_2}. \tag{6}$$

The existence of a solution of problem (1), (2) is based on properties $(P1)$—$(P5)$ of the coefficients $a_{\hat{s}}$ and the function f on the right-hand side of the equation, which maps, by assumption, the space $\mathbf{X}_{d,r}$ into itself.

Write $\Omega_1 = \{(k, m) \in \mathbb{N} \times \mathbb{Z} \colon |\tau(m)| < \|\alpha_k\|\}$, $\Omega_2 = (\mathbb{N} \times \mathbb{Z}) \setminus \Omega_1$.

Let $l \geq d + 2$, $m \geq r + 2$, $C_0 \geq 0$, $C_1 \geq 0$ and $C_2 \geq 0$ be such that conditions $(P1)$—$(P4)$ are satisfied:

(P1) $f \in \mathbf{C}^2(\mathbf{X}_{d,r}; \mathbf{X}_{d,r})$, in particular, f, $\mathrm{D}_u f$, $\mathrm{D}_u^2 f$ are bounded on the ball $K_1 = \{u \in \mathbf{X}_{d,r} \colon \ \|u\|_{d,r} \leq 1\}$ from the space $\mathbf{X}_{d,r}$.

(P2) For any $d' \in [d, l)$, $r' \in [r, m)$ and function $u \in \mathbf{X}_{d',r'}$, the inequality $\|f(u)\|_{d',r'} \leq C_0(1 + \|u\|_{d',r'})$ holds.

(P3) For any $u \in K_1$ and $h \in \mathbf{X}_{d,r}$ there exists $\bar{\bar{d}} > d + 2\delta$, $\bar{\bar{r}} > r + 2\delta$, where $\delta = \kappa + \alpha - n/2$, $\kappa \geq 1/(4\beta_0)$, $\alpha > 1/4$, such that $D_u f(u) \in \mathbf{C}^1\big(\mathbf{X}_{d,r}; \mathbf{X}_{\bar{\bar{d}},\bar{\bar{r}}}\big)$ and $\|\mathrm{D}_u f(u)[h]\|_{\bar{\bar{d}},\bar{\bar{r}}} \leq C_1 \|h\|_{d,r}$.

(P4) For any $d' \in [d, l-2]$, $r' \in [r, m-2]$, $u \in \mathbf{X}_{d',r'} \cap K_1$ and $h \in \mathbf{X}_{d',r'}$ the inequality

$$\|f(u+h) - f(u) - \mathrm{D}_u f(u)h\|_{d',r'} \leq C_2\big(\|u\|_{d',r'}\|h\|_{d,r}^2 + \|h\|_{d,r}\|h\|_{d',r'}\big)$$

holds.

Property $(P4)$ yields

$$\|f(u+h) - f(u) - \mathrm{D}_u f(u)h\|_{d,r} \leq 2C_2\|h\|_{d,r}^2, \quad u \in K_1, \quad h \in \mathbf{X}_{d,r}.$$

Here is an example of a function $f(u)$ that satisfies the conditions $(P1)$–$(P4)$. For any element $u = \sum\limits_{(k,m)\in\mathbb{N}\times\mathbb{Z}} u_{k,m} e^{\tau(m)t} x_k$ from the $\mathbf{X}_{d,r}$ the element $f(u)$ is given by

$$f(u) = \sum_{k+|m|\leq Q} (f_{k,m} u_{k,m})^\sigma e^{\tau(m)t} x_k,$$

where Q is a natural number, $f_{k,m}$ are arbitrary complex numbers, $\sigma > 2$.

This four properties characterize the behavior of a function f in the ball K_1 of the space $\mathbf{X}_{d,r}$. In order to formulate property $(P5)$ characterizing the coefficients $a_{\hat{s}}$ of equation (1), we assume that they belong to the disk $\mathcal{O}_A = \{z \in \mathbb{C} : |z| < A\}$.

Let us introduce the vectors

$$\vec{\varepsilon} = (\operatorname{Re}\varepsilon, \operatorname{Im}\varepsilon), \quad \vec{a} = (\operatorname{Re} a_{\hat{s}(j)}, \operatorname{Im} a_{\hat{s}(j)})_{j=0,1,\dots,p}, \;\; \hat{s}(j) = (\underbrace{0,\dots,0}_{j}, n, 0, \dots, 0) \in \mathbb{Z}_+^p. \tag{7}$$

We now introduce a sequence of integer numbers

$$N_q = N_0^{2^q} (= N_{q-1}^2) \tag{8}$$

with some integer number $N_0 \geq 2$.

We consider the operator L on the set of parameters $\vec{a} \in \mathcal{O}_A^{p+1}$, and all other $a_{\hat{s}}$ to be fixed. In view of conditions $(P3)$ for $\gamma > 0$ we can construct the sequence of sets G_0, $G_1, \dots$, where G_q, $q = 0, 1, \dots$, is the set of those vectors $\vec{a}$ from equation (1), for which the following estimate holds:

$$|\lambda_{k,m}| > \gamma(1 + \|\alpha_k\|^2)^{-\kappa_j}(1 + m^2)^{-\alpha_j} \quad \text{for} \quad (k,m) \in \Omega^{N_q} \cap \Omega_j, \quad j = 1, 2, \tag{9}$$

where $\kappa_j = \kappa - n\delta_{1j}/2$, $\alpha_j = \alpha - n\delta_{2j}/2$, and δ_{ij} is the Kronecker delta.

The definition of the sets G_q yields the embeddings $\mathcal{O}_A^{p+1} \supset G_0 \supseteq G_1 \supseteq \dots$.

Introduce the sets $\mathcal{G}_q$ corresponding to the sets G_q by the formula $\mathcal{G}_q = \mathcal{O}_{\varepsilon_0} \times G_q$, where $q \geq 0$,

$$\varepsilon_0 = \gamma \min\left\{\frac{3}{16C_3}, \frac{1}{2C_0N_0^{2\delta}}, y_1, y_2\right\}, \quad C_3 = \max\{C_0, C_1, 2C_2\}, \tag{10}$$

y_1, y_2 are positive solutions of the equations

$$2C_0N_0^{14\delta}\, y^3 + y^2 = \frac{1}{24}C_3^{-2}, \quad 2C_0N_0^{6\delta}\, y^2 + (2 + N_0^{-8\delta})\, y = \frac{3}{4}C_3^{-1}$$

respectively. Then the embeddings

$$\ldots \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_0 \subset \mathcal{O}_{\varepsilon_0} \times \mathcal{O}_A^{p+1}$$

are obvious.

For any functions $u \in \mathbf{W}^{(N)}$, $h \in \mathbf{W}^{(N)}$, and the parameter $\varepsilon \in \mathbb{C} \setminus \{0\}$, we set

$$\mathcal{L}_N[h] \equiv \mathcal{L}_N(\vec{\varepsilon}, \vec{a}, u)[h] = Lh - \varepsilon \mathrm{P}_N \mathrm{D}_u f(u) h, \quad N \in \mathbb{N},$$

where L stands on the left-hand side of equation (1), and the projector P_N is defined by formula (4).

The following property $(P5)$ concerns the continuity of the inverse operator for the linear operator $\mathcal{L}_{N_q}(\vec{\varepsilon}, \vec{a}, u)\colon \mathbf{W}^{(N_q)} \to \mathbf{W}^{(N_q)}$, where $q \geq 0$:

(P5) For any $u \in \mathbf{W}^{(N_q)} \cap K_1$ and $\gamma > 0$ for all vectors $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_q$ operator $\mathcal{L}_{N_q}(\vec{\varepsilon}, \vec{a}, u)\colon \mathbf{W}^{(N_q)} \to \mathbf{W}^{(N_q)}$ is invertible; in particular, for $\bar{d} \in [d, \bar{\bar{d}} - 2\delta]$, $\bar{r} \in [r, \bar{\bar{r}} - 2\delta]$ and $h \in \mathbf{W}^{(N_q)}$ the estimate

$$\|\mathcal{L}_{N_q}^{-1}(\varepsilon, \vec{a}, u)[h]\|_{\bar{d},\bar{r}} \leq \frac{2}{\gamma} N_q^{2\delta} \|h\|_{\bar{d},\bar{r}}, \quad q \in \mathbb{N}, \tag{11}$$

holds.

