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ON THE GROWTH OF LAPLACE-STIELTJES INTEGRALS

In the paper it is investigated the growth of characteristics of Laplace-Stieltjes integrals $I(\sigma) = \int_0^\infty f(x) dF(x)$, where $F$ is a nonnegative nondecreasing unbounded function continuous on the right on $[0, +\infty)$ and $f$ is a nonnegative on $[0, +\infty)$ function such that there exist $a \geq 0$, $b \geq 0$ and $h > 0$: $\int_{x-a}^{x+b} f(t) dF(t) \geq hf(x)$ for all $x \geq a$. Assume that $\alpha, \beta$ are positive continuously differentiable functions increasing to $+\infty$ on $[0, +\infty)$ such that:

a) $\alpha(cx) = (1 + o(1))\alpha(x)$ ($x \to +\infty$) for any $c > 0$; b) $\beta(x(1 + o(1))) = (1 + o(1))\beta(x)$ ($x \to +\infty$); c) $\frac{d\beta^{-1}(\alpha(x)/o)}{\ln x} = O(1)$ ($x \to +\infty$) for every $o \in (0, +\infty)$. The main results of the paper are contained in Theorems 5 and 7 and are derived from the following two statements of independent interest. If $F$ satisfies condition $\lim F(x) = o\left(\frac{x\beta^{-1}(\alpha(x))}{o}\right)$ ($x \to +\infty$), then $\varrho_{\alpha\beta}(I) = k_{\alpha\beta}(f)$ (Theorem 1). If in additional the function $v(x) = - (\ln f(x))'$ is continuous and increasing on $[x_0, +\infty)$ and $\varrho_{\alpha\beta}(I) < +\infty$, then $\lambda_{\alpha\beta}(I) = \kappa_{\alpha\beta}(f)$ (Theorem 2), where

$$\lim_{\sigma \to +\infty} \frac{\alpha(\ln I(\sigma))}{\beta(\sigma)} := \left\{ \begin{array}{ll} \varrho_{\alpha\beta}(I), & \lambda_{\alpha\beta}(I), \\ \frac{1}{x} \ln \frac{1}{f(x)} & \kappa_{\alpha\beta}(f). \end{array} \right.$$ 

Similar results are proved also for so called the modified generalized order and lower order.

1. Introduction. For an entire function $f(z) = \sum_{n=0}^\infty a_n z^n$ let $\varrho(f)$ be its order and $\sigma(f)$ be its type. Using Hadamard’s formulas for the finding of these characteristics, E.G. Calys ([1]) proved the following theorems.

**Theorem A.** Suppose that entire functions $f_1(z) = \sum_{n=0}^\infty a_{n,1} z^n$ and $f_2(z) = \sum_{n=0}^\infty a_{n,2} z^n$ have finite orders and regular growth (in sense of the equality of order $\varrho(f)$ and lower order $\varrho(f)$) and the sequences $|a_{n,1}/a_{n+1,1}|$ and $|a_{n,2}/a_{n+1,2}|$ are non-decreasing for $n \geq n_0$. If

$$\ln \left(1/|a_n|\right) = (1 + o(1)) \sqrt{\ln \left(1/|a_{n,1}|\right) \ln \left(1/|a_{n,2}|\right)}, \quad n \to \infty,$$

then the function $f$ has regular growth and $\varrho(f) = \sqrt{\varrho(f_1)\varrho(f_2)}$.

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Theorem B. Suppose that functions $f_1$ and $f_2$ from Theorem A have the same order \( \varrho[f_1] = \varrho[f_2] = \varrho \in (0, +\infty) \) and types \( \sigma[f_1] = \sigma_1, \sigma[f_2] = \sigma_2 \). Also suppose that \( a_{n,1} \neq 0 \) and \( |a_{n,2}| \geq |a_{n,1}|/(l(1/|a_{n,1}|)) \) for all \( n \geq n_0 \), where \( l \) is a slowly varying function. If

\[
|a_n| = (1 + o(1))\sqrt{|a_{n,1}||a_{n,2}|}, \quad n \to \infty,
\]

then the function \( f \) has order \( \varrho[f] = \varrho \) and type \( \sigma[f] \leq \sqrt{\sigma_1\sigma_2} \).

We remark that R.S.L. Srivastava ([2,3]) tried to prove Theorem A without assumptions \( a_{n,1} \neq 0 \) and \( |a_{n,2}| \geq |a_{n,1}|/(l(1/|a_{n,1}|)) \) for all \( n \geq n_0 \) and Theorem B without condition of the nondecrease of the sequences \( (|a_{n,1}/a_{n+1,1}|) \) and \( (|a_{n,2}/a_{n+1,2}|) \). On the fallaciousness of such statements it was indicated in Math. Rev., 1963, V.25, N2204, N2206.

In [4] Theorems A and B are transferred on entire Dirichlet series. Here we will obtain such theorems for Laplace-Stieltjes integrals.

Let \( V \) be the class of which are nonnegative nondecreasing unbounded and continuous on the right functions \( F \) on \( [0, +\infty) \).

The Laplace–Stieltjes transform of a real-valued function \( g \) is given, usually, by a Lebesgue–Stieltjes integral of the form \( \int_0^{+\infty} e^{-x} dg(x) \). We write this transformation in a different form. For a nonnegative function \( f \) on \( [0, +\infty) \) the integral

\[
I(\sigma) = \int_0^{+\infty} f(x)e^{x\sigma}dF(x), \quad \sigma \in \mathbb{R},
\]

is called of Laplace-Stieltjes ([5–7]). Integral (1) is a direct generalization of the ordinary Laplace integral \( I(\sigma) = \int_0^{+\infty} f(x)e^{x\sigma}dx \) and of the Dirichlet series \( D(\sigma) = \sum_{n=0}^{+\infty} a_n e^{\lambda_n\sigma} \) with nonnegative coefficients \( a_n \) and exponents \( \lambda_n, 0 \leq \lambda_n \uparrow +\infty (n \to \infty) \), if we choose \( F(x) = n(x) = \sum_{\lambda_n \leq x} 1 \) and \( f(\lambda_n) = a_n \geq 0 \) for all \( n \geq 0 \) (see also [5,8]).

Let

\[
\mu(\sigma) = \mu(\sigma, I) = \max\{f(x)e^{x\sigma} : x \geq 0\}, \quad \sigma \in \mathbb{R},
\]

be the maximum of the integrand, \( \sigma_c \) be the abscissa of convergence of the integral (1) and \( \sigma_\mu \) be the abscissa of maximum of the integrand. Then ([7, p.8])

\[
\sigma_\mu = \lim_{x \to +\infty} \frac{1}{x} \ln \frac{1}{f(x)}
\]

and if either \( \ln F(x) = o(x) \) or \( \ln F(x) = o(\ln f(x)) \) as \( x \to +\infty \) then ([7, p.13]) \( \sigma_c \leq \sigma_\mu \). Also we remark that if \( \ln F(x) = O(x) \) as \( x \to +\infty \) and \( \sigma_\mu = +\infty \) then ([7, p.11]) \( \sigma_c = +\infty \).

To obtain the inequality \( \sigma_c \geq \sigma_\mu \) we introduce as in [7, p.21] the concept of a regular variation of \( f \) in regard to \( F \). We say that a positive function \( f \) has regular variation in regard to \( F \) if there exist \( a \geq 0, b \geq 0 \) and \( h > 0 \) such that for all \( x \geq a \)

\[
\int_{x-a}^{x+b} f(t)dF(t) \geq hf(x).
\]

Then [7, p.21] if \( F \in V \) and \( f \) has regular variation in regard to \( F \) then \( \sigma_c \leq \sigma_\mu \). Thus, if \( F \in V \) and \( f \) has regular variation in regard to \( F \) and either \( \ln F(x) = o(x) \) or \( \ln F(x) = o(\ln f(x)) \) as \( x \to +\infty \) then \( \sigma_c = \sigma_\mu \).
Further we assume that $\sigma_e = \sigma_\mu = +\infty$.

