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R. R. Salimov, B. A. Klishchuk

AN EXTREMAL PROBLEM FOR VOLUME FUNCTIONALS

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We consider the class of ring Q-homeomorphisms with respect to p-modulus in $\mathbb{R}^n$ with $p > n$, and obtain a lower bound for the volume of images of a ball under such mappings. In particular, the following theorem is proved in the paper: Let D be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$ and let $f\colon D \to \mathbb{R}^n$ be a ring Q-homeomorphism with respect to p-modulus at a point $x_0 \in D$ with $p > n$, and the function Q satisfies the condition $q_{x_0}(t) \leqslant q_0\, t^{-\alpha}$, $q_0 \in (0,\infty)$, $\alpha \in [0,\infty)$ for a.e. $t \in (0, d_0)$, $d_0 = \operatorname{dist}(x_0, \partial D)$. Then for all $r \in (0, d_0)$ the estimate

$$m(fB(x_0,r)) \geqslant \Omega_n \left(\frac{p-n}{\alpha+p-n}\right)^{\frac{n(p-1)}{p-n}} q_0^{\frac{n}{n-p}}\, r^{\frac{n(\alpha+p-n)}{p-n}},$$

holds, where Ω_n is the volume of the unit ball in $\mathbb{R}^n$.

In addition, in the paper it is solved an extremal problem on minimizing the volume functional of the image of a ball.

1. Introduction. The study of area distortions under quasiconformal mappings has been originated by B. Bojarskii, see [1]. For other results in this direction, we refer to the works [2]–[4]. An upper bound for the area of the image of a disc under quasiconformal mappings was established by M.A. Lavrentiev in his monograph [5]. In [6, Proposition 3.7] the Lavrentiev inequality was refined in terms of the angular dilatation. Also in [7] and [8] there were obtained upper bounds for the area distortion for the ring and lower Q-homeomorphisms. V. Kruglikov has obtained a bound for the measure of the image of a ball for mappings which are quasiconformal in the mean in $\mathbb{R}^n$ (see Lemma 9 in [9]). In [10] there was established an upper bound for the measure of a ball image for the ring Q-homeomorphisms with respect to p-modulus ($1 < p \leqslant n$). In this paper, we study the ring homeomorphisms with respect to p-modulus in $\mathbb{R}^n$ with $p > n$. For these mappings we derive a lower bound for the measure of a ball image and solve an extremal problem on minimizing its volume functional. Note that the corresponding case for $n = 2$ has been studied in [11]–[13].

Let us recall some definitions. Let Γ be a family of curves γ in $\mathbb{R}^n$, $n \geqslant 2$. A Borel measurable function $\rho\colon \mathbb{R}^n \to [0,\infty]$ is called *admissible* for Γ, (abbr. $\rho \in \operatorname{adm}\Gamma$), if

$$\int_\gamma \rho(x)\, ds \;\geqslant\; 1$$

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for any curve $\gamma \in \Gamma$. Let $p \in (1, \infty)$.

The quantity

$$M_p(\Gamma) \;=\; \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x)\,.$$

is called *p–modulus* of the family Γ.

For arbitrary sets E, F and G of $\mathbb{R}^n$ we denote by $\Delta(E, F, G)$ a set of all continuous curves $\gamma\colon [a, b] \to \mathbb{R}^n$, that connect E and F in G, i. e., such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$.

Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, $x_0 \in D$ and $d_0 = \operatorname{dist}(x_0, \partial D)$. Set

$$\mathbb{A}(x_0, r_1, r_2) = \{x \in \mathbb{R}^n \colon r_1 < |x - x_0| < r_2\}\,,$$
$$S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n \colon\ |x - x_0| = r_i\}\,, \quad i = 1, 2.$$

Let a function $Q\colon D \to [0, \infty]$ be Lebesgue measurable. We say that a homeomorphism $f\colon D \to \mathbb{R}^n$ is ring Q-homeomorphism with respect to p-modulus at $x_0 \in D$, if the relation

$$M_p(\Delta(fS_1, fS_2, fD)) \;\leqslant\; \int\limits_{\mathbb{A}} Q(x)\, \eta^p(|x - x_0|)\, dm(x)$$

holds for any ring $\mathbb{A} = \mathbb{A}(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0$, $d_0 = \operatorname{dist}(x_0, \partial D)$, and for any measurable function $\eta\colon (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1\,.$$

Denote by ω_{n-1} the area of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \colon\ |x| = 1\}$ in $\mathbb{R}^n$ and by

$$q_{x_0}(r) = \frac{1}{\omega_{n-1}\, r^{n-1}} \int\limits_{S(x_0, r)} Q(x)\, d\mathcal{A}$$

the integral mean over the sphere $S(x_0, r) = \{x \in \mathbb{R}^n \colon\ |x - x_0| = r\}$, where $d\mathcal{A}$ is the element of the surface area.

Now we formulate a criterion which guarantees for a homeomorphism to be the ring Q-homeomorphisms with respect to p-modulus for $p > 1$ in $\mathbb{R}^n$, $n \geqslant 2$.

Proposition 1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $Q\colon D \to [0, \infty]$ be a Lebesgue measurable function such that $q_{x_0}(r) \neq \infty$ for a.e. $r \in (0, d_0)$, $d_0 = \operatorname{dist}(x_0, \partial D)$. A homeomorphism $f\colon D \to \mathbb{R}^n$ is ring Q-homeomorphism with respect to p-modulus at a point $x_0 \in D$ if and only if the inequality*

$$M_p\left(\Delta(fS_1, fS_2, f\mathbb{A})\right) \leqslant \frac{\omega_{n-1}}{\left(\int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}}\, q_{x_0}^{\frac{1}{p-1}}(r)}\right)^{p-1}}$$

holds for any $0 < r_1 < r_2 < d_0$ (cf. [10], Theorem 2.3).

Following paper [14], a pair $\mathcal{E} = (A, C)$, where $A \subset \mathbb{R}^n$ is an open set and C is a nonempty compact set contained in A, is called *condenser*. We say that a condenser $\mathcal{E} = (A, C)$ lies in a domain D if $A\subset D$. Clearly, if $f\colon D\to\mathbb{R}^n$ is a homeomorphism and $\mathcal{E}=(A,C)$ is a condenser in D, then (fA,fC) is also condenser in fD. Further, we denote $f\mathcal{E}=(fA,fC)$.

Let $\mathcal{E}=(A,C)$ be a condenser. Denote by $\mathcal{C}_0(A)$ a set of continuous functions $u\colon A\to\mathbb{R}^1$ with compact support. Let $\mathcal{W}_0(\mathcal{E})=\mathcal{W}_0(A,C)$ be a family of nonnegative functions $u\colon A\to\mathbb{R}^1$ such that 1) $u\in\mathcal{C}_0(A)$, 2) $u(x)\geqslant 1$ for $x\in C$ and 3) u belongs to the class ACL and

$$|\nabla u|=\left(\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2\right)^{\frac{1}{2}}.$$

For $p\geqslant 1$ the quantity

$$\operatorname{cap}_p\mathcal{E}=\operatorname{cap}_p(A,C)=\inf_{u\in\mathcal{W}_0(\mathcal{E})}\int\limits_A|\nabla u|^p\,dm(x)$$

is called *p-capacity* of the condenser $\mathcal{E}$. It is known that

$$\operatorname{cap}_p\mathcal{E}=M_p(\Delta(\partial A,\partial C;A\setminus C)),\quad p>1;\tag{1}$$

see in [15, Theorem 1]. For $p\geqslant 1$ the inequality

$$\operatorname{cap}_p\mathcal{E}\geqslant\frac{(\inf\ m_{n-1}\sigma)^p}{[m(A\backslash C)]^{p-1}}\tag{2}$$

holds, where $m_{n-1}\sigma$ denotes the $(n-1)$-dimensional Lebesgue measure of the C^∞-manifold σ that is the boundary $\sigma=\partial U$ of bounded open set U containing C and contained with its closure $\overline{U}$ in A. Here the infimum is taken over all such σ (see [9]).