Proof of property $(P5)$. Given $q \geq 0$; we represent the operator $\mathcal{L}_{N_q}$ in the form $\mathcal{L}_{N_q} = \mathcal{D} - \mathcal{T}_q$, where $\mathcal{D}$ is a diagonal operator, $\mathcal{D} = L$, $\mathcal{T}_q = \varepsilon \mathrm{P}_{N_q} \mathrm{D}_u f$. Then we have

$$\mathcal{L}_{N_q} = |\mathcal{D}|^{1/2} \mathcal{U} |\mathcal{D}|^{1/2} - \mathcal{T}_q = |\mathcal{D}|^{1/2}(\mathcal{U} - \mathcal{R}_1)|\mathcal{D}|^{1/2} = |\mathcal{D}|^{1/2} \mathcal{U}(\mathcal{I} - \mathcal{U}^{-1}\mathcal{R}_1)|\mathcal{D}|^{1/2},$$

where $\mathcal{U} = |\mathcal{D}|^{-1/2} \mathcal{D} |\mathcal{D}|^{-1/2}$, $\mathcal{R}_1 = |\mathcal{D}|^{-1/2} \mathcal{T}_q |\mathcal{D}|^{-1/2}$.

Suppose that

$$h = \sum_{(k,m) \in \mathbb{N} \times \mathbb{Z}} h_{k,m} \varphi_{k,m},$$

then diagonal operators $\mathcal{D}$ and $|\mathcal{D}|^\nu$, $\nu \geq 0$, in the space $\{\mathbf{X}_{d,r}\}_{d,r \in \mathbb{R}}$ are represented by the formulas

$$\mathcal{D}h = \sum_{(k,m) \in \mathbb{N} \times \mathbb{Z}} \lambda_{k,m}\, h_{k,m}\, \varphi_{k,m}, \qquad |\mathcal{D}|^\nu h = \sum_{(k,m) \in \mathbb{N} \times \mathbb{Z}} |\lambda_{k,m}|^\nu\, h_{k,m}\, \varphi_{k,m}.$$

For $\nu < 0$ operators $|\mathcal{D}|^\nu$ exist if the condition $\lambda_{k,m} \neq 0$ holds for any $(k,m) \in \mathbb{N} \times \mathbb{Z}$. Operators $\mathcal{D}$ and $|\mathcal{D}|^\nu$, $\nu \in \mathbb{R}$, in the space $\mathbf{W}^{(N_q)}$ are represented by diagonal matrices and have eigenvalues $\lambda_{k,m}$ and $|\lambda_{k,m}|^\nu$, respectively, and eigenfunctions $\varphi_{k,m} = e^{\tau(m)t} x_k$ for $(k,m) \in \Omega^{N_q}$.

Lemma 1. *For all vectors $\vec{a} \in G_q$, $q \geq 0$, the operator $|\mathcal{D}|$ is invertible in the space $\mathbf{W}^{(N_q)}$ and for any $d^*, r^* \in \mathbb{R}$ and $h \in \mathbf{W}^{(N_q)}$ the relation*

$$\big\| |\mathcal{D}|^{-1/2} h \big\|_{d^*, r^*, \Omega_j} \leq \tfrac{1}{\sqrt{\gamma}} \|h\|_{d^* + \kappa_j, r^* + \alpha_j, \Omega_j}, \quad j = 1, 2,$$

is valid.

The proof of this lemma is given in Section 4.

For the operator $\mathcal{L}_{N_q}^{-1}$ we have a factorization similar to the factorization of $\mathcal{L}_{N_q}$:

$$\mathcal{L}_{N_q}^{-1} = |\mathcal{D}|^{-1/2}(\mathcal{I} - \mathcal{U}^{-1}\mathcal{R}_1)^{-1} \mathcal{U}^{-1} |\mathcal{D}|^{-1/2} = |\mathcal{D}|^{-1/2}(\mathcal{I} - \mathcal{R})^{-1} \mathcal{U}^{-1} |\mathcal{D}|^{-1/2},$$

where $\mathcal{R} = \mathcal{U}^{-1}\mathcal{R}_1$ and $(\mathcal{I} - \mathcal{R})^{-1} = \mathcal{I} + \sum\limits_{r=1}^{\infty} \mathcal{R}^r$.

The last presentation is valid under the assumption of convergence of some geometric progression.

Lemma 2. *Assume that the point spectrum of the operator L does not contain zero. Then, for any $d^*, r^* \in \mathbb{R}$, the operator $\mathcal{U}$ is isometric in the space $\mathbf{X}_{d^*,r^*}$.*

Lemma 3. *For the operator $\mathcal{R}_1\colon \mathbf{W}^{(N_q)} \to \mathbf{W}^{(N_q)}$, $q \geq 0$, for all $\bar{d} \in [d, \bar{\bar{d}} - 2\delta]$, $\bar{r} \in [r, \bar{\bar{r}} - 2\delta]$ and $u \in \mathbf{W}^{(N_q)}$ the estimate*

$$\|\mathcal{R}_1 h\|_{\bar{d},\bar{r}} \leq C_1 \tfrac{|\varepsilon|}{\gamma} \|h\|_{\bar{d},\bar{r}}$$

holds.

The proofs of Lemmas 2 and 3 are given in Section 4.

The formula $\mathcal{R} = \mathcal{U}^{-1}\mathcal{R}_1$ and Lemmas 2 and 3 for all $\bar{d} \in [d, \bar{\bar{d}} - 2\delta]$, $\bar{r} \in [r, \bar{\bar{r}} - 2\delta]$ yield

$$\|\mathcal{R}h\|_{\bar{d},\bar{r}} = \|\mathcal{U}^{-1}\mathcal{R}_1 h\|_{\bar{d},\bar{r}} = \|\mathcal{R}_1 h\|_{\bar{d},\bar{r}} \leq C_1 \frac{|\varepsilon|}{\gamma} \|h\|_{\bar{d},\bar{r}}.$$

We have the estimate $\|(I - \mathcal{R})^{-1} h\|_{\bar{d},\bar{r}} \leq \|h\|_{\bar{d},\bar{r}} + \sum\limits_{r\in\mathbb{N}} \|\mathcal{R}^r h\|_{\bar{d},\bar{r}}$, where

$$\|\mathcal{R}^r h\|_{\bar{d},\bar{r}} = \|\mathcal{R}(\mathcal{R}^{r-1}h)\|_{\bar{d},\bar{r}} \leq C_1 \frac{|\varepsilon|}{\gamma} \|\mathcal{R}^{r-1}h\|_{\bar{d},\bar{r}} \leq \Big(C_1 \frac{|\varepsilon|}{\gamma}\Big)^r \|h\|_{\bar{d},\bar{r}}.$$

Under condition $|\varepsilon| < \varepsilon_0$ and (10), we have $|\varepsilon| < 3\gamma/16C_3 < \gamma/C_1$, and

$$\|(\mathcal{I} - \mathcal{R})^{-1} h\|_{\bar{d},\bar{r}} \leq \|h\|_{\bar{d},\bar{r}} \sum_{r=0}^{\infty} \Big(\frac{C_1|\varepsilon|}{\gamma}\Big)^r = \|h\|_{\bar{d},\bar{r}} \Big(1 - \frac{C_1|\varepsilon|}{\gamma}\Big)^{-1} = \frac{\gamma}{\gamma - C_1|\varepsilon|} \|h\|_{\bar{d},\bar{r}}.$$