2. Generalized orders. Let $L$ be the class of continuous increasing functions $\alpha$ such that $\alpha(x) \geq 0$ for $x \geq x_0$, $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$, and on $[x_0, +\infty)$ the function $\alpha$ increases to $+\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha(x(1+o(1))) = (1+o(1))\alpha(x)$ as $x \to +\infty$; further, $\alpha \in L_{si}$ if $\alpha \in L$ and for any $c > 0$ $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$. It is easy to see that $L_{si} \subset L^0$. Functions from $L_{si}$ are called slowly increasing. In future we will need the next lemma [9].

**Lemma 1.** Let $\beta \in L$ and

$$B(\delta) = \lim_{x \to +\infty} \frac{\beta((1+\delta)x)}{\beta(x)}, \quad \delta > 0.$$  

Then in order that $\beta \in L^0$ it is necessary and sufficient that $B(\delta) \to 1$ as $\delta \to 0$.

Let $\alpha \in L$, $\beta \in L$, and $G$ be an arbitrary function on $[\sigma_0, +\infty)$. The value

$$\varrho_{\alpha\beta}(G) = \lim_{\sigma \to +\infty} \frac{\alpha(G(\sigma))}{\beta(\sigma)}$$  

is called a generalized order of $G$. If we choose $G(\sigma) = \ln I(\sigma)$ then from (2) we obtain the definition of the generalized order $\varrho_{\alpha\beta}(I)$ of the Laplace-Stieltjes integral (1). Also define

$$k_{\alpha\beta}(f) = \lim_{x \to +\infty} \frac{\alpha(x)}{\beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right)}.$$  

First we remark that if the functions $\alpha \in L^0$ and $\beta \in L^0$ are continuously differentiable and for every $\varrho \in (0, +\infty)$

$$\frac{d\beta^{-1}(\alpha(x)/\varrho)}{d\ln x} = O(1), \quad x \to +\infty,$$  

then ([7, p.77]) $\varrho_{\alpha\beta}(\ln \mu) = k_{\alpha\beta}(f)$, and if for every $\varrho \in (0, +\infty)$

$$\ln F(x) = o \left( x\beta^{-1} \left( \frac{\alpha(x)}{\varrho} \right) \right), \quad x \to +\infty,$$  

then ([7, p. 77]) $\varrho_{\alpha\beta}(I) \leq \varrho_{\alpha\beta}(\ln \mu)$. On the other hand, if $f$ has a regular variation in regard to $F$ then ([7, p.81]) $\varrho_{\alpha\beta}(I) \geq \varrho_{\alpha\beta}(\ln \mu)$ for each $\alpha \in L^0$ and $\beta \in L$.

Thus, the following theorem is true.

**Theorem 1.** Let $F \in V$, $f$ have regular variation in regard to $F$ and functions $\alpha \in L_{si}$ and $\beta \in L^0$ satisfy condition (3). If $F$ satisfies condition (4) then $\varrho_{\alpha\beta}(I) = k_{\alpha\beta}(f)$.

Now we put

$$\lambda_{\alpha\beta}(I) = \lim_{\sigma \to +\infty} \frac{\alpha(\ln I(\sigma))}{\beta(\sigma)}, \quad \lambda_{\alpha\beta}(\ln \mu) = \lim_{\sigma \to +\infty} \frac{\alpha(\ln \mu(\sigma))}{\beta(\sigma)}, \quad \kappa_{\alpha\beta}(f) = \lim_{x \to +\infty} \frac{\alpha(x)}{\beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right)}.$$  

**Proposition 1.** If $\alpha \in L$ and $\beta \in L^0$ then $\lambda_{\alpha\beta}(\ln \mu) \geq \kappa_{\alpha\beta}(f)$.
Indeed, if \( \varkappa_{\alpha\beta}(f) > 0 \) then for each \( \varkappa \in (0, \varkappa_{\alpha\beta}(f)) \) and all \( x \geq x_0 = x_0(\varkappa) \) we have
\[
\ln f(x) \geq -x\beta^{-1}(\alpha(x)/\varkappa). \]
Therefore, \( \ln \mu(\sigma) \geq -x\beta^{-1}(\alpha(x)/\varkappa) + x\sigma \) for all \( \sigma \) and \( x \geq x_0 \).

Choosing \( x = \alpha^{-1}(\varkappa\beta(\sigma - 1)) \geq x_0 \) for \( \sigma \geq \sigma_0 \) hence we obtain
\[
\ln \mu(\sigma) \geq a^{-1}(\varkappa\beta(\sigma - 1)) = a^{-1}(\varkappa(1 + o(1))\beta(\sigma)), \quad \sigma \to +\infty.
\]

Therefore, \( \lambda_{\alpha\beta}(\ln \mu) \geq \varkappa \) and in view of the arbitrariness of \( \varkappa \) we have \( \lambda_{\alpha\beta}(\ln \mu) \geq \varkappa_{\alpha\beta}(f) \).

If \( \varkappa_{\alpha\beta}(f) = 0 \) this inequality is obvious.

**Proposition 2.** Let \( \alpha \in L_{si}, \beta \in L^0 \) and condition (3) hold. If the function \( v(x) = -(\ln f(x))' \) is continuous and increasing on \( [x_0, +\infty) \) then \( \lambda_{\alpha\beta}(\ln \mu) \leq \varkappa_{\alpha\beta}(f) \).

Indeed, since \( v(x) = -(\ln f(x))' \) is continuous and increasing on \( [x_0, +\infty) \), the function \( \ln f(x) + x\sigma \) has the unique point \( x \) of the maximum such that \( \sigma = v(x) \), and \( \ln \mu(\sigma) = \ln f(x) + x\sigma, \) where \( \sigma = v(x) \).

Suppose that \( \varkappa_{\alpha\beta}(f) < +\infty \). Then for every \( \varkappa > \varkappa_{\alpha\beta}(f) \) there exists a sequence \( (x_k) \uparrow +\infty \) such that \( \ln f(x_k) \leq -x_k\beta^{-1}(\alpha(x_k)/\varkappa) \). We put \( \mu^*(\sigma) = f(x_k)e^{\sigma x_k} \). Since \( \mu^*(\sigma) = f(x)e^{\sigma x} \) for \( \sigma = v^{-1}(x) \) we have \( \mu(\sigma_k) = \mu^*(\sigma_k) \) for \( \sigma_k = v(x_k) \). Hence
\[
\ln \mu(\sigma_k) = \ln \mu^*(\sigma_k) \leq \max_k \{-x_k\beta^{-1}(\alpha(x_k)/\varkappa) + x_k\sigma_k\} \leq \max_x \{-x\beta^{-1}(\alpha(x)/\varkappa) + x\sigma_k\}.
\]

In view of (3)
\[
(-x\beta^{-1}(\alpha(x)/\varkappa)) + x\sigma_k' = -\beta^{-1}(\alpha(x)/\varkappa) - \frac{d\beta^{-1}(\alpha(x)/k)}{d\ln x} + \sigma_k =
\]
\[
-\beta^{-1}(\alpha(x)/\varkappa) + \sigma_k + O(1), \quad x \to +\infty,
\]
i.e. the function \(-x\beta^{-1}(\alpha(x)/\varkappa)) + x\sigma_k\) attains its maximum at the point
\[
x(\sigma_k) = \alpha^{-1}(\varkappa\beta(\sigma_k + O(1))), \quad x \to +\infty,
\]
\[
\ln \mu(\sigma_k) \leq -\alpha^{-1}(\varkappa\beta(\sigma_k + O(1)))(\sigma_k + O(1))) + \sigma_k\alpha^{-1}(\varkappa\beta(\sigma_k + O(1))) =
\]
\[
= O(\alpha^{-1}(\varkappa\beta(\sigma_k + O(1)))), \quad k \to +\infty.
\]

Since \( \alpha \in L_{si} \) and \( \beta \in L^0 \), hence it follows that \( \lambda_{\alpha\beta}(\ln \mu) \leq \varkappa \). In view of the arbitrariness of \( \varkappa \) we have \( \lambda_{\alpha\beta}(\ln \mu) \leq \varkappa_{\alpha\beta}(f) \). If \( \varkappa_{\alpha\beta}(f) = +\infty \) this inequality is obvious.

