УДК 517.5

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# ON THE GROWTH OF LAPLACE-STIELTJES INTEGRALS

M. M. Sheremeta, A. O. Kuryliak. *On the growth of Laplace-Stieltjes integrals*, Mat. Stud. **50** (2018), 22–35.

In the paper it is investigated the growth of characteristics of Laplace-Stieltjes integrals $I(\sigma)=\int_0^{+\infty} f(x)dF(x)$, where $F$ is a nonnegative nondecreasing unbounded function continuous on the right on $[0,+\infty)$ and $f$ is a nonnegative on $[0,+\infty)$ function such that there exist $a\geq 0$, $b\geq 0$ and $h>0$: $\int_{x-a}^{x+b} f(t)dF(t)\geq hf(x)$ for all $x\geq a$. Assume that $\alpha,\beta$ are positive continuously differentiable functions increasing to $+\infty$ on $[0,+\infty)$ such that: a) $\alpha(cx)=(1+o(1))\alpha(x)$ $(x\to+\infty)$ for any $c>0$; b) $\beta(x(1+o(1)))=(1+o(1))\beta(x)$ $(x\to+\infty)$; c) $\frac{d\beta^{-1}(\alpha(x)/\varrho)}{d\ln x}=O(1)$ $(x\to+\infty)$ for every $\varrho\in(0,+\infty)$. The main results of the paper are contained in Theorems 5 and 7 and are derived from the following two statements of independent interest. If $F$ satisfies condition $\ln F(x)=o\Big(x\beta^{-1}\big(\frac{\alpha(x)}{\varrho}\big)\Big)$ $(x\to+\infty)$, then $\varrho_{\alpha\beta}(I)=k_{\alpha\beta}(f)$ (Theorem 1). If in additional the function $v(x)=-(\ln f(x))'$ is continuous and increasing on $[x_0,+\infty)$ and $\varrho_{\alpha\beta}(I)<+\infty$, then $\lambda_{\alpha\beta}(I)=\varkappa_{\alpha\beta}(f)$ (Theorem 2), where

$$\overline{\underline{\lim}}_{\sigma\to+\infty}\frac{\alpha(\ln I(\sigma))}{\beta(\sigma)}:=\begin{cases}\varrho_{\alpha\beta}(I),\\ \lambda_{\alpha\beta}(I),\end{cases}\qquad \overline{\underline{\lim}}_{x\to+\infty}\frac{\alpha(x)}{\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)}:=\begin{cases}k_{\alpha\beta}(f),\\ \varkappa_{\alpha\beta}(f).\end{cases}$$

Similar results are proved also for so called the modified generalized order and lower order.

**1. Introduction.** For an entire function

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

let $\varrho(f)$ be its order and $\sigma(f)$ be its type. Using Hadamard's formulas fo the finding of these characteristics, E.G. Calys ([1]) proved the following theorems.

**Theorem A.** *Suppose that entire functions $f_1(z)=\sum_{n=0}^{\infty}a_{n,1}z^n$ and $f_2(z)=\sum_{n=0}^{\infty}a_{n,2}z^n$ have finite orders and regular growth (in sense of the equality of order $\varrho[f]$ and lower order $\varrho[f]$) and the sequences $(|a_{n,1}/a_{n+1,1}|)$ and $(|a_{n,2}/a_{n+1,2}|)$ are non-decreasing for $n\geq n_0$. If*

$$\ln(1/|a_n|)=(1+o(1))\sqrt{\ln(1/|a_{n,1}|)\ln(1/|a_{n,2}|)},\quad n\to\infty,$$

*then the function $f$ has regular growth and $\varrho[f]=\sqrt{\varrho[f_1]\varrho[f_2]}$.*

2010 *Mathematics Subject Classification:* 30B50, 30D10, 30D15.
*Keywords:* Laplace–Stieltjes integral; regular growth; asymptotic estimate.
doi:10.15330/ms.50.1.22-35

**Theorem B.** *Suppose that functions $f_1$ and $f_2$ from Theorem A have the same order $\varrho[f_1] = \varrho[f_2] = \varrho \in (0, +\infty)$ and types $\sigma[f_1] = \sigma_1$, $\sigma[f_2] = \sigma_2$. Also suppose that $a_{n,1} \neq 0$ and $|a_{n,2}| \geq |a_{n,1}|/l(1/|a_{n,1}|)$ for all $n \geq n_0$, where $l$ is a slowly varying function. If*

$$|a_n| = (1 + o(1))\sqrt{|a_{n,1}||a_{n,2}|}, \quad n \to \infty,$$

*then the function $f$ has order $\varrho[f] = \varrho$ and type $\sigma[f] \leq \sqrt{\sigma_1\sigma_1}$.*

We remark that R.S.L. Srivastava ( [2,3]) tried to prove Theorem A without assumptions $a_{n,1} \neq 0$ and $|a_{n,2}| \geq |a_{n,1}|/l(1/|a_{n,1}|)$ for all $n \geq n_0$ and Theorem B without condition of the nondecrease of the sequences $(|a_{n,1}/a_{n+1,1}|)$ and $(|a_{n,2}/a_{n+1,2}|)$. On the fallaciousness of such statements it was indicated in Math. Rev., 1963, V.25, №2204, №2206.

In [4] Theorems A and B are transfered on entire Dirichlet series. Here we will obtain such theorems for Laplace-Stieltjes integrals.

Let $V$ be the class of which are nonnegative nondecreasing unbounded and continuous on the right functions $F$ on $[0, +\infty)$.

The Laplace–Stieltjes transform of a real-valued function $g$ is given, usually, by a Lebesgue–Stieltjes integral of the form $\int_0^{+\infty} e^{zx} dg(x)$. We write this transformation in a different form. For a nonnegative function $f$ on $[0, +\infty)$ the integral

$$I(\sigma) = \int_0^{\infty} f(x)e^{x\sigma} dF(x), \quad \sigma \in \mathbb{R}, \tag{1}$$

is called of Laplace-Stieltjes ( [5–7]). Integral (1) is a direct generalization of the ordinary Laplace integral $I(\sigma) = \int_0^{\infty} f(x)e^{x\sigma}dx$ and of the Dirichlet series $D(\sigma) = \sum_{n=0}^{\infty} a_n e^{\lambda_n \sigma}$ with nonnegative coefficients $a_n$ and exponents $\lambda_n$, $0 \leq \lambda_n \uparrow +\infty\,(n \to \infty)$, if we choose $F(x) = n(x) = \sum_{\lambda_n \leq x} 1$ and $f(\lambda_n) = a_n \geq 0$ for all $n \geq 0$ (see also [5,8]).

Let

$$\mu(\sigma) = \mu(\sigma, I) = \max\{f(x)e^{x\sigma} : x \geq 0\}, \quad \sigma \in \mathbb{R},$$

be the maximum of the integrand, $\sigma_c$ be the abscissa of convergence of the integral (1) and $\sigma_\mu$ be the abscissa of maximum of the integrand. Then ( [7, p.8])

$$\sigma_\mu = \varliminf_{x \to +\infty} \frac{1}{x} \ln \frac{1}{f(x)}$$

and if either $\ln F(x) = o(x)$ or $\ln F(x) = o(\ln f(x))$ as $x \to +\infty$ then ( [7, p.13]) $\sigma_c \leq \sigma_\mu$. Also we remark that if $\ln F(x) = O(x)$ as $x \to +\infty$ and $\sigma_\mu = +\infty$ then ( [7, p.11]) $\sigma_c = +\infty$.

To obtain the inequality $\sigma_c \geq \sigma_\mu$ we introduce as in [7, p.21] the concept of a regular variation of $f$ in regard to $F$. We say that a positive function $f$ has regular variation in regard to $F$ if there exist $a \geq 0$, $b \geq 0$ and $h > 0$ such that for all $x \geq a$

$$\int_{x-a}^{x+b} f(t)dF(t) \geq hf(x).$$

Then [7, p.21] if $F \in V$ and $f$ has regular variation in regard to $F$ then $\sigma_c \leq \sigma_\mu$. Thus, if $F \in V$ and $f$ has regular variation in regard to $F$ and either $\ln F(x) = o(x)$ or $\ln F(x) = o(\ln f(x))$ as $x \to +\infty$ then $\sigma_c = \sigma_\mu$.