Returning to the estimate of the norm of operator $\mathcal{L}_{N_q}^{-1}$, from Lemmas 2 and 3 and formulas (6) and (10) we can show that, for any $\bar{d} \in [d, \bar{\bar{d}} - 2\delta]$, $\bar{r} \in [r, \bar{\bar{r}} - 2\delta]$ and vectors $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_q$, the following estimates hold:

$$\|\mathcal{L}_{N_q}^{-1} h\|_{\bar{d},\bar{r},\Omega_j} \leq \frac{1}{\sqrt{\gamma}} \frac{\gamma}{\gamma - C_1|\varepsilon|} \big\| |\mathcal{D}|^{-1/2} h \big\|_{\bar{d}+\kappa_j,\bar{r}+\alpha_j,\Omega_j} \leq \frac{1}{\gamma} \frac{\gamma}{\gamma - C_1|\varepsilon|} \|h\|_{\bar{d}+2\kappa_j,\bar{r}+2\alpha_j,\Omega_j} \leq$$
$$\leq \frac{N_q^{2\delta}}{\gamma - C_1|\varepsilon|} \|h\|_{\bar{d},\bar{r}} = \frac{N_q^{2\delta}}{\gamma/2 - C_1|\varepsilon| + \gamma/2} \|h\|_{\bar{d},\bar{r}} \leq \frac{2}{\gamma} N_q^{2\delta} \|h\|_{\bar{d},\bar{r}}.$$

So, $\|\mathcal{L}_{N_q}^{-1} h\|_{\bar{d},\bar{r}} \leq \frac{2}{\gamma} N_q^{2\delta} \|h\|_{\bar{d},\bar{r}}$, and property $(P5)$ is proved. □

Inequality (11) holds for the vector $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_q$. For any $\gamma > 0$ the limit of the sequence of sets $\{\mathcal{G}_q\}_{q=0,1,\ldots}$ coincides with the set $\mathcal{G}_\infty = \mathcal{G}_\infty(\gamma) = \lim\limits_{q\to\infty} \mathcal{G}_q$. On this set there exists a solution to problem (1), (2).

3. Establishment of the solvability conditions for problem (1)**,** (2)**.** Let us define a sequence $\{u_q\}_{q\geq 0}$ recurrently, where $u_q \in \mathbf{W}^{(N_q)}$ defined on $\mathcal{G}_q$. This sequence converges to solution $u \in \mathbf{X}_{d,r}$ of (1), (2) for all vectors $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_\infty$. We will also prove that the set $\mathcal{G}_\infty$ in the set $\mathcal{O}_{\varepsilon_0} \times \mathcal{O}_A^{p+1}$ of parameters of the problem (1), (2) is large enough and find the lower bound for its measure.

Theorem 1. *Let properties* $(P1)$*—*$(P5)$ *be valid, and let* $\beta = 6\delta$, $\delta = \kappa + \alpha - n/2$. *Then there exists a sequence* $\{u_q\}_{q\geq 0}$, *where* $u_q = \sum_{i=0}^{q} h_i \in \mathbf{W}^{(N_q)}$ *is a solution of the equation*

$$Lu_q - \varepsilon \mathrm{P}_{N_q} f(u_q) = 0, \qquad (P_{N_q})$$

which is defined for $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_q$. *Moreover, if* $B_q = 1 + \|u_q\|_{d+\beta, r+\beta}$ *for* $q \in \{0\} \cup \mathbb{N}$, *then* $B_0 \leq 1 + \frac{|\varepsilon|}{\gamma} 2C_0 N_0^{14\delta}$, $B_q \leq B_0 N_{q+1}^{2\delta}$ *for* $q \in \mathbb{N}$, *and* $\|h_i\|_{d,r} \leq 4C_3 B_0 \frac{|\varepsilon|}{\gamma} N_i^{-2\delta}$ *for* $i \in \mathbb{N}$.

Proof. We use the method of mathematical induction. In view of condition $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_0$ we will find a solution of the equation

$$Lu - \varepsilon \mathrm{P}_{N_0} f(u) = 0. \qquad (P_{N_0})$$

There exists an operator $L^{-1}\colon \mathbf{W}^{(N_0)} \to \mathbf{W}^{(N_0)}$ such that

$$L^{-1} w = \sum_{(k,m)\in\Omega^{N_0}} \lambda_{k,m}^{-1}\, w_{k,m}\, \varphi_{k,m} = \sum_{(k,m)\in\Omega^{N_0}} \frac{w_{k,m}\,\varphi_{k,m}}{\lambda_{k,m}}$$

for any $w = \sum\limits_{(k,m)\in\Omega^{N_0}} w_{k,m}\varphi_{k,m}$. Then estimate (9) yields the inequality

$$\left\|L^{-1}w\right\|^2_{d,r,\Omega_j} = \sum_{(k,m)\in\Omega^{N_0}\cap\Omega_j} \left(1+\|\alpha_k\|^2\right)^d \left(1+m^2\right)^r \frac{|w_{k,m}|^2}{|\lambda_{k,m}|^2} \leq$$
$$\leq \sum_{(k,m)\in\Omega^{N_0}\cap\Omega_j} \frac{1}{\gamma^2}\left(1+\|\alpha_k\|^2\right)^{d+2\kappa_j}\left(1+m^2\right)^{r+2\alpha_j} |w_{k,m}|^2 = \frac{1}{\gamma^2}\|w\|^2_{d+2\kappa_j, r+2\alpha_j, \Omega_j},$$

$j=1,2$. We can reduce equation (P_{N_0}) to the form $u = \varepsilon L^{-1}\mathrm{P}_{N_0} f(u)$ the following estimates hold

$$\left\|L^{-1}w\right\|_{d,r,\Omega_j} \leq \frac{1}{\gamma}\|w\|_{d+2\kappa_j, r+2\alpha_j,\Omega_j} \leq \frac{1}{\gamma} N_0^{2\delta}\|w\|_{d,r,\Omega_j} \leq \frac{1}{\gamma} N_0^{2\delta}\|w\|_{d,r}, \quad j=1,2 \qquad (12)$$

for $w \in \mathbf{W}^{(N_0)}$.

So, we have that $\left\|L^{-1}w\right\|_{d,r} \leq \frac{1}{\gamma} N_0^{2\delta}\|w\|_{d,r}$.

Remark. If $C_1 = 0$, then $f(u) = f(0)$ and $u = \varepsilon L^{-1}\mathrm{P}_{N_0} f(0)$ is a unique solution of equation (P_{N_0}) in the space $\mathbf{W}^{(N_0)}$ for all vectors $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_0$.

We denote by $H_0\colon \mathbf{W}^{(N_0)} \to \mathbf{W}^{(N_0)}$ the operator defined $H_0(u) = \varepsilon L^{-1}\mathrm{P}_{N_0} f(u)$ on element $u \in \mathbf{W}^{(N_0)}$. In order to construct a solution of (P_{N_0}) we must find a fixed point $u \in \mathbf{W}^{(N_0)}$ of the operator.

We will show that for every vector $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_0$ the operator H_0 is a contraction mapping in the domain

$$G_0 = \left\{u \in \mathbf{W}^{(N_0)}\colon\ \|u\|_{d,r} \leq \rho_0 = \frac{|\varepsilon|}{\gamma} 2C_0 N_0^{2\delta}\right\}.$$

Using inequality (12), property $(P2)$, formulas (10) and inequality $|\varepsilon| < \varepsilon_0$, we get the estimate

$$\|H_0(u)\|_{d,r} \leq \frac{|\varepsilon|}{\gamma} N_0^{2\delta}\|f(u)\|_{d,r} \leq \frac{|\varepsilon|}{\gamma} N_0^{2\delta} C_0 (1+\|u\|_{d,r}) \leq \frac{|\varepsilon|}{\gamma} N_0^{2\delta} C_0 (1+\rho_0) =$$

$$= \frac{\rho_0}{2} + \frac{|\varepsilon|}{\gamma} N_0^{2\delta} C_0 \rho_0 \le \rho_0$$

for any $u \in G_0$, i.e., $H_0(G_0) \subset G_0$.