The lower estimates for the image of a ball obtained in this paper generalize the known Gehring result [17, Lemma 7] for a disk $E=B(z_0,r)$.

Lemma G. *Let D and D' be bounded domains in $\mathbb{R}^n$ and let E be an arbitrary Borel measurable subset of D. Assume that $f\colon D\to D'$ is a homeomorphism such that*

$$\operatorname{cap}_p f\mathcal{E}\ \leqslant\ K\operatorname{cap}_p\mathcal{E}$$

for $p>n$, where $\mathcal{E}=\left(B(x_0,r_2),\overline{B(x_0,r_1)}\right)$, $x_0\in D$, $0<r_1<r_2<d_0$, $d_0=\operatorname{dist}(x_0,\partial D)$. Then

$$m(fE)\ \geqslant\ K^{\frac{n}{n-p}}\,m(E)\,.$$

2. Distortion of the volume of a ball. In the following theorem, we provide a lower bound for the volume of the image of a ball under the ring Q-homeomorphisms with respect to p-modulus with $p>n$.

Theorem 1. *Let D be a bounded domain in $\mathbb{R}^n$, $n\geqslant 2$ and let $f\colon D\to\mathbb{R}^n$ be a ring Q-homeomorphism with respect to p-modulus at a point $x_0\in D$ with $p>n$. Assume that the function Q satisfies the condition*

$$q_{x_0}(t)\leqslant q_0\,t^{-\alpha},\ q_0\in(0,\infty)\,,\ \alpha\in[0,\infty)\tag{3}$$

for a.e. $t \in (0, d_0)$, $d_0 = \operatorname{dist}(x_0, \partial D)$. *Then for all* $r \in (0, d_0)$ *the estimate*

$$m(fB(x_0, r)) \geqslant \Omega_n \left(\frac{p-n}{\alpha+p-n}\right)^{\frac{n(p-1)}{p-n}} q_0^{\frac{n}{n-p}} r^{\frac{n(\alpha+p-n)}{p-n}},$$

holds, where Ω_n *is a volume of the unit ball in* $\mathbb{R}^n$.

Proof. Consider a condenser $\mathcal{E} = (A, C)$, where $A = \{x \in D\colon |x - x_0| < t + \Delta t\}$ and $C = \{x \in D\colon |x - x_0| \leqslant t\}$, $t + \Delta t < d_0$. Then $f\mathcal{E} = (fA, fC)$ is a ringlike condenser in fD and by (1)

$$\operatorname{cap}_p f\mathcal{E} = M_p\left(\Delta(\partial fA, \partial fC; f(A \setminus C))\right).$$

In view of (2) we have

$$\operatorname{cap}_p f\mathcal{E} \geqslant \frac{(\inf\ m_{n-1}\sigma)^p}{[m(fA \backslash fC)]^{p-1}}, \tag{4}$$

where $m_{n-1}\sigma$ denotes the $(n-1)$-dimensional Lebesgue measure of the C^∞-manifold σ that is the boundary $\sigma = \partial U$ of a bounded open set U containing fC and contained with its closure $\overline{U}$ in fA, and the infimum is taken over all such σ.