Further we assume that $\sigma_c = \sigma_\mu = +\infty$.

**2. Generalized orders.** Let $L$ be the class of continuous increasing functions $\alpha$ such that $\alpha(x) \geq 0$ for $x \geq x_0$, $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$, and on $[x_0, +\infty)$ the function $\alpha$ increases to $+\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha(x(1+o(1))) = (1+o(1))\alpha(x)$ as $x \to +\infty$; further, $\alpha \in L_{si}$ if $\alpha \in L$ and for any $c > 0$ $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$. It is easy to see that $L_{si} \subset L^0$. Functions from $L_{si}$ are called slowly increasing. In future we will need the next lemma [9].

**Lemma 1.** *Let $\beta \in L$ and*

$$B(\delta) = \overline{\lim_{x\to+\infty}} \frac{\beta((1+\delta)x)}{\beta(x)}, \quad \delta > 0.$$

*Then in order that $\beta \in L^0$ it is necessary and sufficient that $B(\delta) \to 1$ as $\delta \to 0$.*

Let $\alpha \in L, \beta \in L$, and $G$ be an arbitrary function on $[\sigma_0, +\infty)$. The value

$$\varrho_{\alpha\beta}(G) = \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(G(\sigma))}{\beta(\sigma)} \tag{2}$$

is called a generalized order of $G$. If we choose $G(\sigma) = \ln I(\sigma)$ then from (2) we obtain the definition of the generalized order $\varrho_{\alpha\beta}(I)$ of the Laplace-Stieltjes integral (1). Also define

$$k_{\alpha\beta}(f) = \overline{\lim_{x\to+\infty}} \frac{\alpha(x)}{\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)}.$$

First we remark that if the functions $\alpha \in L^0$ and $\beta \in L^0$ are continuously differentiable and for every $\varrho \in (0, +\infty)$

$$\frac{d\beta^{-1}(\alpha(x)/\varrho)}{d\ln x} = O(1), \quad x \to +\infty, \tag{3}$$

then ( [7, p.77]) $\varrho_{\alpha\beta}(\ln\,\mu) = k_{\alpha\beta}(f)$, and if for every $\varrho \in (0, +\infty)$

$$\ln\,F(x) = o\left(x\beta^{-1}\left(\frac{\alpha(x)}{\varrho}\right)\right), \quad x \to +\infty, \tag{4}$$

then ( [7, p. 77]) $\varrho_{\alpha\beta}(I) \leq \varrho_{\alpha\beta}(\ln\,\mu)$. On the other hand, if $f$ has a regular variation in regard to $F$ then ( [7, p.81]) $\varrho_{\alpha\beta}(I) \geq \varrho_{\alpha\beta}(\ln\,\mu)$ for each $\alpha \in L^0$ and $\beta \in L$.

Thus, the following theorem is true.

**Theorem 1.** *Let $F \in V$, $f$ have regular variation in regard to $F$ and functions $\alpha \in L_{si}$ and $\beta \in L^0$ satisfy condition (3). If $F$ satisfies condition (4) then $\varrho_{\alpha\beta}(I) = k_{\alpha\beta}(f)$.*

Now we put

$$\lambda_{\alpha\beta}(I) = \underline{\lim_{\sigma\to+\infty}} \frac{\alpha(\ln\,I(\sigma))}{\beta(\sigma)}, \quad \lambda_{\alpha\beta}(\ln\mu) = \underline{\lim_{\sigma\to+\infty}} \frac{\alpha(\ln\,\mu(\sigma))}{\beta(\sigma)}, \quad \varkappa_{\alpha\beta}(f) = \underline{\lim_{x\to+\infty}} \frac{\alpha(x)}{\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)}.$$

**Proposition 1.** *If $\alpha \in L$ and $\beta \in L^0$ then $\lambda_{\alpha\beta}(\ln\,\mu) \geq \varkappa_{\alpha\beta}(f)$.*

Indeed, if $\varkappa_{\alpha\beta}(f) > 0$ then for each $\varkappa \in (0, \varkappa_{\alpha\beta}(f))$ and all $x \geq x_0 = x_0(\varkappa)$ we have $\ln f(x) \geq -x\beta^{-1}(\alpha(x)/\varkappa)$. Therefore, $\ln \mu(\sigma) \geq -x\beta^{-1}(\alpha(x)/\varkappa))+x\sigma$ for all $\sigma$ and $x \geq x_0$. Choosing $x = \alpha^{-1}(\varkappa\beta(\sigma-1)) \geq x_0$ for $\sigma \geq \sigma_0$ hence we obtain

$$\ln \mu(\sigma) \geq a^{-1}(\varkappa\beta(\sigma-1)) = a^{-1}(\varkappa(1+o(1))\beta(\sigma)), \quad \sigma \to +\infty.$$

Therefore, $\lambda_{\alpha\beta}(\ln \mu) \geq \varkappa$ and in view of the arbitrariness of $\varkappa$ we have $\lambda_{\alpha\beta}(\ln \mu) \geq \varkappa_{\alpha\beta}(f)$. If $\varkappa_{\alpha\beta}(f) = 0$ this inequality is obvious.

**Proposition 2.** *Let $\alpha \in L_{si}$, $\beta \in L^0$ and condition* (3) *hold. If the function $v(x) = -(\ln f(x))'$ is continuous and increasing on $[x_0, +\infty)$ then $\lambda_{\alpha\beta}(\ln \mu) \leq \varkappa_{\alpha\beta}(f)$.*

Indeed, since $v(x) = -(\ln f(x))'$ is continuous and increasing on $[x_0, +\infty)$, the function $\ln f(x) + \sigma x$ has the unique point $x$ of the maximum such that $\sigma = v(x)$, and $\ln \mu(\sigma) = \ln f(x) + \sigma x$, where $\sigma = v(x)$.

Suppose that $\varkappa_{\alpha\beta}(f) < +\infty$. Then for every $\varkappa > \varkappa_{\alpha\beta}(f)$ there exists a sequence $(x_k) \uparrow +\infty$ such that $\ln f(x_k) \leq -x_k\beta^{-1}(\alpha(x_k)/\varkappa))$. We put $\mu^*(\sigma) = f(x_k)e^{\sigma x_k}$. Since $\mu(\sigma) = f(x)e^{\sigma x}$ for $\sigma = v^{-1}(x)$ we have $\mu(\sigma_k) = \mu^*(\sigma_k)$ for $\sigma_k = v(x_k)$. Hence

$$\ln \mu(\sigma_k) = \ln \mu^*(\sigma_k) \leq \max_k\{-x_k\beta^{-1}(\alpha(x_k)/\varkappa) + x_k\sigma_k\} \leq \max_x\{-x\beta^{-1}(\alpha(x)/\varkappa) + x\sigma_k\}.$$

In view of (3)

$$(-x\beta^{-1}(\alpha(x)/\varkappa)) + x\sigma_k)' = -\beta^{-1}(\alpha(x)/\varkappa)) - \frac{d\beta^{-1}(\alpha(x)/k)}{d\ln x} + \sigma_k =$$
$$= -\beta^{-1}(\alpha(x)/\varkappa)) + \sigma_k + O(1), \quad x \to +\infty,$$

i. e. the function $-x\beta^{-1}(\alpha(x)/\varkappa)) + x\sigma_k$ attains its maximum at the point

$$x(\sigma_k) = \alpha^{-1}(\varkappa\beta(\sigma_k + O(1))),\ x \to +\infty,$$
$$\ln \mu(\sigma_k) \leq -\alpha^{-1}(\varkappa\beta(\sigma_k + O(1)))(\sigma_k + O(1)))) + \sigma_k\alpha^{-1}(\varkappa\beta(\sigma_k + O(1))) =$$
$$= O(\alpha^{-1}(\varkappa\beta(\sigma_k + O(1)))), \quad k \to \infty.$$

Since $\alpha \in L_{si}$ and $\beta \in L^0$, hence it follows that $\lambda_{\alpha\beta}(\ln \mu) \leq \varkappa$. In view of the arbitrariness of $\varkappa$ we have $\lambda_{\alpha\beta}(\ln \mu) \leq \varkappa_{\alpha\beta}(f)$. If $\varkappa_{\alpha\beta}(f) = +\infty$ this inequality is obvious.