Using inequality (12), property $(P3)$ and formula $H_0(u) - H_0(u') = \varepsilon L^{-1} \mathrm{P}_{N_0}(f(u) - f(u'))$ for any $u,\ u' \in G_0$, we get the estimate

$$\|H_0(u) - H_0(u')\|_{d,r} \le \frac{|\varepsilon|}{\gamma} N_0^{2\delta} \|f(u) - f(u')\|_{d,r} \le C_1 \frac{|\varepsilon|}{\gamma} N_0^{2\delta} \|u - u'\|_{d,r}.$$

Hence, in view of formula (10) the mapping $H_0\colon \mathbf{W}^{(N_0)} \to \mathbf{W}^{(N_0)}$ is a contraction in G_0. Thus, the fixed point $u = u_0 \in G_0 \subset \mathbf{W}^{(N_0)}$ of operator H_0 is the unique solution of equation (P_{N_0}) and

$$B_0 \le 1 + N_0^{\beta} \rho_0 = 1 + \frac{|\varepsilon|}{\gamma} 2 C_0 N_0^{14\delta}.$$

Hence, the zero step of the induction is finished.

By induction, we construct the following elements of the sequence $\{u_q\}_{q>0}$ as $u_q = \sum\limits_{i=0}^{q} h_i$, where $h_0 = u_0$. In order to estimate their norms in the space $\mathbf{X}_{d+\beta,r+\beta}$ we will use the following lemma, whose proof will be given in the next section.

Lemma 4. *The elements u_q of sequence $\{u_q\}_{q\ge 0}$ satisfy the relation*

$$B_{q+1} \le (1 + N_{q+1}^{2\delta}) B_q. \tag{13}$$

By induction, inequality (13) yields

$$B_q \le B_0 \prod_{i=1}^{q} (1 + N_i^{2\delta}) = B_0 \prod_{i=1}^{q} \Big(1 + \big(N_0^{2^i}\big)^{2\delta}\Big).$$

Since $1 + \big(N_0^{2^i}\big)^{2\delta} \le \big(N_0^{2^i + 2^{-i}}\big)^{2\delta}$, the following inequality holds

$$B_q \le B_0 \prod_{i=1}^{q} \big(N_0^{2^i + 2^{-i}}\big)^{2\delta} = B_0 N_0^{\sum\limits_{i=1}^{q} 2\delta(2^i + 2^{-i})} = B_0 N_0^{\delta 2^{q+2} - 2\delta + \sum\limits_{i=1}^{q} \delta 2^{1-i}} \le B_0 N_{q+1}^{2\delta}.$$

Assuming that $u_0,\, u_1,\, \ldots,\, u_q$ are known, we will prove the existence of a solution $u_{q+1} \in \mathbf{W}^{(N_{q+1})}$ of the equation

$$L u_{q+1} - \varepsilon \mathrm{P}_{N_{q+1}} f(u_{q+1}) = 0. \tag{$P_{N_{q+1}}$}$$

Since $u_{q+1} = u_q + h_{q+1}$, we have $h_{q+1} \in \mathbf{W}^{(N_{q+1})}$ and $B_{q+1} = 1 + \|u_{q+1}\|_{d+\beta,r+\beta} \le B_0 N_{q+2}^{2\delta}$. Let us estimate the norm of h_{q+1} in the spaces $\mathbf{X}_{d,r}$.

According to procedure of linearization of the left-hand side of equation $(P_{N_{q+1}})$, for every $h \in \mathbf{W}^{(N_{q+1})}$ we now write

$$\begin{aligned} L(u_q + h) - \varepsilon \mathrm{P}_{N_{q+1}} f(u_q + h) &= L u_q - \varepsilon \mathrm{P}_{N_{q+1}} f(u_q) + L h - \varepsilon \mathrm{P}_{N_{q+1}} \mathrm{D}_u f(u_q) h + \\ + \varepsilon \mathrm{P}_{N_{q+1}} \mathrm{D}_u f(u_q) h + \varepsilon \mathrm{P}_{N_{q+1}} f(u_q) &- \varepsilon \mathrm{P}_{N_{q+1}} f(u_q + h) = r_q + \mathcal{L}_{N_{q+1}}(\vec{\varepsilon}, \vec{a}, u_q) h - \\ - \varepsilon \mathrm{P}_{N_{q+1}} \big(f(u_q + h) - f(u_q) - \mathrm{D}_u f(u_q) h\big) &= r_q + \mathcal{L}_{N_{q+1}}(\vec{\varepsilon}, \vec{a}, u_q) h + R_q(h), \end{aligned}$$

where $r_q = Lu_q - \varepsilon \mathrm{P}_{N_{q+1}} f(u_q)$, $\quad R_q(h) = -\varepsilon \mathrm{P}_{N_{q+1}}\big(f(u_q+h) - f(u_q) - \mathrm{D}_u f(u_q) h\big)$. As a result, the element $h = h_{q+1}$ is a solution in the space $\mathbf{W}^{(N_{q+1})}$ of the equation

$$r_q + \mathcal{L}_{N_{q+1}}(\vec{\varepsilon}, \vec{a}, u_q) h + R_q(h) = 0.$$

Since u_q is a solution of equation (P_{N_q}), i.e., $Lu_q = \varepsilon \mathrm{P}_{N_q} f(u_q)$, we have

$$r_q = \varepsilon(\mathrm{P}_{N_q} - \mathrm{P}_{N_{q+1}}) f(u_q) = -\varepsilon \mathrm{P}^{\perp}_{N_q} \mathrm{P}_{N_{q+1}} f(u_q) \in \mathbf{W}^{(N_q)\perp} \cap \mathbf{W}^{(N_{q+1})}.$$

Using formula (6) and property $(P2)$, we obtain

$$\|r_q\|_{d,r} = |\varepsilon| N_q^{-2\beta} \|\mathrm{P}_{N_{q+1}} f(u_q)\|_{d+\beta, r+\beta} \le |\varepsilon| C_0 N_q^{-2\beta} B_q \le |\varepsilon| C_0 B_0 N_q^{-2\beta} N_{q+1}^{2\delta}.$$

In order to estimate the norm $\|R_q(h)\|_{d,r}$ of $R_q(h) \in \mathbf{W}^{(N_{q+1})}$, we use property $(P4)$ and get

$$\|R_q(h)\|_{d,r} \le 2|\varepsilon| C_2 \|h\|^2_{d,r}.$$

Since

$$\begin{aligned} R_q(h) - R_q(h') &= -\varepsilon \mathrm{P}_{N_{q+1}}\big(f(u_q+h) - f(u_q+h') - \mathrm{D}_u f(u_q)(h-h')\big) = \\ &= -\varepsilon \mathrm{P}_{N_{q+1}}\big(f(u_q+h'+(h-h')) - f(u_q+h') - \mathrm{D}_u f(u_q)(h-h')\big) \end{aligned}$$

for any functions $h,\ h' \in \mathbf{W}^{(N_{q+1})}$, the following estimate holds

$$\begin{aligned} &\|R_q(h) - R_q(h')\|_{d,r} \le |\varepsilon| \big\| f\big(u_q + h' + (h-h')\big) - f(u_q+h') - \mathrm{D}_u f(u_q)(h-h') \big\|_{d,r} \le \\ &\le |\varepsilon| \big\| f\big(u_q + h' + (h-h')\big) - f(u_q+h') - \mathrm{D}_u f(u_q+h')(h-h') + \mathrm{D}_u f(u_q+h')(h-h') - \\ &- \mathrm{D}_u f(u_q)(h-h') \big\|_{d,r} \le |\varepsilon| \big(2C_2 \|h-h'\|^2_{d,r} + C_1 \|h-h'\|_{d,r} \|h'\|_{d,r}\big) \le \\ &\le |\varepsilon| \big(2C_2 \|h-h'\|_{d,r} \big(\|h\|_{d,r} + \|h'\|_{d,r}\big) + C_1 \|h-h'\|_{d,r} \|h'\|_{d,r}\big) \le \\ &\le C_3 |\varepsilon| \big(\|h\|_{d,r} + 2\|h'\|_{d,r}\big) \|h-h'\|_{d,r}, \end{aligned}$$

by virtue of property $(P4)$.