**Proposition 3.** If \( \alpha \in L^0, \beta \in L \) and \( f \) has regular variation in regard to \( F \) then \( \lambda_{\alpha\beta}(\ln \mu) \leq \lambda_{\alpha\beta}(I) \).

Indeed, if \( f \) has regular variation in regard to \( F \) then ( [7, p.75])
\[
\ln \mu(\sigma) \leq (1 + o(1))\ln I(\sigma), \quad \sigma \to +\infty,
\]
whence \( \lambda_{\alpha\beta}(\ln \mu) \leq \lambda_{\alpha\beta}(I) \).

**Proposition 4.** Let the functions \( \alpha \in L^0 \) and \( \beta \in L^0 \) satisfy condition (3), and the function \( F \in V \) satisfies condition (4). If \( g_{\alpha\beta}(\ln \mu) < +\infty \) then \( \lambda_{\alpha\beta}(\ln \mu) \geq \lambda_{\alpha\beta}(I) \).
Indeed, since \( g_{\alpha\beta}(\ln \mu) < +\infty \), we have \( k_{\alpha\beta}(f) = g_{\alpha\beta}(\ln \mu) < +\infty \), that is \( \ln f(x) \leq -x^\beta^{-1}(\alpha(x)/k) \) for some \( k < +\infty \) and in view of (4) \( \lim_{x \to +\infty} \frac{\ln f(x)}{\ln(1/f(x))} = 0 \). Therefore, [7, p.61]

\[
I(\sigma) \leq K(\varepsilon)\mu(\sigma/(1 - \varepsilon))^{1 - \varepsilon}
\]

for every \( \varepsilon \in (0, 1) \) and all \( \sigma \geq \sigma_0(\varepsilon) \). Hence,

\[
\lambda_{\alpha\beta}(I) \leq \lambda_{\alpha\beta}(\ln \mu) \lim_{\sigma \to +\infty} \frac{\beta(\sigma/(1 - \varepsilon))}{\beta(\sigma)}.
\]

Since \( \beta \in L^0 \) by Lemma 1 \( \lim_{x \to +\infty} \frac{\beta(\sigma/(1 - \varepsilon))}{\beta(\sigma)} \to 1 \) as \( \varepsilon \to 0 \). Thus, \( \lambda_{\alpha\beta}(I) \leq \lambda_{\alpha\beta}(\ln \mu) \).

Combining Propositions 1–4 we get the following theorem.

**Theorem 2.** Let functions \( \alpha \in L_{si} \) and \( \beta \in L^0 \) satisfy condition (3) and the function \( F \in V \) satisfy condition (4). Suppose that the function \( f \) has a regular variation in regard to \( F \) and \( v(x) = -(\ln f(x))' \) is continuous and increasing on \([x_0, +\infty)\). If \( g_{\alpha\beta}(I) < +\infty \) then \( \lambda_{\alpha\beta}(I) = \kappa_{\alpha\beta}(f) \).

### 3. Modified generalized orders.

The values

\[
\ell_{\alpha\beta}^M(I) = \lim_{\sigma \to +\infty} \frac{1}{\beta(\sigma)} \alpha\left(\frac{\ln I(\sigma)}{\sigma}\right), \quad \lambda_{\alpha\beta}^M(I) = \lim_{\sigma \to +\infty} \frac{1}{\beta(\sigma)} \alpha\left(\frac{\ln I(\sigma)}{\sigma}\right)
\]

are called the modified generalized order and the modified lower generalized order of \( I \), respectively. If \( \ln(\mu(\sigma)) \) instead \( I(\sigma) \) then we obtain definitions of \( g_{\alpha\beta}^M(\ln \mu) \) and \( \lambda_{\alpha\beta}^M(\ln \mu) \).

**Proposition 5.** Let either \( \alpha \in L_{si} \) and \( \beta \in L^0 \) or \( \alpha \in L^0 \) and \( \beta \in L_{si} \), and the function \( F \in V \) satisfies condition (4). Then \( g_{\alpha\beta}^M(\ln \mu) = k_{\alpha\beta}(f) \).

**Proof.** Suppose that \( g_{\alpha\beta}^M(\ln \mu) < +\infty \). Then for every \( g > g_{\alpha\beta}^M(\ln \mu) \), all \( \sigma \geq \sigma_0(g) \) and \( x \geq 0 \) we obtain \( \ln f(x) + \sigma x \leq \ln \mu(\sigma) \leq \sigma \alpha^{-1}(g(\beta(\sigma))) \), that is \( \ln f(x) \leq \sigma \alpha^{-1}(g(\beta(\sigma))) - \sigma x \).

We choose \( \sigma = \sigma(x) = \beta^{-1}(\alpha(\delta x)/g) \) for an arbitrary \( \delta \in (0, 1) \). Then \( \sigma(x) \geq \sigma_0(g) \) for \( x \geq x_0 = x_0(\delta, \delta) \) and \( \ln f(x) \leq -(1 - \delta)x^\beta^{-1}(\alpha(\delta x)/g) \) for \( x \geq x_0 \), whence

\[
k_{\alpha\beta}(f) = \lim_{x \to +\infty} \frac{\alpha(x)}{\beta(x)} = \lim_{x \to +\infty} \left( \frac{\alpha(\delta x)}{\beta(1/\delta x)} \right) \frac{\beta(1/\delta x)}{\beta(1/\delta x)} \leq 2 \lim_{x \to +\infty} \frac{\beta(x/(1 - \delta))}{\beta(x)} \lim_{x \to +\infty} \frac{\alpha(x)}{\alpha(\delta x)}.
\]

If \( \alpha \in L_{si} \) and \( \beta \in L^0 \) then \( \lim_{x \to +\infty} \frac{\alpha(x)}{\alpha(\delta x)} = 1 \) and by Lemma 1 \( \lim_{x \to +\infty} \frac{\beta(x/(1 - \delta))}{\beta(x)} \to 1 \) as \( \delta \to 0 \).

