R. R. Salimov, B. A. Klishchuk

AN EXTREMAL PROBLEM FOR VOLUME FUNCTIONALS


We consider the class of ring $Q$-homeomorphisms with respect to $p$-modulus in $\mathbb{R}^n$ with $p > n$, and obtain a lower bound for the volume of images of a ball under such mappings. In particular, the following theorem is proved in the paper: Let $D$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ and let $f: D \to \mathbb{R}^n$ be a ring $Q$-homeomorphism with respect to $p$-modulus at a point $x_0 \in D$ with $p > n$, and the function $Q$ satisfies the condition $q_{x_0}(t) \leq q_0 t^{-\alpha}$, $q_0 \in (0, \infty)$, $\alpha \in [0, \infty)$ for a.e. $t \in (0, d_0)$, $d_0 = \text{dist}(x_0, \partial D)$. Then for all $r \in (0, d_0)$ the estimate

$$m(fB(x_0, r)) \geq \Omega_n \left( \frac{p-n}{\alpha + p-n} \right)^{n(p-1)} q_0^{-n} r^{\frac{n(n+p-n)}{p-n}},$$

holds, where $\Omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

In addition, in the paper it is solved an extremal problem on minimizing the volume functional of the image of a ball.

1. Introduction. The study of area distortions under quasiconformal mappings has been originated by B. Bojarskii, see [1]. For other results in this direction, we refer to the works [2]–[4]. An upper bound for the area of the image of a disc under quasiconformal mappings was established by M.A. Lavrentiev in his monograph [5]. In [6, Proposition 3.7] the Lavrentiev inequality was refined in terms of the angular dilatation. Also in [7] and [8] there were obtained upper bounds for the area distortion for the ring and lower $Q$-homeomorphisms. V. Kruglikov has obtained a bound for the measure of the image of a ball for mappings which are quasiconformal in the mean in $\mathbb{R}^n$ (see Lemma 9 in [9]). In [10] there was established an upper bound for the measure of a ball image for the ring $Q$-homeomorphisms with respect to $p$-modulus ($1 < p \leq n$). In this paper, we study the ring homeomorphisms with respect to $p$-modulus in $\mathbb{R}^n$ with $p > n$. For these mappings we derive a lower bound for the measure of a ball image and solve an extremal problem on minimizing its volume functional. Note that the corresponding case for $n = 2$ has been studied in [11]–[13].

Let us recall some definitions. Let $\Gamma$ be a family of curves $\gamma$ in $\mathbb{R}^n$, $n \geq 2$. A Borel measurable function $\rho: \mathbb{R}^n \to [0, \infty]$ is called admissible for $\Gamma$, (abbr. $\rho \in \text{adm} \Gamma$), if

$$\int_{\gamma} \rho(x) \, ds \geq 1$$

2010 Mathematics Subject Classification: 30C65.

Keywords: ring $Q$-homeomorphism; $p$-modulus of a family of curves; quasiconformal mapping; condenser; $p$-capacity of a condenser; volume functional.

doi:10.15330/ms.50.1.36-43

© R. R. Salimov, B. A. Klishchuk, 2018
for any curve $\gamma \in \Gamma$. Let $p \in (1, \infty)$.

The quantity

$$M_p(\Gamma) = \inf_{\rho \in \adm \Gamma} \int_{\mathbb{R}^n} \rho^p(x) \, dm(x).$$

is called $p$-modulus of the family $\Gamma$.

For arbitrary sets $E$, $F$ and $G$ of $\mathbb{R}^n$ we denote by $\Delta(E, F, G)$ a set of all continuous curves $\gamma: [a, b] \to \mathbb{R}^n$, that connect $E$ and $F$ in $G$, i.e., such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$.

Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$ and $d_0 = \dist(x_0, \partial D)$. Set

$$\mathbb{A}(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\},$$

$$S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}, \quad i = 1, 2.$$

Let a function $Q: D \to [0, \infty]$ be Lebesgue measurable. We say that a homeomorphism $f: D \to \mathbb{R}^n$ is ring $Q$-homeomorphism with respect to $p$-modulus at $x_0 \in D$, if the relation

$$M_p(\Delta(f S_1, f S_2, f D)) \leq \int_{\mathbb{A}} Q(x) \eta^p(|x - x_0|) \, dm(x)$$

holds for any ring $\mathbb{A} = \mathbb{A}(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0$; $d_0 = \dist(x_0, \partial D)$, and for any measurable function $\eta: (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr = 1.$$

Denote by $\omega_{n-1}$ the area of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ in $\mathbb{R}^n$ and by

$$q_{x_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(x_0, r)} Q(x) \, d\mathbb{A}$$

the integral mean over the sphere $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$, where $d\mathbb{A}$ is the element of the surface area.