On the other hand, by Proposition 1,

$$\operatorname{cap}_p f\mathcal{E} \leqslant \frac{\omega_{n-1}}{\left(\int\limits_t^{t+\Delta t} \frac{ds}{s^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(s)}\right)^{p-1}},$$

and applying the growth condition (3) one gets

$$\operatorname{cap}_p f\mathcal{E} \leqslant \omega_{n-1}\, q_0 \left(\frac{p-n+\alpha}{p-1}\right)^{p-1} \left[(t+\Delta t)^{\frac{p-n+\alpha}{p-1}} - t^{\frac{p-n+\alpha}{p-1}}\right]^{1-p}. \tag{5}$$

Combining the inequalities (4) and (5), we obtain

$$\frac{(\inf\ m_{n-1}\sigma)^p}{[m(fA \backslash fC)]^{p-1}} \leqslant \omega_{n-1}\, q_0 \left(\frac{p-n+\alpha}{p-1}\right)^{p-1} \left[(t+\Delta t)^{\frac{p-n+\alpha}{p-1}} - t^{\frac{p-n+\alpha}{p-1}}\right]^{1-p}$$

and, therefore,

$$\inf\ m_{n-1}\sigma \leqslant \omega_{n-1}^{\frac{1}{p}}\, q_0^{\frac{1}{p}} \left(\frac{p-n+\alpha}{p-1}\right)^{\frac{p-1}{p}} \left[(t+\Delta t)^{\frac{p-n+\alpha}{p-1}} - t^{\frac{p-n+\alpha}{p-1}}\right]^{\frac{1-p}{p}} m(fA \backslash fC)^{\frac{p-1}{p}}.$$

Estimating the left-hand side by the isoperimetric inequality

$$\inf\ m_{n-1}\sigma \geqslant n \cdot \Omega_n^{\frac{1}{n}} \left(m(fC)\right)^{\frac{n-1}{n}},$$

we conclude that

$$n \cdot \Omega_n^{\frac{1}{n}} \left(m(fC)\right)^{\frac{n-1}{n}} \leqslant \omega_{n-1}^{\frac{1}{p}}\, q_0^{\frac{1}{p}} \left(\frac{p-n+\alpha}{p-1}\right)^{\frac{p-1}{p}} \left(\frac{m(fA \backslash fC)}{(t+\Delta t)^{\frac{p-n+\alpha}{p-1}} - t^{\frac{p-n+\alpha}{p-1}}}\right)^{\frac{p-1}{p}}. \tag{6}$$

Put $\Phi(t) := m(fB_t)$. Then relation (6) has a form

$$n\cdot\Omega_n^{\frac{1}{n}}\,\Phi^{\frac{n-1}{n}}(t)\leqslant\omega_{n-1}^{\frac{1}{p}}\,q_0^{\frac{1}{p}}\,\left(\frac{p-n+\alpha}{p-1}\right)^{\frac{p-1}{p}}\,\left(\frac{\frac{\Phi(t+\Delta t)-\Phi(t)}{\Delta t}}{\frac{(t+\Delta t)^{\frac{p-n+\alpha}{p-1}}-t^{\frac{p-n+\alpha}{p-1}}}{\Delta t}}\right)^{\frac{p-1}{p}}.$$

Taking into account $\omega_{n-1}=n\,\Omega_n$, the last relation can be rewritten as

$$\frac{n^{\frac{p-1}{p}}\,\Omega_n^{\frac{1}{n}-\frac{1}{p}}}{q_0^{\frac{1}{p}}}\,\Phi^{\frac{n-1}{n}}(t)\leqslant\left(\frac{p-n+\alpha}{p-1}\right)^{\frac{p-1}{p}}\,\left(\frac{\frac{\Phi(t+\Delta t)-\Phi(t)}{\Delta t}}{\frac{(t+\Delta t)^{\frac{p-n+\alpha}{p-1}}-t^{\frac{p-n+\alpha}{p-1}}}{\Delta t}}\right)^{\frac{p-1}{p}}. \tag{7}$$

The function $\Phi(t)$ monotonously increases for a.e. t which implies the existence of derivative $\Phi'(t)$. Therefore, from (7) we conclude as $\Delta t\to 0$

$$\frac{n\,\Omega_n^{\frac{p-n}{n(p-1)}}}{q_0^{\frac{1}{p-1}}}\cdot t^{\frac{\alpha-n+1}{p-1}}\leqslant\frac{\Phi'(t)}{\Phi^{\frac{p(n-1)}{n(p-1)}}(t)}. \tag{8}$$