Hence in view of the arbitrariness of \( g \) we obtain

\[
k_{\alpha\beta}(f) \leq g_{\alpha\beta}^M(\ln \mu).
\]

If \( \beta \in L_{si} \) and \( \alpha \in L^0 \) then \( \lim_{x \to +\infty} \frac{\beta(x/(1 - \delta))}{\beta(x)} = 1 \), and by Lemma 1 \( \lim_{x \to +\infty} \frac{\alpha(x)}{\alpha(\delta x)} \to 1 \) as \( \delta \to 1 \), and we again obtain inequality (9). If \( g_{\alpha\beta}^M(\ln \mu) = +\infty \) inequality (9) is obvious.
Now assume that \( k_{\alpha\beta}(f) \neq g_{\alpha\beta}^M(\ln \mu) \). Then in view of (8) \( k_{\alpha\beta}(f) < g_{\alpha\beta}^M(\ln \mu) \) and if we choose \( k_{\alpha\beta}(f) < \varrho < g_{\alpha\beta}^M(\ln \mu) \) then \( \ln f(x) \leq -x\beta^{-1}(\alpha(x)/\varrho) \) for \( x \geq x_0(\varrho) \), i.e.

\[
\ln \mu(\sigma) \leq \max \left\{ \max_{x \leq x_0(\varrho)} \ln f(x) + x\sigma, \max_{x \geq x_0(\varrho)} (-x\beta^{-1}(\alpha(x)/\varrho) + x\sigma) \right\} \\
\leq \max_{x \geq 0} [x(\sigma - \beta^{-1}(\alpha(x)/\varrho))] + O(\sigma), \quad \sigma \to +\infty.
\]

Since \( \ln \mu(\sigma) \to +\infty \) as \( \sigma \to +\infty \), the function \( x(\sigma - \beta^{-1}(\alpha(x)/\varrho)) \) attains the maximum at the point \( x = x(\sigma) \) such that \( \sigma - \beta^{-1}(\alpha(x)/\varrho) > 0 \), that is \( x(\sigma) \leq \alpha^{-1}(\varrho\beta(\sigma)) \). Therefore,

\[
\ln \mu(\sigma) \leq x(s)(\sigma - \beta^{-1}(\alpha(x(\sigma))/\varrho)) + O(\sigma) \leq \sigma x(\sigma) + O(\sigma) \leq \sigma \alpha^{-1}(\varrho\beta(\sigma)) + O(\sigma), \quad \sigma \to +\infty,
\]

whence it follows that \( g_{\alpha\beta}^M(\ln \mu) \leq \varrho \). It is a contradiction that therefore, Proposition 5 is proved.

**Proposition 6.** Let \( \alpha \in L^0, \beta \in L^0 \), and \( f \) have regular variation in regard to \( F \). Then \( g_{\alpha\beta}^M(\ln \mu) = g_{\alpha\beta}^M(I) \).

Indeed, if \( f \) has regular variation in regard to \( F \) then from (5) we obtain \( g_{\alpha\beta}^M(\ln \mu) \leq g_{\alpha\beta}^M(I) \). On the other hand, if \( g_{\alpha\beta}^M(\ln \mu) < +\infty \) then in view of Proposition 5, as in the proof of Proposition 4, we have (6) for every \( \varepsilon \in (0, 1) \) and all \( \sigma \geq \sigma_0(\varepsilon) \). Since \( \alpha \in L^0 \) and \( \beta \in L^0 \), we have \( g_{\alpha\beta}^M(I) \leq g_{\alpha\beta}^M(\ln \mu) \). Thus, \( g_{\alpha\beta}^M(\ln \mu) = g_{\alpha\beta}^M(I) \).

Uniting Propositions 5 and 6 we get the following theorem.

**Theorem 3.** Let either \( \alpha \in L_{si} \) and \( \beta \in L^0 \) or \( \alpha \in L^0 \) and \( \beta \in L_{si} \), the function \( F \in V \) satisfies condition (4) and \( f \) has regular variation in regard to \( F \). Then \( g_{\alpha\beta}^M(I) = k_{\alpha\beta}(f) \).

**Proposition 7.** Let either \( \alpha \in L_{si} \) and \( \beta \in L^0 \) or \( \alpha \in L^0 \) and \( \beta \in L_{si} \). Then \( \lambda_{\alpha\beta}^M(\ln \mu) \geq \kappa_{\alpha\beta}(f) \). Moreover, if \( v(x) = -(\ln f(x))' \) is continuous and increasing on \( [x_0, +\infty) \) then \( \lambda_{\alpha\beta}^M(\ln \mu) = \kappa_{\alpha\beta}(f) \).

Indeed, if \( \kappa_{\alpha\beta}(f) > 0 \) then for each \( \kappa \in (0, \kappa_{\alpha\beta}(f)) \) and all \( x \geq x_0 = x_0(\kappa) \) as in the proof of Proposition 7 we obtain \( \ln \mu(\sigma) \geq -x\beta^{-1}(\alpha(x)/\kappa) + x\sigma \) for all \( \sigma \) and \( x \geq x_0 \). Choosing \( x = \alpha^{-1}(\kappa\beta(\delta\sigma)) \), where \( 0 < \delta < 1 \), we have \( \ln \mu(\sigma) \geq (1 - \delta)\sigma a^{-1}(\kappa\beta(\delta\sigma)) \). Hence

\[
\lambda_{\alpha\beta}^M(\ln \mu) \geq \lim_{\sigma \to +\infty} \frac{\alpha((1 - \delta)a^{-1}(\kappa\beta(\delta\sigma)))}{\beta(\sigma)}
\]

If \( \alpha \in L_{si} \) and \( \beta \in L^0 \) hence we have as above \( \lambda_{\alpha\beta}^M(\ln \mu) \geq \kappa \lim_{\sigma \to +\infty} \frac{\beta(\delta\sigma)}{\beta(\sigma)} \to \kappa \) as \( \delta \to 1 \). If \( \alpha \in L^0 \) and \( \beta \in L_{si} \) then

\[
\lambda_{\alpha\beta}^M(\ln \mu) \geq \kappa \lim_{\sigma \to +\infty} \frac{\alpha((1 - \delta)a^{-1}(\kappa\beta(\delta\sigma)))}{\kappa\beta(\delta\sigma)} \frac{(\delta\sigma)^{\beta(\delta\sigma)}}{\beta(\sigma)} = \kappa \lim_{t \to +\infty} \frac{\alpha((1 - \delta)a^{-1}(t))}{t} \to \kappa
\]

as \( \delta \to 0 \). Therefore, \( \lambda_{\alpha\beta}^M(\ln \mu) \geq \kappa_{\alpha\beta}(f) \).

On the other hand, as in the proof Proposition 2 we obtain \( \ln \mu(\sigma) = \ln f(x) + \sigma x \) for \( \sigma = v(x) \). Supposing that \( \kappa_{\alpha\beta}(f) < +\infty \), for \( \kappa > \kappa_{\alpha\beta}(f) \) and some sequence \( (x_k) \uparrow +\infty \) as in the proof of Proposition 2 we obtain for \( \sigma_k = v(x_k) \)

\[
\ln \mu(\sigma_k) \leq \max_x \{-x\beta^{-1}(\alpha(x)/\kappa) + x\sigma_k\} = \max_x \{-x(\sigma_k - \beta^{-1}(\alpha(x)/\kappa))\}
\]
Hence, as in the proof of Proposition 5, it follows that
\[ \ln \mu(\sigma_k) \leq \sigma_k \alpha^{-1}(\theta \beta(\sigma_k)) + O(\sigma_k), \quad k \to \infty, \]
whence it follows that \( \lambda_{\alpha \beta}^M(\ln \mu) \leq \kappa. \) In view of the arbitrariness of \( \kappa \) Proposition 7 is proved.

**Proposition 8.** Let \( \alpha \in L^0, \beta \in L^0, \varrho_{\alpha \beta}^M(\ln \mu) < +\infty \), the function \( F \in V \) satisfies condition (4) and \( f \) has regular variation in regard to \( F \). Then \( \lambda_{\alpha \beta}^M(\ln \mu) = \lambda_{\alpha \beta}^M(I). \)

Indeed, since \( f \) has regular variation in regard to \( F \), from (5) we obtain \( \lambda_{\alpha \beta}^M(\ln \mu) \leq \lambda_{\alpha \beta}^M(I). \) On the other hand, since \( \varrho_{\alpha \beta}^M(\ln \mu) < +\infty \), we have (6) for every \( \varepsilon \in (0, 1) \) and all \( \sigma \geq \sigma_0(\varepsilon) \). Since \( \alpha \in L^0 \) and \( \beta \in L^0 \), hence \( \lambda_{\alpha \beta}^M(I) \leq \lambda_{\alpha \beta}^M(\ln \mu) \). Thus, \( \lambda_{\alpha \beta}^M(\ln \mu) = \lambda_{\alpha \beta}^M(I) \).