**Proposition 3.** *If $\alpha \in L^0$, $\beta \in L$ and $f$ has regular variation in regard to $F$ then $\lambda_{\alpha\beta}(\ln \mu) \leq \lambda_{\alpha\beta}(I)$.*

Indeed, if $f$ has regular variation in regard to $F$ then ( [7, p.75])

$$\ln \mu(\sigma) \leq (1 + o(1)) \ln I(\sigma), \quad \sigma \to +\infty, \tag{5}$$

whence $\lambda_{\alpha\beta}(\ln \mu) \leq \lambda_{\alpha\beta}(I)$.

**Proposition 4.** *Let the functions $\alpha \in L^0$ and $\beta \in L^0$ satisfy condition* (3), *and the function $F \in V$ satisfies condition* (4). *If $\varrho_{\alpha\beta}(\ln \mu) < +\infty$ then $\lambda_{\alpha\beta}(\ln \mu) \geq \lambda_{\alpha\beta}(I)$.*

Indeed, since $\varrho_{\alpha\beta}(\ln\mu)<+\infty$, we have $k_{\alpha\beta}(f)=\varrho_{\alpha\beta}(\ln\mu)<+\infty$, that is $\ln f(x)\le -x\beta^{-1}(\alpha(x)/k))$ for some $k<+\infty$ and in view of (4) $\varlimsup\limits_{x\to+\infty}\frac{\ln F(x)}{\ln(1/f(x))}=0$. Therefore, [7, p.61]

$$I(\sigma)\le K(\varepsilon)\mu(\sigma/(1-\varepsilon))^{1-\varepsilon} \tag{6}$$

for every $\varepsilon\in(0,1)$ and all $\sigma\ge\sigma_0(\varepsilon)$. Hence,

$$\lambda_{\alpha\beta}(I)\le\lambda_{\alpha\beta}(\ln\mu)\varlimsup_{\sigma\to+\infty}\frac{\beta(\sigma/(1-\varepsilon))}{\beta(\sigma)}.$$

Since $\beta\in L^0$ by Lemma 1 $\varlimsup\limits_{x\to+\infty}\frac{\beta(\sigma/(1-\varepsilon))}{\beta(\sigma)}\to 1$ as $\varepsilon\to 0$. Thus, $\lambda_{\alpha\beta}(I)\le\lambda_{\alpha\beta}(\ln\mu)$.

Combining Propositions 1–4 we get the following theorem.

**Theorem 2.** *Let functions $\alpha\in L_{si}$ and $\beta\in L^0$ satisfy condition* (3) *and the function $F\in V$ satisfy condition* (4)*. Suppose that the function $f$ has a regular variation in regard to $F$ and $v(x)=-(\ln f(x))'$ is continuous and increasing on $[x_0,+\infty)$. If $\varrho_{\alpha\beta}(I)<+\infty$ then $\lambda_{\alpha\beta}(I)=\varkappa_{\alpha\beta}(f)$.*

**3. Modified generalized orders.** The values

$$\varrho^M_{\alpha\beta}(I)=\varlimsup_{\sigma\to+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\ln I(\sigma)}{\sigma}\right),\quad \lambda^M_{\alpha\beta}(I)=\varliminf_{\sigma\to+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\ln I(\sigma)}{\sigma}\right) \tag{7}$$

are called the modified generalized order and the modified lower generalized order of $I$, respectively. If in (7) we choose $\ln\mu(\sigma)$ instead $I(\sigma)$ then we obtain definitions of $\varrho^M_{\alpha\beta}(\ln\mu)$ and $\lambda^M_{\alpha\beta}(\ln\mu)$.

**Proposition 5.** *Let either $\alpha\in L_{si}$ and $\beta\in L^0$ or $\alpha\in L^0$ and $\beta\in L_{si}$, and the function $F\in V$ satisfies condition* (4)*. Then $\varrho^M_{\alpha\beta}(\ln\mu)=k_{\alpha\beta}(f)$.*

*Proof.* Suppose that $\varrho^M_{\alpha\beta}(\ln\mu)<+\infty$. Then for every $\varrho>\varrho^M_{\alpha\beta}(\ln\mu)$, all $\sigma\ge\sigma_0(\varrho)$ and $x\ge 0$ we obtain $\ln f(x)+\sigma x\le\ln\mu(\sigma)\le\sigma\alpha^{-1}(\varrho\beta(\sigma))$, that is $\ln f(x)\le\sigma\alpha^{-1}(\varrho\beta(\sigma))-\sigma x$. We choose $\sigma=\sigma(x)=\beta^{-1}(\alpha(\delta x)/\varrho)$ for an arbitrary $\delta\in(0,1)$. Then $\sigma(x)\ge\sigma_0(\varrho)$ for $x\ge x_0=x_0(\varrho,\delta)$ and $\ln f(x)\le-(1-\delta)x\beta^{-1}(\alpha(\delta x)/\varrho)$ for $x\ge x_0$, whence

$$\begin{aligned}k_{\alpha\beta}(f)&=\varlimsup_{x\to+\infty}\frac{\alpha(x)}{\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)}=\varlimsup_{x\to+\infty}\left(\frac{\alpha(\delta x)}{\beta\left(\frac{1}{(1-\delta)x}\ln\frac{1}{f(x)}\right)}\frac{\beta\left(\frac{1}{(1-\delta)x}\ln\frac{1}{f(x)}\right)}{\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)}\right)\frac{\alpha(x)}{\alpha(\delta x)}\le\\&\le\varrho\varlimsup_{x\to+\infty}\frac{\beta(x/(1-\delta))}{\beta(x)}\varlimsup_{x\to+\infty}\frac{\alpha(x)}{\alpha(\delta x)}.\end{aligned} \tag{8}$$

If $\alpha\in L_{si}$ and $\beta\in L^0$ then $\varlimsup\limits_{x\to+\infty}\frac{\alpha(x)}{\alpha(\delta x)}=1$ and by Lemma 1 $\varlimsup\limits_{x\to+\infty}\frac{\beta(x/(1-\delta))}{\beta(x)}\to 1$ as $\delta\to 0$. Hence in view of the arbitrariness of $\varrho$ we obtain

$$k_{\alpha\beta}(f)\le\varrho^M_{\alpha\beta}(\ln\mu). \tag{9}$$

If $\beta\in L_{si}$ and $\alpha\in L^0$ then $\varlimsup\limits_{x\to+\infty}\frac{\beta(x/(1-\delta))}{\beta(x)}=1$, and by Lemma 1 $\varlimsup\limits_{x\to+\infty}\frac{\alpha(x)}{\alpha(\delta x)}\to 1$ as $\delta\to 1$, and we again obtain inequality (9). If $\varrho^M_{\alpha\beta}(\ln\mu)=+\infty$ inequality (9) is obvious.