Now we formulate a criterion which guarantees for a homeomorphism to be the ring $Q$-homeomorphisms with respect to $p$-modulus for $p > 1$ in $\mathbb{R}^n$, $n \geq 2$.

**Proposition 1.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q: D \to [0, \infty]$ be a Lebesgue measurable function such that $q_{x_0}(r) \neq \infty$ for a.e. $r \in (0, d_0)$, $d_0 = \dist(x_0, \partial D)$. A homeomorphism $f: D \to \mathbb{R}^n$ is ring $Q$-homeomorphism with respect to $p$-modulus at a point $x_0 \in D$ if and only if the inequality

$$M_p(\Delta(f S_1, f S_2, f \mathbb{A})) \leq \frac{\omega_{n-1}}{\int_{r_1}^{r_2} \frac{dr}{q_{x_0}^{p-1}(r)}}$$

holds for any $0 < r_1 < r_2 < d_0$ (cf. [10], Theorem 2.3).

Following paper [14], a pair $\mathcal{E} = (A, C)$, where $A \subset \mathbb{R}^n$ is an open set and $C$ is a nonempty compact set contained in $A$, is called condenser. We say that a condenser $\mathcal{E} = (A, C)$ lies
in a domain $D$ if $A \subseteq D$. Clearly, if $f : D \to \mathbb{R}^n$ is a homeomorphism and $\mathcal{E} = (A, C)$ is a condenser in $D$, then $(fA, fC)$ is also condenser in $fD$. Further, we denote $f\mathcal{E} = (fA, fC)$.

Let $\mathcal{E} = (A, C)$ be a condenser. Denote by $\mathcal{C}_0(A)$ a set of continuous functions $u : A \to \mathbb{R}^1$ with compact support. Let $\mathcal{W}_0(\mathcal{E}) = \mathcal{W}_0(A, C)$ be a family of nonnegative functions $u : A \to \mathbb{R}^1$ such that 1) $u \in \mathcal{C}_0(A)$, 2) $u(x) > 1$ for $x \in C$ and 3) $u$ belongs to the class ACL and

$$|\nabla u| = \left( \sum_{i=1}^n \left( \frac{\partial u}{\partial x_i} \right)^2 \right)^{\frac{1}{2}}.$$ 

For $p \geq 1$ the quantity

$$\text{cap}_p \mathcal{E} = \text{cap}_p (A, C) = \inf_{u \in \mathcal{W}_0(\mathcal{E})} \int_A |\nabla u|^p \, dm(x)$$

is called $p$-capacity of the condenser $\mathcal{E}$. It is known that

$$\text{cap}_p \mathcal{E} = M_p(\Delta(A, C; A \setminus C)), \quad p > 1; \tag{1}$$

see in [15, Theorem 1]. For $p \geq 1$ the inequality

$$\text{cap}_p \mathcal{E} \geq \left( \inf m_{n-1}\sigma \right)^p \left[ m(A \setminus C) \right]^{p-1} \tag{2}$$

holds, where $m_{n-1}\sigma$ denotes the $(n - 1)$-dimensional Lebesgue measure of the $C^\infty$-manifold $\sigma$ that is the boundary $\sigma = \partial U$ of bounded open set $U$ containing $C$ and contained with its closure $\overline{U}$ in $A$. Here the infimum is taken over all such $\sigma$ (see [9]).

The lower estimates for the image of a ball obtained in this paper generalize the known Gehring result [17, Lemma 7] for a disk $E = B(x_0, r)$.

**Lemma G.** Let $D$ and $D'$ be bounded domains in $\mathbb{R}^n$ and let $E$ be an arbitrary Borel measurable subset of $D$. Assume that $f : D \to D'$ is a homeomorphism such that

$$\text{cap}_p f\mathcal{E} \leq K \text{cap}_p \mathcal{E}$$

for $p > n$, where $\mathcal{E} = \left( B(x_0, r_2), \overline{B(x_0, r_1)} \right)$, $x_0 \in D$, $0 < r_1 < r_2 < d_0$, $d_0 = \text{dist}(x_0, \partial D)$. Then

$$m(fE) \geq K^{\frac{n}{n-p}} m(E).$$

2. **Distortion of the volume of a ball.** In the following theorem, we provide a lower bound for the volume of the image of a ball under the ring $Q$-homeomorphisms with respect to $p$-modulus with $p > n$.