From Propositions 7 and 8 we obtain the following theorem.

**Theorem 4.** Let \( \alpha \in L_{si} \) and \( \beta \in L^0 \) or \( \alpha \in L^0 \) and \( \beta \in L_{si} \) and the function \( F \in V \) satisfies condition (4). Suppose that \( f \) has regular variation in regard to \( F \) and \( v(x) = -(\ln f(x))^\prime \) is continuous and increasing on \( [x_0, \infty) \). Then \( \lambda_{\alpha \beta}^M(I) = \kappa_{\alpha \beta}(f) \).

**4. Analogues of Theorem A.** Let \( LS(F) \) be the class of Laplace-Stieltjes integrals for which \( \sigma_c = \sigma_\mu = +\infty \). Suppose that \( I_j \in LS(F), 1 \leq j \leq m, \) and
\[ I_j(\sigma) = \int_{0}^{\infty} f_j(x)e^{\sigma x} dF(x), \quad \sigma \in \mathbb{R}. \] (10)

The following theorem is an analogue of Theorem A.

**Theorem 5.** Let functions \( \alpha \in L_{si} \) and \( \beta \in L^0 \) satisfy condition (3) and the function \( F \in V \) satisfy condition (4). Suppose that all functions \( f_j \) have regular variation in regard to \( F \) and \( v_j(x) = -(\ln f_j(x))^\prime \) is continuous and increasing on \( [x_0, \infty) \). Also suppose that \( f \) have regular variation in regard to \( F \) and
\[ \beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right) = (1 + o(1)) \prod_{j=1}^{m} \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right)^{\omega_j}, \quad x \to +\infty, \] (11)
where \( \omega_j > 0 \) for \( 1 \leq j \leq m \) and \( \sum_{1 \leq j \leq m} \omega_j = 1 \).

If all integrals (10) have regular \( \alpha \beta \)-growth (i.e. \( \lambda_{\alpha \beta}(I_j) = \varrho_{\alpha \beta}(I_j) < +\infty \)) then integral (1) has regular \( \alpha \beta \)-growth and \( \varrho_{\alpha \beta}(I) = \prod_{j=1}^{m} \varrho_{\alpha \beta}(I_j)^{\omega_j}. \)

**Proof.** By Theorem 1 \( \varrho_{\alpha \beta}(I_j) = k_{\alpha \beta}(f_j) \) and by Theorem 2 \( \lambda_{\alpha \beta}(I_j) = \kappa_{\alpha \beta}(f_j) \). Since \( \lambda_{\alpha \beta}(I_j) = \varrho_{\alpha \beta}(I_j) = q_j < 1 \), we have \( k_{\alpha \beta}(f_j) = \kappa_{\alpha \beta}(f_j) = q_j \), that is
\[ \lim_{x \to +\infty} \frac{1}{x} \ln \frac{1}{f_j(x)} = q_j. \] (12)

Therefore, from (11) we obtain
\[ \lim_{x \to +\infty} \frac{1}{x} \ln \frac{1}{f(x)} = \lim_{x \to +\infty} \frac{1}{x} \ln \frac{1}{f_j(x)} \prod_{j=1}^{m} \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right)^{\omega_j} = \]
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\[ = \lim_{x \to +\infty} \prod_{j=1}^{m} \left( \frac{1}{\alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \right)^{\omega_j} = \prod_{j=1}^{m} \lim_{x \to +\infty} \left( \frac{1}{\alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \right)^{\omega_j} = \prod_{j=1}^{m} \left( \frac{1}{g_j} \right)^{\omega_j}, \]

i.e.

\[ \prod_{j=1}^{m} g_j^{\omega_j} = \lim_{x \to +\infty} \frac{\alpha(x)}{\beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right)} = k_{\alpha \beta}(f) = \varphi_{\alpha \beta}(f). \] (13)

By Propositions 2 and 1 \( \lambda_{\alpha \beta}(\ln I) \geq \lambda_{\alpha \beta}(\ln \mu) \geq \varphi_{\alpha \beta}(f) \) and by Theorem 1 \( g_{\alpha \beta}(\ln I) = k_{\alpha \beta}(f) \). Therefore, \( g_{\alpha \beta}(I) = \lambda_{\alpha \beta}(I) = \prod_{j=1}^{m} (r_{\alpha \beta}(I_j))^{\omega_j} \).

If we choose \( \alpha(x) = \ln x \) and \( \beta(x) = x \) for \( x \geq x_0 \) then from the definitions of \( g_{\alpha \beta}(I) \) and \( \lambda_{\alpha \beta}(I) \) we obtain the definitions of the R-order \( g_R \) and lower R-order \( \lambda_R \), respectively. Choosing some more \( m = 2 \) and \( \omega_1 = \omega_2 = 1/2 \), we get the following statement.

**Corollary 1.** Let \( F \in V \) and \( \ln F(x) = o(x \ln x) \) as \( x \to +\infty \). Suppose that the functions \( f_j, j = 1, 2, \) have regular variation in regard to \( F \) and \( v_j(x) = - (\ln f_j(x))^t \) is continuous and increasing on \( [x_0, +\infty) \). Also suppose that \( f \) have regular variation in regard to \( F \) and

\[ \ln \frac{1}{f(x)} = (1 + o(1)) \sqrt{\ln \frac{1}{f_1(x)} \ln \frac{1}{f_2(x)}}, \quad x \to +\infty. \]

If integrals \( I_1 \) and \( I_2 \) have regular R-growth (i.e. \( \lambda_R(I_j) = g_R(I_j) < +\infty \) for \( j = 1, 2 \)) then integral (1) has regular R-growth and \( g_R(I) = \sqrt{g_R(I_1)g_R(I_2)}. \)

Using modified generalized orders we get the following theorem.

**Theorem 6.** Let either \( \alpha \in L_{s_1} \) and \( \beta \in L^0 \) or \( \alpha \in L^0 \) and \( \beta \in L_{s_2} \), and the function \( F \in V \) satisfy condition (4). Suppose that \( f \) has regular variation in regard to \( F \) and \( v(x) = - (\ln f(x))^t \) is continuous and increasing on \( [x_0, +\infty) \). Also suppose that \( f \) have regular variation in regard to \( F \) and (11) holds.

If all integrals (10) have regular modified \( \alpha \beta \)-growth (i.e. \( \lambda_{\alpha \beta}^{M}(I_j) = g_{\alpha \beta}^{M}(I_j) < +\infty \)) then integral (1) has regular modified \( \alpha \beta \)-growth and \( g_{\alpha \beta}^{M}(I) = \prod_{j=1}^{m} (g_{\alpha \beta}^{M}(I_j))^{\omega_j}. \)

**Proof.** By Theorem 3 \( g_{\alpha \beta}^{M}(I_j) = k_{\alpha \beta}(f_j) \) and by Theorem 4 \( \lambda_{\alpha \beta}^{M}(I_j) = \varphi_{\alpha \beta}(f_j) \).