Now assume that $k_{\alpha\beta}(f)\neq\varrho^M_{\alpha\beta}(\ln\,\mu)$. Then in view of (8) $k_{\alpha\beta}(f)<\varrho^M_{\alpha\beta}(\ln\,\mu)$ and if we choose $k_{\alpha\beta}(f)<\varrho<\varrho^M_{\alpha\beta}(\ln\,\mu)$ then $\ln\,f(x)\leq -x\beta^{-1}(\alpha(x)/\varrho))$ for $x\geq x_0(\varrho)$, i. e.

$$\ln\,\mu(\sigma)\leq\max\left\{\max_{x\leq x_0(\varrho)}(\ln\,f(x)+x\sigma),\ \max_{x\geq x_0(\varrho)}(-x\beta^{-1}(\alpha(x)/\varrho))+x\sigma)\right\}\leq$$
$$\leq\max_{x\geq 0}(x(\sigma-\beta^{-1}(\alpha(x)/\varrho)))+O(\sigma),\quad\sigma\to+\infty.$$

Since $\ln\,\mu(\sigma)\to+\infty$ as $\sigma\to+\infty$, the function $x(\sigma-\beta^{-1}(\alpha(x)/\varrho))$ attains the maximum at the point $x=x(\sigma)$ such that $\sigma-\beta^{-1}(\alpha(x)/\varrho)>0$, that is $x(\sigma)\leq\alpha^{-1}(\varrho\beta(\sigma))$. Therefore,

$$\ln\mu(\sigma)\leq x(s)(\sigma-\beta^{-1}(\alpha(x(\sigma))/\varrho))+O(\sigma)\leq\sigma x(\sigma)+O(\sigma)\leq\sigma\alpha^{-1}(\varrho\beta(\sigma))+O(\sigma),\quad\sigma\to+\infty,$$

whence it follows that $\varrho^M_{\alpha\beta}(\ln\,\mu)\leq\varrho$. It is a contradiction that therefore, Proposition 5 is proved. □

**Proposition 6.** *Let $\alpha\in L^0$, $\beta\in L^0$, and $f$ have regular variation in regard to $F$. Then $\varrho^M_{\alpha\beta}(\ln\,\mu)=\varrho^M_{\alpha\beta}(I)$.*

Indeed, if $f$ has regular variation in regard to $F$ then from (5) we obtain $\varrho^M_{\alpha\beta}(\ln\,\mu)\leq$ $\leq\varrho^M_{\alpha\beta}(I)$. On the other hand, if $\varrho^M_{\alpha\beta}(\ln\,\mu)<+\infty$ then in view of Proposition 5, as in the proof of Proposition 4, we have (6) for every $\varepsilon\in(0,\,1)$ and all $\sigma\geq\sigma_0(\varepsilon)$. Since $\alpha\in L^0$ and $\beta\in L^0$, we have $\varrho^M_{\alpha\beta}(I)\leq\varrho^M_{\alpha\beta}(\ln\,\mu)$. Thus, $\varrho^M_{\alpha\beta}(\ln\,\mu)=\varrho^M_{\alpha\beta}(I)$.

Uniting Propositions 5 and 6 we get the following theorem.

**Theorem 3.** *Let either $\alpha\in L_{si}$ and $\beta\in L^0$ or $\alpha\in L^0$ and $\beta\in L_{si}$, the function $F\in V$ satisfies condition (4) and $f$ has regular variation in regard to $F$. Then $\varrho^M_{\alpha\beta}(I)=k_{\alpha\beta}(f)$.*

**Proposition 7.** *Let either $\alpha\in L_{si}$ and $\beta\in L^0$ or $\alpha\in L^0$ and $\beta\in L_{si}$. Then $\lambda^M_{\alpha\beta}(\ln\,\mu)\geq\varkappa_{\alpha\beta}(f)$. Moreover, if $v(x)=-(\ln\,f(x))'$ is continuous and increasing on $[x_0,+\infty)$ then $\lambda^M_{\alpha\beta}(\ln\,\mu)=\varkappa_{\alpha\beta}(f)$.*

Indeed, if $\varkappa_{\alpha\beta}(f)>0$ then for each $\varkappa\in(0,\,\varkappa_{\alpha\beta}(f))$ and all $x\geq x_0=x_0(\varkappa)$ as in the proof of Proposition 7 we obtain $\ln\,\mu(\sigma)\geq -x\beta^{-1}(\alpha(x)/\varkappa))+x\sigma$ for all $\sigma$ and $x\geq x_0$. Choosing $x=\alpha^{-1}(\varkappa\beta(\delta\sigma))$, where $0<\delta<1$, we have $\ln\,\mu(\sigma)\geq(1-\delta)\sigma a^{-1}(\varkappa\beta(\delta\sigma))$. Hence

$$\lambda^M_{\alpha\beta}(\ln\,\mu)\geq\varliminf_{\sigma\to+\infty}\frac{\alpha((1-\delta)a^{-1}(\varkappa\beta(\delta\sigma)))}{\beta(\sigma)}$$

If $\alpha\in L_{si}$ and $\beta\in L^0$ hence we have as above $\lambda^M_{\alpha\beta}(\ln\,\mu)\geq\varkappa\varliminf_{\sigma\to+\infty}\frac{\beta(\delta\sigma)}{\beta(\sigma)}\to\varkappa$ as $\delta\to 1$. If $\alpha\in L^0$ and $\beta\in L_{si}$ then

$$\lambda^M_{\alpha\beta}(\ln\,\mu)\geq\varkappa\varliminf_{\sigma\to+\infty}\frac{\alpha((1-\delta)a^{-1}(\varkappa\beta(\delta\sigma)))}{\varkappa\beta(\delta\sigma)}\frac{\beta(\delta\sigma)}{\beta(\sigma)}=\varkappa\varliminf_{t\to+\infty}\frac{\alpha((1-\delta)a^{-1}(t)}{t}\to\varkappa$$

as $\delta\to 0$. Therefore, $\lambda^M_{\alpha\beta}(\ln\,\mu)\geq\varkappa_{\alpha\beta}(f)$.

On the other hand, as in the proof Proposition 2 we obtain $\ln\,\mu(\sigma)=\ln\,f(x)+\sigma x$ for $\sigma=v(x)$. Supposing that $\varkappa_{\alpha\beta}(f)<+\infty$, for $\varkappa>\varkappa_{\alpha\beta}(f)$ and some sequence $(x_k)\uparrow+\infty$ as in the proof of Proposition 2 we obtain for $\sigma_k=v(x_k)$

$$\ln\,\mu(\sigma_k)\leq\max_x\{-x\beta^{-1}(\alpha(x)/\varkappa))+x\sigma_k\}=\max_x\{-x(\sigma_k-\beta^{-1}(\alpha(x)/\varkappa)))\}.$$

Hence, as in the proof of Proposition 5, it follows that

$$\ln\,\mu(\sigma_k) \le \sigma_k\alpha^{-1}(\varrho\beta(\sigma_k)) + O(\sigma_k), \quad k\to\infty,$$

whence it follows that $\lambda^M_{\alpha\beta}(\ln\,\mu) \le \varkappa$. In view of the arbitrariness of $\varkappa$ Proposition 7 is proved.

**Proposition 8.** *Let $\alpha\in L^0$, $\beta\in L^0$, $\varrho^M_{\alpha\beta}(\ln\,\mu) < +\infty$, the function $F\in V$ satisfies condition* (4) *and $f$ has regular variation in regard to $F$. Then $\lambda^M_{\alpha\beta}(\ln\,\mu) = \lambda^M_{\alpha\beta}(I)$.*

Indeed, since $f$ has regular variation in regard to $F$, from (5) we obtain $\lambda^M_{\alpha\beta}(\ln\,\mu) \le \lambda^M_{\alpha\beta}(I)$. On the other hand, since $\varrho^M_{\alpha\beta}(\ln\,\mu) < +\infty$, we have (6) for every $\varepsilon\in(0,\,1)$ and all $\sigma\ge\sigma_0(\varepsilon)$. Since $\alpha\in L^0$ and $\beta\in L^0$, hence $\lambda^M_{\alpha\beta}(I) \le \lambda^M_{\alpha\beta}(\ln\,\mu)$. Thus, $\lambda^M_{\alpha\beta}(\ln\,\mu) = = \lambda^M_{\alpha\beta}(I)$.

From Propositions 7 and 8 we obtain the following theorem.