By property $(P5)$ the operator $\mathcal{L}_{N_{q+1}}(\vec{\varepsilon}, \vec{a}, u_q)$ is invertible for vectors $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_{q+1}$, $u_q \in \mathbf{W}^{(N_q)}$ and $\|\mathcal{L}^{-1}_{N_{q+1}}(\vec{\varepsilon}, \vec{a}, u_q)[h]\|_{d,r} \le \frac{2}{\gamma} N_{q+1}^{2\delta} \|h\|_{d,r}$. We denote the value of the operator $H_{q+1}\colon \mathbf{W}^{(N_{q+1})} \to \mathbf{W}^{(N_{q+1})}$ on the element $h \in \mathbf{W}^{(N_{q+1})}$ by

$$H_{q+1}(h) = -\mathcal{L}^{-1}_{N_{q+1}}(\vec{\varepsilon}, \vec{a}, u_q)\big(r_q + R_q(h)\big).$$

Then a solution of equation $(\mathrm{P}_{N_{q+1}})$ is a point $h = h_{q+1} \in \mathbf{W}^{(N_{q+1})}$ of operator H_{q+1}, i.e., a solution of the equation $h = H_{q+1}(h)$.

We now formulate lemma which will be proved in Section 4.

Lemma 5. *For every vector $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_{q+1}$ and $q \ge 0$, the operator H_{q+1} is a contraction mapping in domain*

$$G_{q+1} = \Big\{ h \in \mathbf{W}^{(N_{q+1})}\colon\ \|h\|_{d,r} \le \rho_{q+1} = 4C_3 B_0 \frac{|\varepsilon|}{\gamma} N_{q+1}^{-2\delta} \Big\}.$$

This lemma yields the existence of the unique solution $h_{q+1} \in \mathbf{W}^{(N_{q+1})}$ of the equation $h = H_{q+1}(h)$ for all $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_{q+1}$, that satisfies the relation $\|h_{q+1}\|_{d,r} \le \rho_{q+1} = 4C_3B_0\frac{|\varepsilon|}{\gamma}N_{q+1}^{-2\delta}$. Thus, the function $u_{q+1} = u_q + h_{q+1}$ is a solution of equation $(P_{N_{q+1}})$ in the space $\mathbf{W}^{(N_{q+1})}$ defined for all vectors $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_{q+1} \subseteq \mathcal{G}_0$ and $u_{q+1} = \sum_{i=0}^{q+1} h_i$, where $h_i \in \mathbf{W}^{(N_i)}$.

Moreover, $\|h_i\|_{d,r} \le 4C_3B_0\frac{|\varepsilon|}{\gamma}N_i^{-2\delta}$ for any $i = 0, 1, \ldots, q+1$. □

Let us find the measure of set $\mathcal{G}_\infty$, whose elements satisfy Theorem 1 for any $q \ge 0$.

Theorem 2. *The set $\mathcal{G}_\infty$ is defined by the formula $\mathcal{G}_\infty = \bigcap_{q\ge 0} \mathcal{G}_q$. Its Lebesgue measure satisfies the estimate*

$$\operatorname{meas} \mathcal{G}_\infty \ge \varepsilon_0^2 A^{2p+2}\pi^{p+2}\left(1 - \gamma^2\frac{\tilde{C}p^{2n}\zeta_1(2\kappa)\zeta_2(2\alpha)}{A^2}\right), \quad \tilde{C} = 2^n/\min\left\{1, C^{2n}\right\} > 0.$$

Proof. Since $\mathcal{G}_q = \bigcap_{l=0}^{q} \mathcal{G}_l$, we have $\mathcal{G}_\infty = \lim_{q\to\infty} \mathcal{G}_q = \bigcap_{q=0}^{\infty} \mathcal{G}_q = \bigcap_{q\ge 0} \mathcal{G}_q$. Note that

$$\operatorname{meas} \mathcal{G}_\infty = \varepsilon_0^2 A^{2p+2}\pi^{p+2} - \operatorname{meas}\overline{\mathcal{G}_\infty}, \qquad \overline{\mathcal{G}_\infty} = \bigcup_{q=0}^{\infty} \overline{\mathcal{G}_q} = \bigcup_{q=0}^{\infty} \left(\mathcal{O}_{\varepsilon_0} \times \overline{G_q}\right),$$

where $\operatorname{meas}\overline{\mathcal{G}_\infty} = \lim_{q\to\infty} \operatorname{meas}\overline{\mathcal{G}_q}$, and the bar stand for the complement of a set in the set $\mathcal{O}_{\varepsilon_0} \times \mathcal{O}_A^{p+1}$ or $\mathcal{O}_A^{p+1}$. Let us find the measure of the set $\overline{\mathcal{G}_\infty}$.

To this end, we now estimate the measure of the set $\overline{G_q} = \bigcup_{(k,m)\in\Omega^{N_q}} \overline{G_q(k,m)}$, where $\overline{G_q(k,m)}$ is the set of vectors $\vec{a}$ such that the relation

$$|\lambda_{k,m}| < \gamma\left(1 + \|\alpha_k\|^2\right)^{-\kappa_j}\left(1 + m^2\right)^{-\alpha_j}, \qquad j = 1, 2,$$

is satisfied at the fixed vector $(k,m) \in \Omega^{N_q}$.

Consider the case $j = 1$, i.e., $(k,m) \in \Omega_1$. Then $|\tau(m)| < \|\alpha_k\|$ and

$$|\lambda_{k,m}| = |\alpha_{\eta k}|^n\left|a_{\hat{s}(\eta)} + \lambda^*_{k,m}/\alpha^n_{\eta k}\right| < \gamma\left(1 + \|\alpha_k\|^2\right)^{-\kappa_1}\left(1 + m^2\right)^{-\alpha_1},$$

where η is the minimal number in equality $|\alpha_{\eta k}| = \max_{j=1,\ldots,p} |\alpha_{jk}|, \quad \lambda^*_{k,m} = \lambda_{k,m} - a_{\hat{s}(\eta)}\alpha^n_{\eta k}$. Hence,

$$\left|a_{\hat{s}(\eta)} + \lambda^*_{k,m}/\alpha^n_{\eta k}\right| < \frac{\gamma}{|\alpha_{\eta k}|^n}\left(1 + \|\alpha_k\|^2\right)^{-\kappa_1}\left(1 + m^2\right)^{-\alpha_1}.$$

Since $p|\alpha_{\eta k}|^2 \ge \|\alpha_k\|^2 \ge C^2$, we have $|\alpha_{\eta k}|^n \ge p^{-n/2}\|\alpha_k\|^n$. The inequality

$$\|\alpha_k\|^2 \ge \frac{C^2 + \|\alpha_k\|^2}{2} \ge \frac{\min\{1, C^2\}}{2}\left(1 + \|\alpha_k\|^2\right),$$

implies $\|\alpha_k\|^n \ge \left(\frac{\min\{1,C^2\}}{2}\right)^{n/2}\left(1 + \|\alpha_k\|^2\right)^{n/2}$, $C > 0$. Thus, we have $\left|a_{\hat{s}(\eta)} + \lambda^*_{k,m}/\alpha^n_{\eta k}\right| < r^*_{k,m}$, where

$$r^*_{k,m} = \gamma p^{n/2}\left(\frac{\min\{1, C^2\}}{2}\right)^{-n/2}\left(1 + \|\alpha_k\|^2\right)^{-\kappa_1 - n/2}\left(1 + m^2\right)^{-\alpha_1}.$$