**Theorem 1.** Let $D$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ and let $f : D \to \mathbb{R}^n$ be a ring $Q$-homeomorphism with respect to $p$-modulus at a point $x_0 \in D$ with $p > n$. Assume that the function $Q$ satisfies the condition

$$q_{x_0}(t) \leq q_0 \, t^{-\alpha}, \quad q_0 \in (0, \infty), \quad \alpha \in [0, \infty) \tag{3}$$
for a.e. \( t \in (0, d_0) \), \( d_0 = \text{dist}(x_0, \partial D) \). Then for all \( r \in (0, d_0) \) the estimate

\[
m(fB(x_0, r)) \succeq \Omega_n \left( \frac{p-n}{\alpha + p-n} \right) \frac{n(p-1)^{n(p-1)}}{q_0^{p-n} r^{n(p-n)}} ,
\]

holds, where \( \Omega_n \) is a volume of the unit ball in \( \mathbb{R}^n \).

**Proof.** Consider a condenser \( \mathcal{E} = (A, C) \), where \( A = \{ x \in D : |x - x_0| < t + \Delta t \} \) and \( C = \{ x \in D : |x - x_0| \leq t \} \), \( t + \Delta t < d_0 \). Then \( f\mathcal{E} = (fA, fC) \) is a ringlike condenser in \( fD \) and by (1)

\[
\text{cap}_p f\mathcal{E} = M_p(\Delta(\partial fA, \partial fC; f(A \setminus C))).
\]

In view of (2) we have

\[
\text{cap}_p f\mathcal{E} \geq \frac{(\inf m_{n-1}\sigma)^p}{[m(fA \setminus fC)]^{p-1}}, \tag{4}
\]

where \( m_{n-1}\sigma \) denotes the \((n-1)\)-dimensional Lebesgue measure of the \( C^\infty\)- manifold \( \sigma = \partial U \) of a bounded open set \( U \) containing \( fC \) and contained with its closure \( \overline{U} \) in \( fA \), and the infimum is taken over all such \( \sigma \).

On the other hand, by Proposition 1,

\[
\text{cap}_p f\mathcal{E} \leq \frac{\omega_{n-1}}{(t + \Delta t \int_t^{t + \Delta t} \frac{ds}{s^{p-1} q_0^{p-1}(s)}})^{p-1},
\]

and applying the growth condition (3) one gets

\[
\text{cap}_p f\mathcal{E} \leq \omega_{n-1} q_0 \left( \frac{p-n+\alpha}{p-1} \right)^{p-1} \left( t + \Delta t \right)^{p-n+\alpha \over p-1} - t^{p-n+\alpha \over p-1} \right]^{1-p}. \tag{5}
\]

Combining the inequalities (4) and (5), we obtain

\[
\frac{(\inf m_{n-1}\sigma)^p}{[m(fA \setminus fC)]^{p-1}} \leq \omega_{n-1} q_0 \left( \frac{p-n+\alpha}{p-1} \right)^{p-1} \left( t + \Delta t \right)^{p-n+\alpha \over p-1} - t^{p-n+\alpha \over p-1} \right]^{1-p}.
\]

and, therefore,

\[
\inf m_{n-1}\sigma \leq \omega_{n-1} q_0 \left( \frac{p-n+\alpha}{p-1} \right)^{p-1} \left( t + \Delta t \right)^{p-n+\alpha \over p-1} - t^{p-n+\alpha \over p-1} \right]^{1-p} \left( t + \Delta t \right)^{p-n+\alpha \over p-1} \right]^{1-p} m(fA \setminus fC)^{p-1}.
\]

Estimating the left-hand side by the isoperimetric inequality

\[
\inf m_{n-1}\sigma \geq n \cdot \Omega_n^{\frac{1}{n}} (m(fC))^{\frac{n-1}{n}},
\]

we conclude that

\[
n \cdot \Omega_n^{\frac{1}{n}} (m(fC))^{\frac{n-1}{n}} \leq \omega_{n-1} q_0 \left( \frac{p-n+\alpha}{p-1} \right)^{p-1} \left( t + \Delta t \right)^{p-n+\alpha \over p-1} - t^{p-n+\alpha \over p-1} \right]^{1-p} \left( t + \Delta t \right)^{p-n+\alpha \over p-1} \right]^{1-p} m(fA \setminus fC)^{p-1} \left( t + \Delta t \right)^{p-n+\alpha \over p-1} - t^{p-n+\alpha \over p-1} \right]^{1-p}. \tag{6}
\]