**Theorem 4.** *Let either $\alpha\in L_{si}$ and $\beta\in L^0$ or $\alpha\in L^0$ and $\beta\in L_{si}$ and the function $F\in V$ satisfies condition* (4). *Suppose that $f$ has regular variation in regard to $F$ and $v(x) = -(\ln\,f(x))'\uparrow+\infty$ as $x_0\le x\to+\infty$. Then $\lambda^M_{\alpha\beta}(I) = \varkappa_{\alpha\beta}(f)$.*

**4. Analogues of Theorem A**. Let $LS(F)$ be the class of Laplace-Stieltjes integrals for which $\sigma_c = \sigma_\mu = +\infty$. Suppose that $I_j\in LS(F)$, $1\le j\le m$, and

$$I_j(\sigma) = \int_0^\infty f_j(x)e^{x\sigma}dF(x), \quad \sigma\in\mathbb{R}. \tag{10}$$

The following theorem is an analogue of Theorem A.

**Theorem 5.** *Let functions $\alpha\in L_{si}$ and $\beta\in L^0$ satisfy condition* (3) *and the function $F\in V$ satisfy condition* (4). *Suppose that all functions $f_j$ have regular variation in regard to $F$ and $v_j(x) = -(\ln\,f_j(x))'$ is continuous and increasing on $[x_0,+\infty)$. Also suppose that $f$ have regular variation in regard to $F$ and*

$$\beta\left(\frac{1}{x}\ln\,\frac{1}{f(x)}\right) = (1+o(1))\prod_{j=1}^m \beta\left(\frac{1}{x}\ln\,\frac{1}{f_j(x)}\right)^{\omega_j}, \quad x\to+\infty, \tag{11}$$

*where $\omega_j>0$ for $1\le j\le m$ and $\sum_{1\le j\le m}\omega_j = 1$.*

*If all integrals* (10) *have regular $\alpha\beta$-growth (i.e. $\lambda_{\alpha\beta}(I_j) = \varrho_{\alpha\beta}(I_j) < +\infty$) then integral* (1) *has regular $\alpha\beta$-growth and $\varrho_{\alpha\beta}(I) = \prod_{j=1}^m(\varrho_{\alpha\beta}(I_j))^{\omega_j}$.*

*Proof.* By Theorem 1 $\varrho_{\alpha\beta}(I_j) = k_{\alpha\beta}(f_j)$ and by Theorem 2 $\lambda_{\alpha\beta}(I_j) = \varkappa_{\alpha\beta}(f_j)$. Since $\lambda_{\alpha\beta}(I_j) = \varrho_{\alpha\beta}(I_j) = \varrho_j < 1$, we have $k_{\alpha\beta}(f_j) = \varkappa_{\alpha\beta}(f_j) = \varrho_j$, that is

$$\lim_{x\to+\infty}\frac{\alpha(x)}{\beta\left(\frac{1}{x}\ln\,\frac{1}{f_j(x)}\right)} = \varrho_j. \tag{12}$$

Therefore, from (11) we obtain

$$\lim_{x\to+\infty}\frac{1}{\alpha(x)}\beta\left(\frac{1}{x}\ln\,\frac{1}{f(x)}\right) = \lim_{x\to+\infty}\frac{1}{\alpha(x)}\prod_{j=1}^m\beta\left(\frac{1}{x}\ln\,\frac{1}{f_j(x)}\right)^{\omega_j} =$$

$$= \lim_{x\to+\infty}\prod_{j=1}^{m}\left(\frac{1}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right)^{\omega_j} = \prod_{j=1}^{m}\lim_{x\to+\infty}\left(\frac{1}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right)^{\omega_j} = \prod_{j=1}^{m}\left(\frac{1}{\varrho_j}\right)^{\omega_j},$$

i. e.

$$\prod_{j=1}^{m}\varrho_j^{\omega_j} = \lim_{x\to+\infty}\frac{\alpha(x)}{\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)} = k_{\alpha\beta}(f) = \varkappa_{\alpha\beta}(f). \tag{13}$$

By Propositions 2 and 1 $\lambda_{\alpha\beta}(\ln I) \geq \lambda_{\alpha\beta}(\ln \mu) \geq \varkappa_{\alpha\beta}(f)$ and by Theorem 1 $\varrho_{\alpha\beta}(\ln I) = k_{\alpha\beta}(f)$. Therefore, $\varrho_{\alpha\beta}(I) = \lambda_{\alpha\beta}(I) = \prod_{j=1}^{m}(r_{\alpha\beta}(I_j))^{\omega_j}$.

If we choose $\alpha(x) = \ln x$ and $\beta(x) = x$ for $x \geq x_0$ then from the definitions of $\varrho_{\alpha\beta}(I)$ and $\lambda_{\alpha\beta}(I)$ we obtain the definitions of the $R$-order $\varrho_R$ and lower $R$-order $\lambda_R$, respectively. Choosing some more $m = 2$ and $\omega_1 = \omega_2 = 1/2$, we get the following statement.

**Corollary 1.** *Let $F \in V$ and $\ln F(x) = o(x\ln x)$ as $x \to +\infty$. Suppose that the functions $f_j$, $j = 1, 2$, have regular variation in regard to $F$ and $v_j(x) = -(\ln f_j(x))'$ is continuous and increasing on $[x_0, +\infty)$. Also suppose that $f$ have regular variation in regard to $F$ and*

$$\ln\frac{1}{f(x)} = (1 + o(1))\sqrt{\ln\frac{1}{f_1(x)}\ln\frac{1}{f_2(x)}}, \quad x \to +\infty.$$

*If integrals $I_1$ and $I_2$ have regular $R$-growth (i.e. $\lambda_R(I_j) = \varrho_R(I_j) < +\infty$ for $j = 1, 2$) then integral (1) has regular $R$-growth and $\varrho_R(I) = \sqrt{\varrho_R(I_1)\varrho_R(I_1)}$.*

Using modified generalized orders we get the following theorem.

**Theorem 6.** *Let either $\alpha \in L_{si}$ and $\beta \in L^0$ or $\alpha \in L^0$ and $\beta \in L_{si}$, and the function $F \in V$ satisfy condition (4). Suppose that $f$ has regular variation in regard to $F$ and $v(x) = -(\ln f(x))'$ is continuous and increasing on $[x_0, +\infty)$. Also suppose that $f$ have regular variation in regard to $F$ and (11) holds.*

*If all integrals (10) have regular modified $\alpha\beta$-growth (i.e. $\lambda^M_{\alpha\beta}(I_j) = \varrho^M_{\alpha\beta}(I_j) < +\infty$) then integral (1) has regular modified $\alpha\beta$-growth and $\varrho^M_{\alpha\beta}(I) = \prod_{j=1}^{m}(\varrho^M_{\alpha\beta}(I_j))^{\omega_j}$.*

*Proof.* By Theorem 3 $\varrho^M_{\alpha\beta}(I_j) = k_{\alpha\beta}(f_j)$ and by Theorem 4 $\lambda^M_{\alpha\beta}(I_j) = \varkappa_{\alpha\beta}(f_j)$. Since $\lambda^M_{\alpha\beta}(I_j) = \varrho^M_{\alpha\beta}(I_j) = \varrho_j < 1$, we have $k_{\alpha\beta}(f_j) = \varkappa_{\alpha\beta}(f_j) = \varrho_j$, that is (12) holds. Therefore, as in the proof of Theorem 5 we obtain (13) from (11). By Propositions 7 and 6 $\lambda^M_{\alpha\beta}(\ln I) \geq \lambda^M_{\alpha\beta}(\ln \mu) \geq \varkappa_{\alpha\beta}(f)$ and by Theorem 3 $\varrho^M_{\alpha\beta}(\ln I) = k_{\alpha\beta}(f)$. Therefore, $\varrho^M_{\alpha\beta}(I) = \lambda_{\alpha\beta}(I) = \prod_{j=1}^{m}(r^M_{\alpha\beta}(I_j))^{\omega_j}$.