The set of numbers $a_{\hat{s}(\eta)} \in \mathcal{O}_A$, satisfying this inequality, belongs to a disk of radius $r^*_{k,m}$. So its measure does not exceed $\pi(r^*_{k,m})^2$. For the measure meas $\overline{G_q(k,m)}$ of the set $\overline{G_q(k,m)}$ for $(k,m) \in \Omega_1$ the estimate

$$\operatorname{meas} \overline{G_q(k,m)} \le \pi^{p+1} A^{2p} \gamma^2 p^n \Big(\frac{2}{\min\{1, C^2\}}\Big)^n \big(1 + \|\alpha_k\|^2\big)^{-2\kappa_1 - n} \big(1 + m^2\big)^{-2\alpha_1} \tag{14}$$

hold.

Consider the case $j = 2$, i.e., $(k,m) \in \Omega_2$. Then $|\tau(m)| \ge \|\alpha_k\|$ and

$$|\lambda_{k,m}| = |\tau(m)|^n \Big| a_{\hat{s}(0)} + \sum_{\hat{s} \ne \hat{s}(0)} a_{\hat{s}} \tau^{s_0}(m) \alpha_k^s \Big| < \gamma \big(1 + \|\alpha_k\|^2\big)^{-\kappa_2} \big(1 + m^2\big)^{-\alpha_2}.$$

Hence,

$$\Big| a_{\hat{s}(0)} + \sum_{\hat{s} \ne \hat{s}(0)} a_{\hat{s}} \tau^{s_0}(m) \alpha_k^s \Big| < \frac{\gamma}{|\tau(m)|^n} \big(1 + \|\alpha_k\|^2\big)^{-\kappa_2} \big(1 + m^2\big)^{-\alpha_2}.$$

Since $|\tau(m)|^2 \ge \frac{C^2 + m^2}{2} \ge \frac{\min\{1, C^2\}}{2} (1 + m^2)$, the following inequality

$$|\tau(m)|^n \ge \Big(\frac{\min\{1, C^2\}}{2}\Big)^{n/2} \big(1 + m^2\big)^{n/2}, \ C > 0,$$

is valid. Hence, we deduce the estimate

$$\Big| a_{\hat{s}(0)} + \sum_{\hat{s} \ne \hat{s}(0)} a_{\hat{s}} \tau^{s_0}(m) \alpha_k^s \Big| < \gamma \Big(\frac{\min\{1, C^2\}}{2}\Big)^{-n/2} \big(1 + \|\alpha_k\|^2\big)^{-\kappa_2} \big(1 + m^2\big)^{-\alpha_2 - n/2}.$$

For the measure meas $\overline{G_q(k,m)}$ of the set $\overline{G_q(k,m)}$ for $(k,m) \in \Omega_2$ the estimate

$$\operatorname{meas} \overline{G_q(k,m)} \le \pi^{p+1} A^{2p} \gamma^2 \Big(\frac{2}{\min\{1, C^2\}}\Big)^n \big(1 + \|\alpha_k\|^2\big)^{-2\kappa_2} \big(1 + m^2\big)^{-2\alpha_2 - n} \tag{15}$$

holds.

Using the estimates (14) and (15), we get the estimate for the measure meas $\overline{G_q(k,m)}$ of the set $\overline{G_q(k,m)}$ for any vector (k,m):

$$\operatorname{meas} \overline{G_q(k,m)} \le \tilde{C} \pi^{p+1} A^{2p} p^n \gamma^2 \big(1 + \|\alpha_k\|^2\big)^{-2\kappa} \big(1 + m^2\big)^{-2\alpha}.$$

Then

$$\operatorname{meas} \overline{G_q} \le \sum_{(k,m) \in \Omega^{N_q}} \operatorname{meas} \overline{G_q(k,m)} \le \tilde{C} p^{2n} \pi^{p+1} A^{2p} \gamma^2 \sum_{(k,m) \in \Omega^{N_q}} \big(1 + \|\alpha_k\|^2\big)^{-2\kappa} \big(1 + m^2\big)^{-2\alpha} \le$$
$$\le \tilde{C} p^{2n} \pi^{p+1} A^{2p} \gamma^2 \sum_{k \in \mathbb{N}} \big(1 + \|\alpha_k\|^2\big)^{-2\kappa} \sum_{m \in \mathbb{Z}} \big(1 + m^2\big)^{-2\alpha} \le \tilde{C} p^{2n} \pi^{p+1} A^{2p} \gamma^2 \zeta_1(2\kappa) \zeta_2(2\alpha).$$

Since $2\kappa > 1/2\beta_0$ and $2\alpha > 1/2$, we have $\zeta_1(2\kappa) < \infty$ and $\zeta_2(2\alpha) < \infty$. For the measure meas $\overline{\mathcal{G}_q}$ of the set $\overline{\mathcal{G}_q}$ we obtain the estimate

$$\operatorname{meas} \overline{\mathcal{G}_q} \le \pi \varepsilon_0^2 \operatorname{meas} \overline{G_q} \le \tilde{C} \pi^{p+2} \varepsilon_0^2 p^{2n} A^{2p} \gamma^2 \zeta_1(2\kappa) \zeta_2(2\alpha).$$

Let us find the measure $\operatorname{meas}\overline{\mathcal{G}_\infty} = \lim_{q\to\infty} \operatorname{meas}\overline{\mathcal{G}_q}$, so

$$\operatorname{meas}\overline{\mathcal{G}_\infty} \le \tilde{C}\pi^{p+2}\varepsilon_0^2 p^{2n} A^{2p}\gamma^2\zeta_1(2\kappa)\zeta_2(2\alpha).$$

The measure of the set $\mathcal{G}_\infty$ has the asymptotics $\operatorname{meas}\mathcal{G}_\infty = \operatorname{meas}(\mathcal{O}_{\varepsilon_0} \times \mathcal{O}_A^{p+1}) + O(\gamma^2)$, $\gamma \to 0_+$, in particular,

$$\operatorname{meas}\mathcal{G}_\infty \ge \varepsilon_0^2 A^{2p+2}\pi^{p+2} -$$
$$-\tilde{C}\pi^{p+2}\varepsilon_0^2 p^{2n} A^{2p}\zeta_1(2\kappa)\zeta_2(2\alpha)\gamma^2 = \varepsilon_0^2 A^{2p+2}\pi^{p+2}\left(1 - \gamma^2\frac{\tilde{C}p^{2n}\zeta_1(2\kappa)\zeta_2(2\alpha)}{A^2}\right).$$

□

Theorem 3. *For all vectors $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_\infty$, any $\gamma > 0$, and a natural number $N_0 \ge 2$ the series $\sum_{i\ge0} h_i$ converges in the space $\mathbf{X}_{d,r}$ to a solution u of problem* (1), (2). *Its norm is determined by the inequality*

$$\|u\|_{d,r} \le \tfrac{8C_3B_0}{N_0^{2\delta}}\tfrac{|\varepsilon|}{\gamma}.$$

Proof. By Theorem 1, for all vectors $(\vec{\varepsilon}, \vec{a}) \in \mathcal{G}_\infty$ for the majorizing series $\sum\limits_{i\ge0}\|h_i\|_{d,r}$ the relation

$$\sum_{i\ge0}\|h_i\|_{d,r} \le \sum_{i\ge0} 4C_3B_0\frac{|\varepsilon|}{\gamma}N_i^{-2\delta}$$

holds.