Put \( \Phi(t) = m(fB_t) \). Then relation (6) has a form
Taking into account \( \omega_{n-1} = n \Omega_n \), the last relation can be rewritten as

\[
\frac{n}{q_0^\frac{1}{p}} \frac{\frac{1}{p} \Omega_{n-1}^{\frac{1}{p}}}{} \Phi^{\frac{n-1}{p}}(t) \leq \left( \frac{p - n + \alpha}{p - 1} \right)^{\frac{p-1}{p}} \left( \frac{\Phi((t + \Delta t) - \Phi(t))}{\Delta t} \right) \frac{p-1}{p}.
\]

The function \( \Phi(t) \) monotonously increases for a.e. \( t \) which implies the existence of derivative \( \Phi'(t) \). Therefore, from (7) we conclude as \( \Delta t \to 0 \)

\[
\frac{n}{q_0^\frac{1}{p}} \frac{\frac{1}{p} \Omega_{n-1}^{\frac{1}{p}}}{} \Phi^{\frac{n-1}{p}}(t) \leq \left( \frac{p - n + \alpha}{p - 1} \right)^{\frac{p-1}{p}} \left( \frac{\Phi((t + \Delta t) - \Phi(t))}{\Delta t} \right) \frac{p-1}{p}.
\]

Inequality (8) easily implies that

\[
\frac{n}{q_0^\frac{1}{p}} \frac{\frac{1}{p} \Omega_{n-1}^{\frac{1}{p}}}{n} \cdot t^{\frac{p-1}{n}} \leq \left( \frac{\Phi^{(p,n-1)}(t)}{n} \right)'.
\]

Note that the function

\[
g(t) = \frac{\Phi^{\frac{p-n}{n(p-1)}}(t)}{\frac{p-n}{n(p-1)}}
\]

is nondecreasing for \( p > n \). Integrating both parts of the last inequality over \( t \in [\varepsilon, r] \) and taking into account that

\[
\int_\varepsilon^r \left( \frac{\Phi^{\frac{p-n}{n(p-1)}}(t)}{n} \right)' \, dt = \int_\varepsilon^r g'(t) \, dt \leq g(r) - g(\varepsilon) \leq \frac{\Phi^{\frac{p-n}{n(p-1)}}(r) - \Phi^{\frac{p-n}{n(p-1)}}(\varepsilon)}{n}
\]

see, e.g., [16, Theorem IV. 7.4], we obtain

\[
\frac{n}{q_0^\frac{1}{p}} \frac{\frac{1}{p} \Omega_{n-1}^{\frac{1}{p}}}{n} \cdot t^{\frac{p-1}{n}} \leq \frac{\Phi^{\frac{p-n}{n(p-1)}}(r) - \Phi^{\frac{p-n}{n(p-1)}}(\varepsilon)}{n}
\]

Putting \( \varepsilon \to 0 \) in (9) and taking into account that \( \Phi(0) = 0 \), we get the estimate

\[
\Phi(r) \geq \Omega_n \left( \frac{p - n}{\alpha + p - n} \right)^{\frac{n(p-1)}{p-n}} q_0^\frac{n}{p} r^{\frac{n}{n(p-1)}}
\]

Finally, denoting \( \Phi(r) = m(fB(x_0, r)) \) in (10), one gives

\[
m(fB(x_0, r)) \geq \Omega_n \left( \frac{p - n}{\alpha + p - n} \right)^{\frac{n(p-1)}{p-n}} q_0^\frac{n}{p} r^{\frac{n}{n(p-1)}}
\]

This completes the proof of Theorem 1.

Let \( Q : \mathbb{B}^n \rightarrow [0, \infty] \) be a Lebesgue measurable function, \( p > n, \ q_0 \in (0, \infty) \) and \( \alpha \in [0, \infty) \). Assume that \( \mathcal{H} = \mathcal{H}(p, q_0, \alpha) \) be a set of all ring \( Q \)-homeomorphisms \( f : \mathbb{B}^n \rightarrow \mathbb{R}^n \) with respect to the \( p \)-modulus at the point \( x_0 = 0 \) satisfying the condition

\[
q(t) = \frac{1}{\omega_{n-1} t^{n-1}} \int_{S_t} Q(x) \, d\mathcal{A} \leq q_0 \ t^{-\alpha}
\]

for a.e. \( t \in (0, 1) \). Consider a volume functional \( V_r(f) = m(fB_r) \) on the class \( \mathcal{H} \). We prove a result on minimization of the functional \( V_r(f) \).