If we choose $\alpha(x) = \ln x$ and $\beta(x) = \ln x$ for $x \geq x_0$ then from the definitions of $\varrho_{\alpha\beta}(I)$ and $\lambda_{\alpha\beta}(I)$ we obtain the definitions of the logarithmic order $\varrho_l$ and lower logarithmic order $\lambda_l$, respectively. Since

$$\frac{1}{\ln\sigma}\ln\left(\frac{\ln I(\sigma))}{\sigma}\right) = \frac{\ln\ln I(\sigma)}{\ln\sigma} - 1,$$

for such function we have $\varrho^M_{\alpha\beta}(I) = \varrho_l(I)$ and $\lambda^M_{\alpha\beta}(I) = \lambda_l(I)$. Therefore, choosing $m = 2$ and $\omega_1 = \omega_2 = 1/2$, we get the following statement.

**Corollary 2.** *Let $F \in V$ and $\ln F(x) = o(x)$ as $x \to +\infty$. Suppose that the functions $f_j$, $j = 1, 2$, have regular variation in regard to $F$ and $v_j(x) = -(\ln f_j(x))'$ is continuous and increasing on $[x_0, +\infty)$. Also suppose that $f$ has regular variation in regard to $F$ and*

$$\ln\left(\frac{1}{x}\ln\frac{1}{f(x)}\right) = (1+o(1))\sqrt{\ln\left(\frac{1}{x}\ln\frac{1}{f_1(x)}\right)\ln\left(\frac{1}{x}\ln\frac{1}{f_2(x)}\right)}, \ x \to +\infty. \qquad (14)$$

*If integrals $I_1$ and $I_2$ have regular logarithmic growth (i.e. $\lambda_l(I_j) = \varrho_l(I_j) \in (1, +\infty)$ for $j = 1, 2$) then integral (1) has regular logarithmic growth and $\varrho_l(I) = \sqrt{(\varrho_l(I_1) - 1)(\varrho_l(I_1) - 1}$.*

Finally, if we choose $\alpha(x) = x$ and $\beta(x) = \ln x$ for $x \geq x_0$ then $\varrho^M_{\alpha\beta}(I) = T(I) := = \overline{\lim}_{\sigma\to+\infty} \frac{\ln I(\sigma)}{\sigma \ln \sigma}$ and $\lambda^M_{\alpha\beta}(I) = t(I) := \underline{\lim}_{\sigma\to+\infty} \frac{\ln I(\sigma)}{\sigma \ln \sigma}$, and we obtain the next corollary.

**Corollary 3.** *Let $F \in V$ and $\ln F(x) = o(x^2)$ as $x \to +\infty$. Suppose that the functions $f_j$, $j = 1, 2$, have regular variation in regard to $F$ and $v_j(x) = -(\ln f_j(x))'$ is continuous and increasing on $[x_0, +\infty)$. Also suppose that $f$ has regular variation in regard to $F$ and (14) holds. If $t(I_j) = T(I_j) < +\infty$ for $j = 1, 2$ then for integral (1) $t(I) = T(I) = \sqrt{T(I_1)T(I_1)}$.*

**5. Analogues of Theorem B.** Since $\varrho_{\alpha\beta}(I) = \overline{\lim}_{\sigma\to+\infty} \frac{\ln \exp\{\alpha(\ln I(\sigma))\}}{\ln \exp\{\beta(\sigma)\}}$, we define the generalized type $T_{\alpha\beta}(I)$ of integral (1) by the formula

$$T_{\alpha\beta}(I) = \overline{\lim_{\sigma\to+\infty}} \frac{\exp\{\alpha(\ln I(\sigma))\}}{\exp\{\varrho\beta(\sigma)\}}, \quad (\varrho = \varrho_{\alpha\beta}(I)).$$

Theorem 1 implies the following lemma.

**Lemma 2.** *Suppose that the functions $\alpha \in L$ and $\beta \in L$ are continuously differentiable, $x\alpha'(x) = o(1)$, $x\beta'(x) = O(1)$ as $x \to +\infty$, and for every $c \in (-\infty, +\infty)$*

$$\frac{d\beta^{-1}(\alpha(x) + c)}{d \ln x} = O(1), \quad x \to +\infty. \qquad (15)$$

*If $F \in V$, $f$ has regular variation in regard to $F$ and for every $c \in (-\infty, +\infty)$*

$$\ln F(x) = o(x\beta^{-1}(\alpha(x) + c)), \quad x \to +\infty, \qquad (16)$$

*then*

$$T_{\alpha\beta}(I) = \overline{\lim_{x\to+\infty}} \frac{\exp\{\alpha(x)\}}{\exp\left\{\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)\right\}}.$$

Indeed, if $\alpha_1 \in L$ and $\frac{x\alpha_1'(x)}{\alpha_1(x)} = o(1)$ as $x \to +\infty$ then $\alpha_1 \in L_{si}$, and if $\beta_1 \in L$ and $\frac{x\beta_1'(x)}{\beta_1(x)} = O(1)$ as $x \to +\infty$ then $\beta_1 \in L^0$. Hence it follows that if $\alpha_1(x) = e^{\alpha(x)}$, $\beta_1(x) = e^{\beta(x)}$ and $x\alpha'(x) = o(1)$, $x\beta'(x) = O(1)$ as $x \to +\infty$ then $\alpha_1 \in L_{si}$ and $\beta_1 \in L^0$. From (15) condition (3) follows with $\alpha_1$ and $\beta_1$ instead of $\alpha$ and $\beta$. Condition (18) implies (4) with $\alpha_1$ and $\beta_1$ instead of $\alpha$ and $\beta$. Therefore, Theorem 1 implies Lemma 2.

**Theorem 7.** *Let functions $\alpha \in L$ and $\beta \in L$ be continuously differentiable, $x\alpha'(x) = o(1)$, $x\beta'(x) = O(1)$ as $x \to +\infty$ and (3) and (15) hold. Let $F \in V$, $f$ and $f_j$ have a regular variation in regard to $F$ and (18) holds. Suppose that all integrals (10) have the same generalized order $\varrho_{\alpha\beta}(I_j) = \varrho \in (0, +\infty)$ and the generalized types $T_{\alpha\beta}(I_j) \in (0, +\infty)$. Suppose also that $f_1(x) > 0$ for all $x \geq x_0$ and for all $2 \leq j \leq m$*

$$\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right) \leq (1 + o(1))\beta\left(\frac{1}{x}\ln\frac{1}{f_1(x)}\right), \quad x \to +\infty. \tag{17}$$

*If $\omega_j > 0$ for $1 \leq j \leq m$, $\sum_{1\leq j\leq m}\omega_j = 1$ and*

$$\exp\left\{\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)\right\} = (1 + o(1))\prod_{j=1}^{m}\exp\left\{\omega_j\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right\}, \quad x \to +\infty, \tag{18}$$

*then integral (1) has the generalized order $\varrho_{\alpha\beta}(I) = \varrho$ and the generalized type $T_{\alpha\beta}(I) \leq \prod_{j=1}^{m} T_{\alpha\beta}(I_j)^{\omega_j}$.*

*Proof.* At first, we remark that from the conditions $x\alpha'(x) = o(1)$, $x\beta'(x) = O(1)$ as $x \to +\infty$ it follows that $\alpha \in L_{si}$ and $\beta \in L^0$, and (18) implies (4). Thus, the functions $\alpha$, $\beta$, and $F$ satisfy the assumptions of Theorem 1.

From (18) we have

$$\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right) = \sum_{j=1}^{m}\omega_j\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right) + o(1), \quad x \to +\infty. \tag{19}$$

Therefore, by Theorem 1

$$\frac{1}{\varrho_{\alpha\beta}(I)} = \varliminf_{x\to+\infty}\frac{1}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right) \geq \sum_{j=1}^{m}\varliminf_{x\to+\infty}\frac{\omega_j}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right) = \frac{1}{\varrho}.$$

On the other hand, in view of (17) we obtain from (19)

$$\frac{1}{\varrho_{\alpha\beta}(I)} \leq \sum_{j=1}^{m}\varliminf_{x\to+\infty}\frac{\omega_j}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f_1(x)}\right) = \frac{1}{\varrho}.$$

Thus, $\varrho_{\alpha\beta}(I) = \varrho$.