Then the series $\sum\limits_{i\ge0} h_i$ converges in the space $\mathbf{X}_{d,r}$ to some function $u \in \mathbf{X}_{d,r}$, because

$$\|u\|_{d,r} \le \sum_{i\ge0}\|h_i\|_{d,r} \le 4C_3B_0\frac{|\varepsilon|}{\gamma}\sum_{i\ge0}\left(N_0^{2^i}\right)^{-2\delta} \le 4C_3B_0\frac{|\varepsilon|}{\gamma}\frac{2}{N_0^{2\delta}} = \frac{8C_3B_0}{N_0^{2\delta}}\frac{|\varepsilon|}{\gamma}.$$

We now show that $Lu = \varepsilon f(u)$. The function u_q is a solution of equation (P_{N_q}). Then

$$Lu_q = \varepsilon P_{N_q} f(u_q) = \varepsilon f(u_q) - \varepsilon P_{N_q}^\perp f(u_q). \tag{16}$$

With regard for the property (6) of the projector $P_{N_q}^\perp$, property $(P2)$ and the estimate for B_q in the proof of Theorem 1, we have

$$\|P_{N_q}^\perp f(u_q)\|_{d,r} \le N_q^{-2\beta}\|f(u_q)\|_{d+\beta,r+\beta} \le C_0N_q^{-2\beta}B_q \le C_0B_0N_q^{-24\delta}N_{q+1}^{2\delta} =$$
$$= C_0B_0N_q^{-24\delta}N_q^{4\delta} = C_0B_0N_q^{-20\delta} = C_0B_0N_0^{-20\delta2^q}.$$

Thus, $P_{N_q}^\perp f(u_q) \to 0$ as $q \to \infty$. Then, by virtue of property $(P1)$ the right-hand side of (16) converges to $\varepsilon f(u)$ in the space $\mathbf{X}_{d,r}$. The continuity of the operator L implies that the left-hand side of (16) Lu_q converges to Lu as $q \to \infty$ in the sense of distributions. □

4. Proof of auxiliary lemmas.

Proof of Lemma 1. For all vectors $\vec{a} \in G_q$ the operator $|\mathcal{D}|^{-1/2}$ exists. Since

$$h = \sum_{(k,m)\in\Omega^{N_q}} e^{\tau(m)t}\, x_k\, h_{k,m},$$

we have

$$|\mathcal{D}|^{-1/2}h = \sum_{(k,m)\in\Omega^{N_q}} \frac{h_{k,m}\varphi_{k,m}}{\sqrt{|\lambda_{k,m}|}}.$$

Moreover, inequality (9) yields the estimate

$$\big\||\mathcal{D}|^{-1/2}h\big\|^2_{d^*,r^*,\Omega_j} = \sum_{(k,m)\in\Omega^{N_q}} \big(1+\|\alpha_k\|^2\big)^{d^*}\big(1+m^2\big)^{r^*}\frac{|h_{k,m}|^2}{|\lambda_{k,m}|} \le$$

$$\le \sum_{(k,m)\in\Omega^{N_q}} \frac{1}{\gamma}\big(1+\|\alpha_k\|^2\big)^{d^*+\kappa_j}\big(1+m^2\big)^{r^*+\alpha_j}|h_{k,m}|^2 = \frac{1}{\gamma}\|h\|^2_{d^*+\kappa_j,r^*+\alpha_j,\Omega_j}.$$

□

Proof of Lemma 2. Since $\mathcal{U} = |\mathcal{D}|^{-1/2}\mathcal{D}|\mathcal{D}|^{-1/2}$, its action on function $h \in \mathbf{X}_{d^*,r^*}$ is given by the formula

$$\mathcal{U}h = \sum_{(k,m)\in\mathbb{N}\times\mathbb{Z}} \frac{\lambda_{k,m}}{|\lambda_{k,m}|}\; h_{k,m}\;\varphi_{k,m},$$

where $\lambda_{k,m}$ are eigenvalues of the operator $\mathcal{D}$. Using the definitions of the space $\mathbf{X}_{d^*,r^*}$ and the norm in this space, we obtain $\|\mathcal{U}^{-1}h\|_{d^*,r^*} = \|h\|_{d^*,r^*}$. □

Proof of Lemma 3. We note that $\mathcal{R}_1 h = \varepsilon|\mathcal{D}|^{-1/2}\mathrm{P}_{N_q}\big(D_u f|\mathcal{D}|^{-1/2}h\big)$. Then, in view of the Lemma 1, property $(P3)$ and inequalities (5), we obtain

$$\|\mathcal{R}_1 h\|_{\bar d,\bar r,\Omega_j} \le \frac{|\varepsilon|}{\sqrt{\gamma}}\big\|\mathrm{P}_{N_q}\mathrm{D}_u f|\mathcal{D}|^{-1/2}h\big\|_{\bar d+\kappa_j,\bar r+\alpha_j,\Omega_j} \le \frac{|\varepsilon|}{\sqrt{\gamma}}N_q^{-\delta}\big\|\mathrm{D}_u f|\mathcal{D}|^{-1/2}h\big\|_{\bar d+2\kappa_j,\bar r+2\alpha_j,\Omega_j} \le$$

$$\le \frac{|\varepsilon|}{\sqrt{\gamma}}N_q^{-\delta}C_1\big\||\mathcal{D}|^{-1/2}h\big\|_{\bar d,\bar r,\Omega_j} \le C_1\frac{|\varepsilon|}{\gamma}N_q^{-\delta}\|h\|_{\bar d+\kappa_j,\bar r+\alpha_j,\Omega_j} \le C_1\frac{|\varepsilon|}{\gamma}\|h\|_{\bar d,\bar r,\Omega_j} \le C_1\frac{|\varepsilon|}{\gamma}\|h\|_{\bar d,\bar r},$$

$j=1,2$. So, $\|\mathcal{R}_1 h\|_{\bar d,\bar r} \le C_1\frac{|\varepsilon|}{\gamma}\|h\|_{\bar d,\bar r}$. □

Proof of Lemma 4. Since $u_{q+1} = u_q + h_{q+1}$ i $h_{q+1}\in G_{q+1}$, we have

$$B_{q+1} = 1 + \|u_{q+1}\|_{d+\beta,r+\beta} \le 1 + \|u_q\|_{d+\beta,r+\beta} + \|h_{q+1}\|_{d+\beta,r+\beta} = B_q + \|h_{q+1}\|_{d+\beta,r+\beta}.$$

The norm of the solution h_{q+1} of the equation $h_{q+1} = -\mathcal{L}^{-1}_{N_{q+1}}(\vec\varepsilon,\vec a,u_q)\big(r_q + R_q(h_{q+1})\big)$ in the space $\mathbf{X}_{d+\beta,r+\beta}$ satisfies the estimate

$$\|h_{q+1}\|_{d+\beta,r+\beta} \le \frac{2}{\gamma}N^{2\delta}_{q+1}\big(\|r_q\|_{d+\beta,r+\beta} + \|R_q(h_{q+1})\|_{d+\beta,r+\beta}\big). \tag{17}$$

We note that $r_q = -\varepsilon\mathrm{P}^{\perp}_{N_q}\mathrm{P}_{N_{q+1}}f(u_q)$. Therefore, we can write, by using $(P2)$, the estimate for the norm $\|r_q\|_{d+\beta,r+\beta}$ as follows:

$$\|r_q\|_{d+\beta,r+\beta} \le |\varepsilon|\|f(u_q)\|_{d+\beta,r+\beta} \le |\varepsilon|C_0(1+\|u_q\|_{d+\beta,r+\beta}) = |\varepsilon|C_0B_q. \tag{18}$$