Inequality (8) easily implies that

$$\frac{n\,\Omega_n^{\frac{p-n}{n(p-1)}}}{q_0^{\frac{1}{p-1}}}\cdot t^{\frac{\alpha-n+1}{p-1}}\leqslant\left(\frac{\Phi^{\frac{p-n)}{n(p-1)}}(t)}{\frac{p-n}{n(p-1)}}\right)'.$$

Note that the function

$$g(t)=\frac{\Phi^{\frac{p-n}{n(p-1)}}(t)}{\frac{p-n}{n(p-1)}}$$

is nondecreasing for $p>n$. Integrating both parts of the last inequality over $t\in[\varepsilon,r]$ and taking into account that

$$\int\limits_{\varepsilon}^{r}\left(\frac{\Phi^{\frac{p-n}{n(p-1)}}(t)}{\frac{p-n}{n(p-1)}}\right)'dt=\int\limits_{\varepsilon}^{r}g'(t)\,dt\leqslant g(r)-g(\varepsilon)\leqslant\frac{\Phi^{\frac{p-n}{n(p-1)}}(r)-\Phi^{\frac{p-n}{n(p-1)}}(\varepsilon)}{\frac{p-n}{n(p-1)}},$$

see, e.g., [16, Theorem IV. 7.4], we obtain

$$\frac{n\,\Omega_n^{\frac{p-n}{n(p-1)}}}{q_0^{\frac{1}{p-1}}}\int\limits_{\varepsilon}^{r}t^{\frac{\alpha-n+1}{p-1}}dt\leqslant\frac{\Phi^{\frac{p-n}{n(p-1)}}(r)-\Phi^{\frac{p-n}{n(p-1)}}(\varepsilon)}{\frac{p-n}{n(p-1)}}. \tag{9}$$

Putting $\varepsilon\to 0$ in (9) and taking into account that $\Phi(0)=0$, we get the estimate

$$\Phi(r)\geqslant\Omega_n\left(\frac{p-n}{\alpha+p-n}\right)^{\frac{n(p-1)}{p-n}}q_0^{\frac{n}{n-p}}\,r^{\frac{n(\alpha+p-n)}{p-n}}. \tag{10}$$

Finally, denoting $\Phi(r)=m(fB(x_0,r))$ in (10), one gives

$$m(fB(x_0,r))\geqslant\Omega_n\left(\frac{p-n}{\alpha+p-n}\right)^{\frac{n(p-1)}{p-n}}q_0^{\frac{n}{n-p}}\,r^{\frac{n(\alpha+p-n)}{p-n}}.$$

This completes the proof of Theorem 1. □

3. Extremal problem for the volume functional.

Let $Q\colon \mathbb{B}^n \to [0,\infty]$ be a Lebesgue measurable function, $p > n$, $q_0 \in (0,\infty)$ and $\alpha \in [0,\infty)$. Assume that $\mathcal{H} = \mathcal{H}(p, q_0, \alpha)$ be a set of all ring Q-homeomorphisms $f\colon \mathbb{B}^n \to \mathbb{R}^n$ with respect to the p-modulus at the point $x_0 = 0$ satisfying the condition

$$q(t) = \frac{1}{\omega_{n-1}\, t^{n-1}} \int\limits_{S_t} Q(x)\, d\mathcal{A} \leqslant q_0\, t^{-\alpha}$$

for a.e. $t \in (0,1)$. Consider a volume functional $\mathbf{V}_r(f) = m(fB_r)$ on the class $\mathcal{H}$. We prove a result on minimization of the functional $\mathbf{V}_r(f)$.