Since \( \lambda_{\alpha \beta}^{M}(I_j) = g_{\alpha \beta}^{M}(I_j) = \varphi_{\alpha \beta}(f_j) = g_j \), that is (12) holds. Therefore, as in the proof of Theorem 5 we obtain (13) from (11). By Propositions 7 and 6 \( \lambda_{\alpha \beta}(\ln \mu) \geq \lambda_{\alpha \beta}(\ln I) \geq \lambda_{\alpha \beta}(\ln \mu) \geq \varphi_{\alpha \beta}(f) \) and by Theorem 3 \( g_{\alpha \beta}^{M}(\ln I) = k_{\alpha \beta}(f) \). Therefore, \( g_{\alpha \beta}^{M}(I) = \lambda_{\alpha \beta}(I) = \prod_{j=1}^{m} (g_{\alpha \beta}^{M}(I_j))^{\omega_j}. \)

If we choose \( \alpha(x) = \ln x \) and \( \beta(x) = \ln x \) for \( x \geq x_0 \) then from the definitions of \( g_{\alpha \beta}(I) \) and \( \lambda_{\alpha \beta}(I) \) we obtain the definitions of the logarithmic order \( g_l \) and lower logarithmic order \( \lambda_l \), respectively. Since

\[ \frac{1}{\ln \sigma} \ln \left( \frac{\ln I(\sigma)}{\sigma} \right) = \frac{\ln \ln I(\sigma)}{\ln \sigma} - 1, \]

for such function we have \( g_{\alpha \beta}^{M}(I) = g_l(I) \) and \( \lambda_{\alpha \beta}^{M}(I) = \lambda_l(I) \). Therefore, choosing \( m = 2 \) and \( \omega_1 = \omega_2 = 1/2 \), we get the following statement.
Corollary 2. Let $F \in V$ and $\ln F(x) = o(x)$ as $x \to +\infty$. Suppose that the functions $f_j$, $j = 1, 2$, have regular variation in regard to $F$ and $v_j(x) = -\ln f_j(x)'$ is continuous and increasing on $[x_0, +\infty)$. Also suppose that $f$ has regular variation in regard to $F$ and

$$
\ln \left( \frac{1}{x} \ln f(x) \right) = (1 + o(1)) \sqrt{\ln \left( \frac{1}{x} \ln f_1(x) \right) \ln \left( \frac{1}{x} \ln f_2(x) \right)}, \quad x \to +\infty.
$$

(14)

If integrals $I_1$ and $I_2$ have regular logarithmic growth (i.e. $\lambda(I_j) = \varrho(I_j) \in (1, +\infty)$ for $j = 1, 2$) then integral (1) has regular logarithmic growth and $\varrho(I) = \sqrt{\varrho(I_1) - 1}(\varrho(I_1) - 1).

Finally, if we choose $\alpha(x) = x$ and $\beta(x) = \ln x$ for $x \geq x_0$ then $\varrho(I) = T(I) := \lim_{\sigma \to +\infty} \frac{\ln \varrho(\sigma)}{\sigma}$ and $\lambda(I) = t(I) := \lim_{\sigma \to +\infty} \frac{\ln \lambda(\sigma)}{\sigma}$, and we obtain the next corollary.

Corollary 3. Let $F \in V$ and $\ln F(x) = o(x^2)$ as $x \to +\infty$. Suppose that the functions $f_j$, $j = 1, 2$, have regular variation in regard to $F$ and $v_j(x) = -\ln f_j(x)'$ is continuous and increasing on $[x_0, +\infty)$. Also suppose that $f$ has regular variation in regard to $F$ and (14) holds. If $t(I_j) = T(I_j) < +\infty$ for $j = 1, 2$ then for integral (1) $t(I) = T(I) = \sqrt{T(I_1)T(I_2)}$.

5. Analogues of Theorem B. Since $\varrho(I) = \lim_{\sigma \to +\infty} \frac{\ln \varrho(\sigma)}{\sigma}$, we define the generalized type $T_{\alpha\beta}(I)$ of integral (1) by the formula

$$
T_{\alpha\beta}(I) = \lim_{\sigma \to +\infty} \frac{\exp\{\alpha(\ln I(\sigma))\}}{\exp\{\varrho(\sigma)\}}, \quad (\varrho = \varrho_{\alpha\beta}(I)).
$$

Theorem 1 implies the following lemma.

Lemma 2. Suppose that the functions $\alpha \in L$ and $\beta \in L$ are continuously differentiable, $x\alpha'(x) = o(1)$, $x\beta'(x) = O(1)$ as $x \to +\infty$, and for every $c \in (-\infty, +\infty)$

$$
\frac{d\beta^{-1}(\alpha(x) + c)}{d\ln x} = O(1), \quad x \to +\infty.
$$

(15)

If $F \in V$, $f$ has regular variation in regard to $F$ and for every $c \in (-\infty, +\infty)$

$$
\ln F(x) = o(x\beta^{-1}(\alpha(x) + c)), \quad x \to +\infty,
$$

(16)

then

$$
T_{\alpha\beta}(I) = \lim_{x \to +\infty} \frac{\exp\{\alpha(x)\}}{\exp\left\{\beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right) \right\}}.
$$

Indeed, if $\alpha \in L$ and $x\alpha'(x) = o(1)$ as $x \to +\infty$ then $\alpha \in L_{si}$, and if $\beta \in L$ and $x\beta'(x) = O(1)$ as $x \to +\infty$ then $\beta \in L^0$. Hence it follows that if $\alpha(x) = e^{\alpha(x)}$, $\beta(x) = e^{\beta(x)}$ and $x\alpha'(x) = o(1)$, $x\beta'(x) = O(1)$ as $x \to +\infty$ then $\alpha \in L_{si}$ and $\beta \in L^0$. From (15) condition (3) follows with $\alpha$ and $\beta$ instead of $\alpha$ and $\beta$. Condition (18) implies (4) with $\alpha$ and $\beta$ instead of $\alpha$ and $\beta$. Therefore, Theorem 1 implies Lemma 2.
Theorem 7. Let functions $\alpha \in L$ and $\beta \in L$ be continuously differentiable, $x\alpha'(x) = o(1)$, $x\beta'(x) = O(1)$ as $x \to +\infty$ and (3) and (15) hold. Let $F \in V$, $f$ and $f_j$ have a regular variation in regard to $F$ and (19) holds. Suppose that all integrals (10) have the same generalized order $\varrho_{\alpha\beta}(I_j) = \varrho \in (0, +\infty)$ and the generalized types $T_{\alpha\beta}(I_j) \in (0, +\infty)$. Suppose also that $f_1(x) > 0$ for all $x \geq x_0$ and for all $2 \leq j \leq m$

\[
\beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \leq (1 + o(1)) \beta \left( \frac{1}{x} \ln \frac{1}{f_1(x)} \right), \quad x \to +\infty.
\] (17)

If $\omega_j > 0$ for $1 \leq j \leq m$, $\sum_{1 \leq j \leq m} \omega_j = 1$ and

\[
\exp \left\{ \beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right) \right\} = (1 + o(1)) \prod_{j=1}^{m} \exp \left\{ \omega_j \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \right\}, \quad x \to +\infty,
\] (18)

then integral (1) has the generalized order $\varrho_{\alpha\beta}(I) = \varrho$ and the generalized type $T_{\alpha\beta}(I) \leq \prod_{j=1}^{m} T_{\alpha\beta}(I_j)^{\omega_j}$.

Proof. At first, we remark that from the conditions $x\alpha'(x) = o(1)$, $x\beta'(x) = O(1)$ as $x \to +\infty$ it follows that $\alpha \in L_{st}$ and $\beta \in L^0$, and (18) implies (4). Thus, the functions $\alpha$, $\beta$, and $F$ satisfy the assumptions of Theorem 1.

From (18) we have

\[
\beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right) = \sum_{j=1}^{m} \omega_j \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) + o(1), \quad x \to +\infty.
\] (19)

Therefore, by Theorem 1

\[
\frac{1}{\varrho_{\alpha\beta}(I)} = \lim_{x \to +\infty} \frac{1}{\alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right) \geq \sum_{j=1}^{m} \lim_{x \to +\infty} \frac{\omega_j}{\alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) = \frac{1}{\varrho}.
\]

On the other hand, in view of (17) we obtain from (19)

\[
\frac{1}{\varrho_{\alpha\beta}(I)} \leq \sum_{j=1}^{m} \lim_{x \to +\infty} \frac{\omega_j}{\alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) = \frac{1}{\varrho}.
\]

Thus, $\varrho_{\alpha\beta}(I) = \varrho$.