From (18) and Lemma 2 it follows that

$$\frac{1}{T_{\alpha\beta}(I)} = \varliminf_{x\to+\infty}\frac{1}{\exp\alpha(x)}\prod_{j=1}^{m}\exp\left\{\varrho\omega_j\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right\} \geq$$
$$\geq \prod_{j=1}^{m}\varliminf_{x\to+\infty}\left(\frac{\exp\left\{\varrho\omega_j\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right\}}{\exp\alpha(x)}\right)^{\omega_j} = \prod_{j=1}^{m}\left(\frac{1}{T_{\alpha\beta}(I)}\right)^{\omega_j}.$$

□

If we choose $\alpha(x) = \ln\ln\ln x$, $\beta(x) = \ln x$ for $x \geq x_0$, $m = 2$ and $\omega_j = 1/2$ then from Theorem 7 we obtain the following statement.

**Corollary 4.** *Let* $F \in V$, $\ln F(x) = o(x \ln\ln x)$ *as* $x \to +\infty$ *and the functions* $f$ *and* $f_j$ $(j = 1, 2)$ *have regular variation in regard to* $F$. *Suppose that* $f_1(x) > 0$ *for all* $x \geq x_0$ *and* $\ln\left(\frac{1}{x}\ln\frac{1}{f_2(x)}\right) \leq (1+o(1))\ln\left(\frac{1}{x}\ln\frac{1}{f_1(x)}\right)$ *as* $x \to +\infty$.

*If* $\varlimsup\limits_{\sigma\to+\infty} \frac{\ln\ln\ln\ln I_j(\sigma)}{\ln\sigma} = \varrho \in (0, +\infty)$ *for* $j = 1, 2$ *and*

$$\ln\frac{1}{f(x)} = (1+o(1))\sqrt{\ln\frac{1}{f_1(x)}\ln\frac{1}{f_2(x)}}, \quad x \to +\infty,$$

*then* $\varlimsup\limits_{\sigma\to+\infty} \frac{\ln\ln\ln\ln I(\sigma)}{\ln\sigma} = \varrho$ *and*

$$\varlimsup_{x\to+\infty} \frac{x}{\exp_3\left\{\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)^{\varrho}\right\}} \leq \sqrt{\varlimsup_{x\to+\infty} \frac{x}{\exp_3\left\{\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)^{\varrho}\right\}} \varlimsup_{x\to+\infty} \frac{x}{\exp_3\left\{\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)^{\varrho}\right\}}},$$

*where* $\exp_3 x = \exp\{\exp\{e^x\}\}$.

For integral (1) of finite modified generalized orders we define the generalized type $T^M_{\alpha\beta}(I)$ by the formula

$$T^M_{\alpha\beta}(I) = \varlimsup_{\sigma\to+\infty} \frac{\ln I(\sigma))}{\sigma\alpha^{-1}(\varrho\beta(\sigma))}, \quad (\varrho = \varrho^M_{\alpha\beta}(I)).$$

Then the following lemma is true.

**Lemma 3.** *Let* $\beta \in L$, $\beta_1(x) = \alpha^{-1}(\varrho\beta(x)) \in L_{si}$, *the function* $F \in V$ *satisfy condition* $\ln F(x) = o(x\beta_1^{-1}(cx))$ *as* $x \to +\infty$ *for every* $c \in (0, +\infty)$, *and* $f$ *have regular variation in regard to* $F$. *Then*

$$T^M_{\alpha\beta}(I) = \varlimsup_{x\to+\infty} \frac{x}{\alpha^{-1}\left(\varrho\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)\right)}.$$

Indeed, $\beta_1^{-1}(cx)) = \beta^{-1}(c\alpha(x/\varrho)) \leq \beta^{-1}(\alpha(x)/\varrho_1)$, because from the condition $\alpha \in L^0$ it follows that $\alpha(cx) \leq K(c)\alpha(cx)$ for every $c \in (0, +\infty)$. Therefore, if we choose $\alpha_1(x) \equiv x$ for $x \geq x_0$, then from Theorem 3 with $\alpha_1$ and $\beta_1$ instead of $\alpha$ and $\beta$ we deduce Lemma 3.

**Theorem 8.** *Let* $\beta \in L_{si}$, $\alpha(x) = (1+o(1))\ln x$ *as* $x \to +\infty$ *and* $\beta_1(x) = \alpha^{-1}(\varrho\beta(x)) \in L_{si}$. *Let* $F \in V$, $\ln F(x) = o(x\beta_1^{-1}(x))$ *as* $x \to +\infty$ *and* $f$ *and* $f_j$ *have regular variation in regard to* $F$. *Suppose that all integrals* (10) *have the same modified generalized order* $\varrho^M_{\alpha\beta}(I_j) = \varrho \in (0, +\infty)$ *and the modified generalized types* $T^M_{\alpha\beta}(I_j) \in (0, +\infty)$. *Suppose also that* $f_1(x) > 0$ *for all* $x \geq x_0$ *and* (17) *holds.*

*If* $\omega_j > 0$ *for* $1 \leq j \leq m$, $\sum_{1\leq j\leq m}\omega_j = 1$, *and*

$$\alpha^{-1}\left(\varrho\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)\right) = (1+o(1))\prod_{j=1}^{m}\left(\alpha^{-1}\left(\varrho\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right)\right)^{\omega_j}, \quad x \to +\infty, \quad (20)$$

*then integral* (1) *has the modified generalized order* $\varrho^M_{\alpha\beta}(I) = \varrho$ *and the modified generalized type* $T^M_{\alpha\beta}(I) \leq \prod_{j=1}^{m} T^M_{\alpha\beta}(I_j)^{\omega_j}$.

*Proof.* At first, we remark that the condition $\ln F(x)=o(x\beta_1^{-1}(x))$ as $x\to+\infty$ implies (4), because $\beta_1^{-1}(x))=\beta^{-1}(\alpha(t)/\varrho))$. Thus, the functions $\alpha$, $\beta$, and $F$ satisfy the conditions of Theorem 3.

Since $\alpha(x)=(1+o(1))\ln x$ as $x\to+\infty$, from (20) we obtain (19). As above from (19) by Theorem 3 $1/\varrho^M_{\alpha\beta}(I)\ge 1/\varrho$. On the other hand, from (17) and (19) as above we get $1/\varrho^M_{\alpha\beta}(I)\le 1/\varrho$. Thus, $\varrho^M_{\alpha\beta}(I)=\varrho$.

From (20) and Lemma 3 it follows that

$$\frac{1}{T^M_{\alpha\beta}(I)}=\varliminf_{x\to+\infty}\frac{1}{x}\alpha^{-1}\left(\varrho\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)\right)=\varliminf_{x\to+\infty}\frac{1}{x}\prod_{j=1}^m\left(\alpha^{-1}\left(\varrho\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right)\right)^{\omega_j}\ge$$
$$=\prod_{j=1}^m\varliminf_{x\to+\infty}\left(\frac{1}{x}\alpha^{-1}\left(\varrho\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right)\right)^{\omega_j}=\prod_{j=1}^m\left(\frac{1}{T^M_{\alpha\beta}(I)}\right)^{\omega_j}.$$

□

If we choose $\alpha(x)=\ln x$, $\beta(x)=\ln\ln x$ for $x\ge x_0$, m=2 and $\omega_j=1/2$ then from Theorem 8 we obtain the following statement.