Theorem 2. *The equality*

$$\min_{f\in\mathcal{H}} \mathbf{V}_r(f) = \Omega_n \left(\frac{p-n}{\alpha+p-n}\right)^{\frac{n(p-1)}{p-n}} q_0^{\frac{n}{n-p}}\, r^{\frac{n(\alpha+p-n)}{p-n}},$$

holds for all $r \in [0,1]$, where Ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. The estimate

$$\mathbf{V}_r(f) \geqslant \Omega_n \left(\frac{p-n}{\alpha+p-n}\right)^{\frac{n(p-1)}{p-n}} q_0^{\frac{n}{n-p}}\, r^{\frac{n(\alpha+p-n)}{p-n}}. \tag{11}$$

is an immediate consequence from Theorem 1.

Let us construct a homeomorphism $f_0 \in \mathcal{H}$ at which the functional $\mathbf{V}_r(f)$ attains its minimum.

Define $f_0\colon \mathbb{B}^n \to \mathbb{R}^n$ by

$$f_0(x) = \begin{cases} q_0^{\frac{1}{n-p}} \left(\frac{p-n}{\alpha+p-n}\right)^{\frac{p-1}{p-n}} |x|^{\frac{\alpha+p-n}{p-n}} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0\,. \end{cases}$$

It can be easily seen that the estimate (11) is sharp and it becomes an equality for the mapping f_0.

Let us show that the mapping f_0 is a ring Q-homeomorphism with respect to p-modulus with the function $Q(x) = q_0\,|x|^{-\alpha}$ at the point $x_0 = 0$. Clearly, $q_{x_0}(t) = q_0\, t^{-\alpha}$. Consider a ring $\mathbb{A}(0, r_1, r_2)$, $0 < r_1 < r_2 < 1$. Note that the mapping f_0 maps the ring $\mathbb{A}(0, r_1, r_2)$ onto the ring $\widetilde{\mathbb{A}}(0, \widetilde{r}_1, \widetilde{r}_2)$, where

$$\widetilde{r}_i = q_0^{\frac{1}{n-p}} \left(\frac{p-n}{\alpha+p-n}\right)^{\frac{p-1}{p-n}} r_i^{\frac{\alpha+p-n}{p-n}}, \quad i = 1,\, 2.$$

Denote by Γ a set of all curves joining the spheres $S(0, r_1)$ and $S(0, r_2)$ in the ring $\mathbb{A}(0, r_1, r_2)$. Then we can calculate p-modulus of the family of curves $f_0\Gamma$ in implicit form:

$$M_p(f_0\Gamma) = \omega_{n-1} \left(\frac{p-n}{p-1}\right)^{p-1} \left(\widetilde{r}_2^{\frac{p-n}{p-1}} - \widetilde{r}_1^{\frac{p-n}{p-1}}\right)^{1-p}.$$

(see, e.g., relation (2) in [17]).

Substituting in the above equality the values $\widetilde{r}_1$ and $\widetilde{r}_2$, defined above, we get

$$M_p(f_0\Gamma) = \omega_{n-1}\, q_0 \left(\frac{\alpha+p-n}{p-1}\right)^{p-1} \left(r_2^{\frac{\alpha+p-n}{p-1}} - r_1^{\frac{\alpha+p-n}{p-1}}\right)^{1-p} .$$

Note that the last equality can be written by

$$M_p(f_0\Gamma) = \frac{\omega_{n-1}}{\left(\int\limits_{r_1}^{r_2} \frac{dt}{t^{\frac{n-1}{p-1}}\, q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{p-1}} ,$$

where $q_{x_0}(t) = q_0\, t^{-\alpha}$.

Hence, by Proposition 1, the homeomorphism f_0 is a ring Q-homeomorphism with respect to p-modulus for $p > n$ with the function $Q(x) = q_0\, |x|^{-\alpha}$ at the point $x_0 = 0$. This completes the proof. □

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Institute of Mathematics of the NAS of Ukraine
ruslan.salimov1@gmail.com

Institute of Mathematics of the NAS of Ukraine
kban1988@gmail.com

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