From (18) and Lemma 2 it follows that

\[
\frac{1}{T_{\alpha\beta}(I)} = \lim_{x \to +\infty} \frac{1}{\exp \alpha(x)} \prod_{j=1}^{m} \exp \left\{ \omega_j \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \right\} \geq \prod_{j=1}^{m} \lim_{x \to +\infty} \left( \frac{\exp \{ \omega_j \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \} }{\exp \alpha(x)} \right)^{\omega_j} = \prod_{j=1}^{m} \left( \frac{1}{T_{\alpha\beta}(I)} \right)^{\omega_j}.
\]

If we choose $\alpha(x) = \ln \ln \ln x$, $\beta(x) = \ln x$ for $x \geq x_0$, $m = 2$ and $\omega_j = 1/2$ then from Theorem 7 we obtain the following statement.
Corollary 4. Let \( F \in V \), \( \ln F(x) = o(x \ln \ln x) \) as \( x \to +\infty \) and the functions \( f \) and \( f_j \) \((j = 1, 2)\) have regular variation in regard to \( F \). Suppose that \( f_1(x) > 0 \) for all \( x \geq x_0 \) and
\[
\ln \left( \frac{1}{x} \ln \frac{1}{f_2(x)} \right) \leq (1 + o(1)) \ln \left( \frac{1}{x} \ln \frac{1}{f_1(x)} \right) \text{ as } x \to +\infty.
\]
If \( \lim_{\sigma \to +\infty} \ln \frac{\ln \ln \ln f(x)}{\ln \sigma} = \sigma \in (0, +\infty) \) for \( j = 1, 2 \) and
\[
\ln \frac{1}{f(x)} = (1 + o(1)) \sqrt{\ln \frac{1}{f_1(x)} \ln \frac{1}{f_2(x)}}, \quad x \to +\infty,
\]
then \( \lim_{\sigma \to +\infty} \ln \frac{\ln \ln \ln f(x)}{\ln \sigma} = \sigma \) and
\[
\lim_{x \to +\infty} \exp_3 \left\{ \left( \frac{x}{\ln f(x)} \right)^\sigma \right\} \leq \sqrt{\lim_{x \to +\infty} \exp_3 \left\{ \left( \frac{x}{\ln f(x)} \right)^\sigma \right\} \lim_{x \to +\infty} \exp_3 \left\{ \left( \frac{x}{\ln f(x)} \right)^\sigma \right\}},
\]
where \( \exp_3 x = \exp\{\exp\{\exp\{e^x\}\}\} \).

For integral (1) of finite modified generalized orders we define the generalized type \( T_{\alpha \beta}^M(I) \) by the formula
\[
T_{\alpha \beta}^M(I) = \lim_{\sigma \to +\infty} \frac{\ln I(\sigma)}{\ln \sigma \alpha^{-1}(\sigma \beta(\sigma))}, \quad (\sigma = \sigma_{\alpha \beta}^M(I)).
\]

Then the following lemma is true.

**Lemma 3.** Let \( \beta \in L \), \( \beta_1(x) = \alpha^{-1}(\sigma \beta(\sigma)) \in L_{si} \), the function \( F \in V \) satisfy condition
\( \ln F(x) = o(x \beta_1^{-1}(cx)) \) as \( x \to +\infty \) for every \( c \in (0, +\infty) \), and \( f \) have regular variation in regard to \( F \). Then
\[
T_{\alpha \beta}^M(I) = \lim_{x \to +\infty} \frac{x}{\alpha^{-1}}} \left( \frac{1}{\ln f(x)} \right)^{\sigma}).
\]

Indeed, \( \beta_1^{-1}(cx)) = \beta^{-1}(\alpha(\sigma)) \leq \beta^{-1}(\alpha(\sigma)) \), because from the condition \( \alpha \in L^0 \) it follows that \( \alpha(cx) \leq K(c)\alpha(cx) \) for every \( c \in (0, +\infty) \). Therefore, if we choose \( \alpha_1(x) \equiv x \) for \( x \geq x_0 \), then from Theorem 3 with \( \alpha_1 \) and \( \beta_1 \) instead of \( \alpha \) and \( \beta \) we deduce Lemma 3.

**Theorem 8.** Let \( \beta \in L_{si} \), \( \alpha(x) = (1 + o(1)) \ln x \) as \( x \to +\infty \) and \( \beta_1(x) = \alpha^{-1}(\sigma \beta(\sigma)) \in L_{si} \). Let \( F \in V \), \( \ln F(x) = o(x \beta_1^{-1}(cx)) \) as \( x \to +\infty \) and \( f \) and \( f_j \) have regular variation in regard to \( F \). Suppose that all integrals (10) have the same modified generalized order \( \sigma_{\alpha \beta}^M(I_j) = \sigma \in (0, +\infty) \) and the modified generalized types \( T_{\alpha \beta}^M(I_j) \in (0, +\infty) \). Suppose also that \( f_1(x) > 0 \) for all \( x \geq x_0 \) and (17) holds.

If \( \omega_j > 0 \) for \( 1 \leq j \leq m \), \( \sum_{1 \leq j \leq m} \omega_j = 1 \), and
\[
\alpha^{-1} \left( \frac{1}{\ln f(x)} \right)^{\omega_j} = (1 + o(1)) \prod_{j=1}^{m} \alpha^{-1} \left( \frac{1}{\ln f_j(x)} \right)^{\omega_j}, \quad x \to +\infty,
\]
then integral (1) has the modified generalized order \( \sigma_{\alpha \beta}^M(I) = \sigma \) and the modified generalized type \( T_{\alpha \beta}^M(I) \leq \prod_{j=1}^{m} T_{\alpha \beta}^M(I_j)^{\omega_j} \).
Proof. At first, we remark that the condition \( \ln F(x) = o(x\beta^{-1}(x)) \) as \( x \to +\infty \) implies (4), because \( \beta^{-1}(x) = \beta^{-1}(\alpha(t)/\varrho)) \). Thus, the functions \( \alpha, \beta, \) and \( F \) satisfy the conditions of Theorem 3.

Since \( \alpha(x) = (1 + o(1)) \ln x \) as \( x \to +\infty \), from (20) we obtain (19). As above from (19) by Theorem 3 \( 1/\varrho_{\alpha\beta}^M(I) \geq 1/\varrho \). On the other hand, from (17) and (19) as above we get \( 1/\varrho_{\alpha\beta}^M(I) \leq 1/\varrho \). Thus, \( \varrho_{\alpha\beta}^M(I) = \varrho \).

From (20) and Lemma 3 it follows that

\[
\frac{1}{T_{\alpha\beta}^M(I)} = \lim_{x \to +\infty} \frac{1}{x} \alpha^{-1} \left( \beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right) \right) = \lim_{x \to +\infty} \frac{1}{x} \prod_{j=1}^{m} \left( \alpha^{-1} \left( \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \right) \right)^{\omega_j} \\
= \prod_{j=1}^{m} \lim_{x \to +\infty} \left( \frac{1}{x} \alpha^{-1} \left( \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \right) \right)^{\omega_j} = \prod_{j=1}^{m} \left( \frac{1}{T_{\alpha\beta}^M(I)} \right)^{\omega_j}.
\]

If we choose \( \alpha(x) = \ln x, \beta(x) = \ln \ln x \) for \( x \geq x_0, m=2 \) and \( \omega_j = 1/2 \) then from Theorem 8 we obtain the following statement.