**Corollary 5.** *Let $F\in V$, $\ln F(x)=o(x\exp\{e^x\})$ as $x\to+\infty$ and the functions $f$ and $f_j$, $j=1,2$, have regular variation in regard to $F$. Suppose that that $f_1(x)>0$ for all $x\ge x_0$ and $\ln\ln\left(\frac{1}{x}\ln\frac{1}{f_2(x)}\right)\le(1+o(1))\ln\ln\left(\frac{1}{x}\ln\frac{1}{f_1(x)}\right)$ as $x\to+\infty$.*

*If $\varlimsup\limits_{\sigma\to+\infty}\frac{\ln\ln I_j(\sigma)-\ln\sigma}{\ln\ln\sigma}=\varrho\in(0,+\infty)$ for $j=1,2$ and*

$$\ln\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)=(1+o(1))\sqrt{\ln\left(\frac{1}{x}\ln\frac{1}{f_1(x)}\right)\ln\left(\frac{1}{x}\ln\frac{1}{f_2(x)}\right)},\quad x\to+\infty,$$

*then $\varlimsup\limits_{\sigma\to+\infty}\frac{\ln\ln I(\sigma)-\ln\sigma}{\ln\ln\sigma}=\varrho$ and $\varlimsup\limits_{\sigma\to+\infty}\frac{\ln I(\sigma)}{\sigma\ln^{\varrho}\sigma}\le\sqrt{\varlimsup\limits_{\sigma\to+\infty}\frac{\ln I_1(\sigma)}{\sigma\ln^{\varrho}\sigma}\varlimsup\limits_{\sigma\to+\infty}\frac{\ln I_2(\sigma)}{\sigma\ln^{\varrho}\sigma}}$.*

Finally, we prove a theorem, which supplements Theorems 6 and 8.

**Theorem 9.** *Let either $\alpha\in L_{si}$ and $\beta\in L^0$ or $\alpha\in L^0$ and $\beta\in L_{si}$, $F\in V$, $\ln F(x)==o(x\beta^{-1}(\alpha^c(x)))$ as $x\to+\infty$ for every $c\in(0,+\infty)$ and integrals (10) have modified general orders $\varrho^M_{\alpha\beta}(I_j)\in(0,+\infty)$. Suppose that $f$ has regular variation in regard to $F$ and (11) holds. Then:*

*1) if $f_1(x)>0$ for all $x\ge x_0$ and for all $2\le j\le m_0$*

$$\ln\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\le(1+o(1))\ln\beta\left(\frac{1}{x}\ln\frac{1}{f_1(x)}\right),\quad x\to+\infty.\tag{21}$$

*then*

$$\varlimsup_{\sigma\to+\infty}\frac{1}{\ln\beta(\sigma)}\ln\alpha\left(\frac{\ln I(\sigma)}{\sigma}\right)=1\tag{22}$$

*and $\varrho^M_{\alpha\beta}(I)\le\prod_{j=1}^m(\varrho^M_{\alpha\beta}(I_j))^{\omega_j}$;*

*2) if $v(x)=-(\ln f(x))'$ is continuous and increasing on $[x_0,+\infty)$ and all integrals (10) have regular modified $\alpha\beta$-growth then integral (1) has regular modified $\alpha\beta$-growth and*

$$\varrho^M_{\alpha\beta}(I)=\prod_{j=1}^m(\varrho^M_{\alpha\beta}(I_j))^{\omega_j}.$$

*Proof.* Since $\varrho^M_{\alpha\beta}(I_j) \in (0, +\infty)$, we have

$$\overline{\lim_{\sigma\to+\infty}} \frac{1}{\ln\ \beta(\sigma)} \ln\ \alpha\left(\frac{\ln\ I_j(\sigma)}{\sigma}\right) = 1.$$

It is known ([9]) that if $h \in L^0$ then $h$ is a *RO*-increase function ([10, p.86]), that is for every $\lambda \in [1, +\infty)$ and all $x \geq x_0$ the inequalities $1 \leq h(\lambda x)/h(x) \leq M(\lambda) < +\infty$, whence it follows that $\ln\ h \in L_{si}$. Therefore, using Theorem 3 with $\ln\ \alpha$ and $\ln\ \beta$ instead of $\alpha$ and $\beta$ (the condition $\ln\ F(x) = o(x\beta^{-1}(\alpha^c(x)))$ as $x \to +\infty$ for every $c \in (0, +\infty)$ implies condition (4)), we obtain

$$\underline{\lim_{x\to+\infty}} \frac{1}{\ln\ \alpha(x)} \ln\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right) = 1$$

for each $j = 1, 2, \ldots, m$, and in view of (11)

$$\underline{\lim_{x\to+\infty}} \frac{1}{\ln\ \alpha(x)} \ln\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right) = \underline{\lim_{x\to+\infty}} \frac{1}{\ln\ \alpha(x)} \sum_{j=1}^{m} \omega_j \ln\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right) \geq$$
$$\geq \sum_{j=1}^{m} \omega_j \underline{\lim_{x\to+\infty}} \frac{1}{\ln\ \alpha(x)} \left(\frac{1}{x}\ln\frac{1}{f_1(x)}\right) = 1.$$

On the other hand by virtue of (21)

$$\underline{\lim_{x\to+\infty}} \frac{1}{\ln\ \alpha(x)} \ln\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right) =$$
$$= \underline{\lim_{x\to+\infty}} \frac{1}{\ln\ \alpha(x)} \left(\omega_1 \ln\beta\left(\frac{1}{x}\ln\frac{1}{f_1(x)}\right) + \sum_{j=2}^{m} \omega_j \ln\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right) \leq$$
$$\leq \underline{\lim_{x\to+\infty}} \frac{1}{\ln\ \alpha(x)} \sum_{j=1}^{m} \omega_j \ln\beta\left(\frac{1}{x}\ln\frac{1}{f_1(x)}\right) = 1,$$

i. e.

$$\overline{\lim_{x\to+\infty}} \frac{\ln\ \alpha(x)}{\ln\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)} = 1$$

and by Theorem 3 equality (22) is true.

The condition $\ln\ F(x) = o(x\beta^{-1}(\alpha^c(x)))$ as $x \to +\infty$ for every $c \in (0, +\infty)$ implies the condition $\ln\ F(x) = o(x\beta^{-1}(\alpha(cx)))$ as $x \to +\infty$ for every $c \in (0, +\infty)$. Therefore, by Theorem 3 in view of (11)

$$\frac{1}{\varrho^M_{\alpha\beta}(I)} = \underline{\lim_{x\to+\infty}} \frac{1}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right) = \underline{\lim_{x\to+\infty}} \prod_{j=1}^{m}\left(\frac{1}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right)^{\omega_j} \geq$$
$$\geq \prod_{j=1}^{m} \underline{\lim_{x\to+\infty}} \left(\frac{1}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right)\right)^{\omega_j} = \prod_{j=1}^{m}(\varrho^M_{\alpha\beta}(I_j))^{\omega_j},$$

i. e. the statement 1) is proved.

Now, if $\varrho^M_{\alpha\beta}(I_j) = \varrho^M_{\alpha\beta}(I_j)$ then by Theorem 4 for each $j = 1, 2, \dots m$

$$\frac{1}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right) \to \frac{1}{\varrho^M_{\alpha\beta}(I_j)}, \quad x \to +\infty,$$

and from (11) we get

$$\frac{1}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f(x)}\right) = (1+o(1))\prod_{j=1}^{m}\frac{1}{\alpha(x)}\beta\left(\frac{1}{x}\ln\frac{1}{f_j(x)}\right) =$$
$$= (1+o(1))\prod_{j=1}^{m}\left(\frac{1}{\varrho^M_{\alpha\beta}(I_j)}\right)^{\omega_j}, \quad x \to +\infty.$$

Hence by Theorems 3 and 4 it follows that integral (1) has regular modified $\alpha\beta$-growth and $\varrho^M_{\alpha\beta}(I) = \prod_{j=1}^{m}(\varrho^M_{\alpha\beta}(I_j))^{\omega_j}$. □

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*Received 22.12.2017*