**Theorem 2.** The equality

\[
\min_{f \in \mathcal{H}} V_r(f) = \Omega_n \left( \frac{p - n}{\alpha + p - n} \right)^{\frac{n(p-1)}{p-n}} q_0^{\frac{n}{p-n}} r^{\frac{n(\alpha + p - n)}{p-n}},
\]

holds for all \( r \in [0, 1] \), where \( \Omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \).

**Proof.** The estimate

\[
V_r(f) \geq \Omega_n \left( \frac{p - n}{\alpha + p - n} \right)^{\frac{n(p-1)}{p-n}} q_0^{\frac{n}{p-n}} r^{\frac{n(\alpha + p - n)}{p-n}}.
\]

is an immediate consequence from Theorem 1.

Let us construct a homeomorphism \( f_0 \in \mathcal{H} \) at which the functional \( V_r(f) \) attains its minimum.

Define \( f_0 : \mathbb{B}^n \rightarrow \mathbb{R}^n \) by

\[
f_0(x) = \begin{cases} 
q_0^{\frac{1}{n-p}} \left( \frac{p-n}{\alpha + p - n} \right)^{\frac{p-1}{p-n}} |x|^{\frac{\alpha + p - n}{p-n}} \frac{x}{|x|}, & x \neq 0 \\
0, & x = 0. 
\end{cases}
\]

It can be easily seen that the estimate (11) is sharp and it becomes an equality for the mapping \( f_0 \).

Let us show that the mapping \( f_0 \) is a ring \( Q \)-homeomorphism with respect to \( p \)-modulus with the function \( Q(x) = q_0 |x|^{-\alpha} \) at the point \( x_0 = 0 \). Clearly, \( q_{2x_0}(t) = q_0 t^{-\alpha} \). Consider a ring \( \mathcal{A}(0, r_1, r_2), 0 < r_1 < r_2 < 1 \). Note that the mapping \( f_0 \) maps the ring \( \mathcal{A}(0, r_1, r_2) \) onto the ring \( \mathcal{A}(0, \tilde{r}_1, \tilde{r}_2) \), where

\[
\tilde{r}_i = q_0^{\frac{1}{n-p}} \left( \frac{p-n}{\alpha + p - n} \right)^{\frac{p-1}{p-n}} r_i^{\frac{\alpha + p - n}{p-n}}, \quad i = 1, 2.
\]

Denote by \( \Gamma \) a set of all curves joining the spheres \( S(0, r_1) \) and \( S(0, r_2) \) in the ring \( \mathcal{A}(0, r_1, r_2) \). Then we can calculate \( p \)-modulus of the family of curves \( f_0 \Gamma \) in implicit form:

\[
M_p(f_0 \Gamma) = \omega_{n-1} \left( \frac{p - n}{p - 1} \right)^{p-1} \left( \frac{p-n}{\tilde{r}_2^{p-1} - \tilde{r}_1^{p-1}} \right)^{1-p}.
\]

(see, e.g., relation (2) in [17]).
Substituting in the above equality the values $e_r^1$ and $e_r^2$, defined above, we get

$$M_p(f_0\Gamma) = \omega_{n-1} q_0 \left( \frac{\alpha + p - n}{p - 1} \right)^{p-1} \left( \frac{r_2^{\alpha + p - n}}{r_1^{p - 1}} - r_1^{\alpha + p - n} \right)^{-1-p}.$$

Note that the last equality can be written by

$$M_p(f_0\Gamma) = \omega_{n-1} \left( \frac{r_2^p}{\int_{r_1}^{r_2} \frac{dt}{t^{\alpha + p - n} q_0(t)}} \right)^{p-1},$$

where $q_{x_0}(t) = q_0 t^{-\alpha}$.

Hence, by Proposition 1, the homeomorphism $f_0$ is a ring $Q$-homeomorphism with respect to $p$-modulus for $p > n$ with the function $Q(x) = q_0 |x|^{-\alpha}$ at the point $x_0 = 0$. This completes the proof.

The publication contains the results of studies conducted by President’s of Ukraine grant for competitive projects Ф75/30308 of the State Fund for Fundamental Research.

REFERENCES


Institute of Mathematics of the NAS of Ukraine
ruslan.salimov1@gmail.com

Institute of Mathematics of the NAS of Ukraine
kban1988@gmail.com

Received 27.05.2018