Corollary 5. Let \( F \in V, \ln F(x) = o(x\exp(x^e)) \) as \( x \to +\infty \) and the functions \( f \) and \( f_j, j = 1, 2 \), have regular variation in regard to \( F \). Suppose that that \( f_i(x) > 0 \) for all \( x \geq x_0 \) and \( \ln \left( \frac{1}{x} \ln \frac{1}{f_i(x)} \right) \leq (1 + o(1)) \ln \left( \frac{1}{x} \ln \frac{1}{f_1(x)} \right) \) as \( x \to +\infty \).

If \( \lim_{\sigma \to +\infty} \frac{\ln \ln I(\sigma) - \ln \sigma}{\ln \sigma} = \varrho \in (0, +\infty) \) for \( j = 1, 2 \) and

\[
\ln \left( \frac{1}{x} \ln \frac{1}{f(x)} \right) = (1 + o(1)) \sqrt{\ln \left( \frac{1}{x} \ln \frac{1}{f_1(x)} \right) \ln \left( \frac{1}{x} \ln \frac{1}{f_2(x)} \right)}, \quad x \to +\infty,
\]

then \( \lim_{\sigma \to +\infty} \frac{\ln \ln I(\sigma) - \ln \sigma}{\ln \sigma} = \varrho \) and \( \lim_{\sigma \to +\infty} \frac{\ln I(\sigma) - \ln \sigma}{\ln \sigma} \leq \sqrt{\lim_{\sigma \to +\infty} \ln \frac{I(\sigma)}{\sigma \ln \sigma} \lim_{\sigma \to +\infty} \ln \frac{I(\sigma)}{\sigma \ln \sigma}}. \)

Finally, we prove a theorem, which supplements Theorems 6 and 8.

Theorem 9. Let either \( \alpha \in L_{si} \) and \( \beta \in L^0 \) or \( \alpha \in L^0 \) and \( \beta \in L_{si} \), \( F \in V, \ln F(x) = o(x\beta^{-1}(\alpha^t(x))) \) as \( x \to +\infty \) for every \( c \in (0, +\infty) \) and integrals (10) have modified general orders \( \varrho_{\alpha\beta}^M(I_j) \in (0, +\infty) \). Suppose that \( f \) has regular variation in regard to \( F \) and (11) holds. Then:

1) if \( f_1(x) > 0 \) for all \( x \geq x_0 \) and for all \( 2 \leq j \leq m_0 \)

\[
\ln \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \leq (1 + o(1)) \ln \beta \left( \frac{1}{x} \ln \frac{1}{f_1(x)} \right), \quad x \to +\infty.
\]

(21)

then

\[
\lim_{\sigma \to +\infty} \frac{1}{\ln \beta(\sigma)} \ln \alpha \left( \frac{\ln I(\sigma)}{\sigma} \right) = 1
\]

(22)

and \( \varrho_{\alpha\beta}^M(I) \leq \prod_{j=1}^{m} \left( \varrho_{\alpha\beta}^M(I_j) \right)^{\omega_j} \).

2) if \( v(x) = -(\ln f(x))^t \) is continuous and increasing on \( [x_0, +\infty) \) and all integrals (10) have regular modified \( \alpha\beta \)-growth then integrals (1) has regular modified \( \alpha\beta \)-growth and

\[
\varrho_{\alpha\beta}^M(I) = \prod_{j=1}^{m} \left( \varrho_{\alpha\beta}^M(I_j) \right)^{\omega_j}.
\]
Proof. Since \( g_{\alpha \beta}^M(I_j) \in (0, +\infty) \), we have
\[
\lim_{\sigma \to +\infty} \frac{1}{\ln \beta(\sigma)} \ln \alpha \left( \frac{\ln I_j(\sigma)}{\sigma} \right) = 1.
\]

It is known ([9]) that if \( h \in L^0 \) then \( h \) is a RO-increase function ([10, p.86]), that is for every \( \lambda \in [1, +\infty) \) and all \( x \geq x_0 \) the inequalities
1. \( 1 \leq h(\lambda x)/h(x) \leq M(\lambda) < +\infty \), whence it follows that \( \ln h \in L_{si} \). Therefore, using Theorem 3 with \( \ln \alpha \) and \( \ln \beta \) instead of \( \alpha \) and \( \beta \) (the condition \( \ln F(x) = o(x\beta^{-1}(\alpha c(x))) \) as \( x \to +\infty \) for every \( c \in (0, +\infty) \) implies condition (4)), we obtain
\[
\lim_{x \to +\infty} \frac{1}{\ln \alpha(x)} \ln \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) = 1
\]
for each \( j = 1, 2, \ldots, m \), and in view of (11)
\[
\lim_{x \to +\infty} \frac{1}{\ln \alpha(x)} \ln \beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right) = \lim_{x \to +\infty} \frac{1}{\ln \alpha(x)} \sum_{j=1}^{m} \omega_j \ln \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \geq
\]
\[
\geq \sum_{j=1}^{m} \omega_j \lim_{x \to +\infty} \frac{1}{\ln \alpha(x)} \left( \frac{1}{x} \ln \frac{1}{f_1(x)} \right) = 1.
\]

On the other hand by virtue of (21)
\[
\lim_{x \to +\infty} \frac{1}{\ln \alpha(x)} \ln \beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right) = \frac{1}{\ln \alpha(x)} \left( \omega_1 \ln \beta \left( \frac{1}{x} \ln \frac{1}{f_1(x)} \right) + \sum_{j=2}^{m} \omega_j \ln \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \right) \leq
\]
\[
\leq \lim_{x \to +\infty} \frac{1}{\ln \alpha(x)} \sum_{j=1}^{m} \omega_j \ln \beta \left( \frac{1}{x} \ln \frac{1}{f_1(x)} \right) = 1,
\]
i. e.
\[
\lim_{x \to +\infty} \frac{\ln \alpha(x)}{\ln \beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right)} = 1
\]
and by Theorem 3 equality (22) is true.

The condition \( \ln F(x) = o(x\beta^{-1}(\alpha c(x))) \) as \( x \to +\infty \) for every \( c \in (0, +\infty) \) implies the condition \( \ln F(x) = o(x\beta^{-1}(\alpha c(x))) \) as \( x \to +\infty \) for every \( c \in (0, +\infty) \). Therefore, by Theorem 3 in view of (11)
\[
1 = \lim_{x \to +\infty} \frac{1}{\ln \alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f_1(x)} \right) = \lim_{x \to +\infty} \prod_{j=1}^{m} \left( \frac{1}{\alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \right)^{\omega_j} \geq
\]
\[
\geq \prod_{j=1}^{m} \lim_{x \to +\infty} \left( \frac{1}{\alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \right)^{\omega_j} = \prod_{j=1}^{m} (g_{\alpha \beta}^M(I_j))^{\omega_j},
\]
i. e. the statement 1) is proved.
Now, if $\varrho_{\alpha\beta}^M(I_j) = \varrho_{\alpha\beta}^M(I_j)$ then by Theorem 4 for each $j = 1, 2, \ldots m$

\[
\frac{1}{\alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) \rightarrow \frac{1}{\varrho_{\alpha\beta}^M(I_j)}, \quad x \rightarrow +\infty,
\]

and from (11) we get

\[
\frac{1}{\alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f(x)} \right) = (1 + o(1)) \prod_{j=1}^m \frac{1}{\alpha(x)} \beta \left( \frac{1}{x} \ln \frac{1}{f_j(x)} \right) =
\]

\[
= (1 + o(1)) \prod_{j=1}^m \left( \frac{1}{\varrho_{\alpha\beta}^M(I_j)} \right)^{\omega_j}, \quad x \rightarrow +\infty.
\]

Hence by Theorems 3 and 4 it follows that integral (1) has regular modified $\alpha\beta$-growth and $\varrho_{\alpha\beta}^M(I) = \prod_{j=1}^m (\varrho_{\alpha\beta}^M(I_j))^{\omega_j}$.

REFERENCES