To estimate the quantities $\|R_q(h_{q+1})\|_{d+\beta,r+\beta}$ we use Lemma 2 and property $(P4)$ and get

$$\|R_q(h_{q+1})\|_{d+\beta,r+\beta} \le |\varepsilon|C_2\big(\|u_q\|_{d+\beta,r+\beta}\|h_{q+1}\|^2_{d,r} + \|h_{q+1}\|_{d,r}\|h_{q+1}\|_{d+\beta,r+\beta}\big) \le$$

$$\le |\varepsilon| C_2 \big(B_q \rho_{q+1}^2 + \rho_{q+1} \|h_{q+1}\|_{d+\beta,r+\beta}\big). \tag{19}$$

Substituting estimates (18) and (19) in inequality (17), we have

$$\|h_{q+1}\|_{d+\beta,r+\beta} \le \frac{2}{\gamma} N_{q+1}^{2\delta} \Big(|\varepsilon| C_0 B_q + |\varepsilon| C_2 \big(B_q \rho_{q+1}^2 + \rho_{q+1} \|h_{q+1}\|_{d+\beta,r+\beta}\big)\Big) =$$
$$= 2C_0 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} B_q + 2C_2 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} \rho_{q+1}^2 B_q + 2C_2 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} \rho_{q+1} \|h_{q+1}\|_{d+\beta,r+\beta} <$$
$$< 2C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} B_q + C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} \rho_{q+1}^2 B_q + C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} \rho_{q+1} \|h_{q+1}\|_{d+\beta,r+\beta}.$$

In view of Lemma 5 and equality (10) for $|\varepsilon| < \varepsilon_0$, from the inequality

$$4C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} \rho_{q+1} < 4C_3 \frac{\varepsilon_0}{\gamma} N_{q+1}^{2\delta} \rho_{q+1} \le 4 \Big(\frac{2C_3 \varepsilon_0}{\gamma}\Big)^2 = \frac{2}{3} < 1$$

we obtain the following estimate for $q \ge 2$

$$\|h_{q+1}\|_{d+\beta,r+\beta} \le 2C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} B_q + \frac{\rho_{q+1}}{4} B_q + \frac{1}{4} \|h_{q+1}\|_{d+\beta,r+\beta} <$$
$$< 2C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} B_q + C_3 B_0 \frac{|\varepsilon|}{\gamma} N_{q+1}^{-2\delta} B_q + \frac{1}{4} \|h_{q+1}\|_{d+\beta,r+\beta} =$$
$$= 2C_3 \frac{|\varepsilon|}{\gamma} B_q \big(N_{q+1}^{2\delta} + \frac{1}{2} B_0 N_{q+1}^{-2\delta}\big) + \frac{1}{4} \|h_{q+1}\|_{d+\beta,r+\beta} \le 4C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} B_q + \frac{1}{4} \|h_{q+1}\|_{d+\beta,r+\beta}.$$

Then $\frac{3}{4} \|h_{q+1}\|_{d+\beta,r+\beta} \le 4C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} B_q$ for $q \ge 2$. This result and relation (10) for $|\varepsilon| < \varepsilon_0$ yield

$$\|h_{q+1}\|_{d+\beta,r+\beta} \le \frac{16}{3} C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} B_q \le N_{q+1}^{2\delta} B_q, \quad q \ge 2,$$
$$\|h_1\|_{d+\beta,r+\beta} \le \frac{8}{3} C_3 \frac{|\varepsilon|}{\gamma} (1+\alpha_1) N_1^{2\delta} B_0, \quad \|h_2\|_{d+\beta,r+\beta} \le \frac{8}{3} C_3 \frac{|\varepsilon|}{\gamma} (1+\alpha_2) N_2^{2\delta} B_1,$$

where $\alpha_1 = \frac{1}{2} B_0 N_0^{-8\delta}$, $\alpha_2 = \frac{1}{2} B_0 N_0^{-16\delta}$.

From (10) we have $|\varepsilon| < y_2$, so

$$|\varepsilon| < \frac{3\gamma}{8C_3 \Big(1 + \frac{1}{2} N_0^{-8\delta} \Big(1 + \frac{|\varepsilon|}{\gamma} 2C_0 N_0^{14\delta}\Big)\Big)}.$$

Moreover, $|\varepsilon| < \frac{3\gamma}{8C_3(1+\alpha_1)}$. Then $\|h_1\|_{d+\beta,r+\beta} \le N_1^{2\delta} B_0$.

Similarly, using the formula (10), we have $|\varepsilon| < y_2$, so

$$|\varepsilon| < \frac{3\gamma}{8C_3 \Big(1 + \frac{1}{2} N_0^{-16\delta} \Big(1 + \frac{|\varepsilon|}{\gamma} 2C_0 N_0^{14\delta}\Big)\Big)},$$

then $|\varepsilon| < \frac{3\gamma}{8C_3(1+\alpha_2)}$ and $\|h_2\|_{d+\beta,r+\beta} \le N_2^{2\delta} B_1$. Hence, $B_{q+1} \le B_q + N_{q+1}^{2\delta} B_q = (1 + N_{q+1}^{2\delta}) B_q$, which finishes the proof. □

Proof of Lemma 5. For any $h \in G_{q+1}$ with regard for property $(P5)$, we have

$$\|H_{q+1}(h)\|_{d,r} \le \frac{2}{\gamma} N_{q+1}^{2\delta} \big(\|r_q\|_{d,r} + \|R_q(h)\|_{d,r}\big) \le 2C_0B_0 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} N_q^{-2\beta} N_{q+1}^{2\delta} + 4C_2 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} \|h\|_{d,r}^2.$$

In view of the value of β, equality $N_{q+1} = N_q^2$, and formula (10) and $|\varepsilon| < \varepsilon_0$, we can write the inequality

$$\|H_{q+1}(h)\|_{d,r} \le 2C_0B_0 \frac{|\varepsilon|}{\gamma} N_{q+1}^{-2\delta} + 4C_2 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} \rho_{q+1}^2 \le$$
$$\le 2C_3B_0 \frac{|\varepsilon|}{\gamma} N_{q+1}^{-2\delta} + 2C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} \rho_{q+1}^2 \le \frac{\rho_{q+1}}{2} + 8B_0 \Big(C_3 \frac{|\varepsilon|}{\gamma}\Big)^2 \rho_{q+1} \le \rho_{q+1}.$$

Therefore, the operator H_{q+1} maps G_{q+1} into itself.

Any functions $h,\ h' \in G_{q+1}$ satisfy the equality

$$H_{q+1}(h) - H_{q+1}(h') = -\mathcal{L}_{N_{q+1}}^{-1}(\vec{\varepsilon}, \vec{a}, u_q)\big(R_q(h) - R_q(h')\big),$$

then we can write the estimate

$$\|H_{q+1}(h) - H_{q+1}(h')\|_{d,r} \le \frac{2}{\gamma} N_{q+1}^{2\delta} \big\|R_q(h) - R_q(h')\big\|_{d,r} \le$$
$$\le 2C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} (\|h\|_{d,r} + 2\|h'\|_{d,r}) \|h - h'\|_{d,r} \le$$
$$\le 2C_3 \frac{|\varepsilon|}{\gamma} N_{q+1}^{2\delta} 12C_3B_0 \frac{|\varepsilon|}{\gamma} N_{q+1}^{-2\delta} \|h - h'\|_{d,r} \le 24B_0 \Big(C_3 \frac{|\varepsilon|}{\gamma}\Big)^2 \|h - h'\|_{d,r}.$$

Formula (10) implies that $|\varepsilon| < y_1$ and

$$|\varepsilon| < \frac{\gamma}{2\sqrt{6}C_3 \sqrt{1 + \frac{|\varepsilon|}{\gamma} 2C_0 N_0^{14\delta}}}.$$

Hence, $|\varepsilon| < \frac{\gamma}{2C_3\sqrt{6B_0}}$. Thus, the operator H_{q+1} is contractive. □

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