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LEIBNIZ'S WELL-FOUNDED FICTIONS AND THEIR INTERPRETATIONS

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Leibniz used the term *fiction* in conjunction with infinitesimals. What kind of fictions they were exactly is a subject of scholarly dispute. The position of Bos and Mancosu contrasts with that of Ishiguro and Arthur. Leibniz's own views, expressed in his published articles and correspondence, led Bos to distinguish between two methods in Leibniz's work: (A) one exploiting classical 'exhaustion' arguments, and (B) one exploiting *inassignable* infinitesimals together with a law of continuity.

Of particular interest is evidence stemming from Leibniz's work *Nouveaux Essais sur l'Entendement Humain* as well as from his correspondence with Arnauld, Bignon, Dagincourt, Des Bosses, and Varignon. A careful examination of the evidence leads us to the opposite conclusion from Arthur's.

We analyze a hitherto unnoticed objection of Rolle's concerning the lack of justification for extending axioms and operations in geometry and analysis from the ordinary domain to that of infinitesimal calculus, and reactions to it by Saurin and Leibniz.

A newly released 1705 manuscript by Leibniz (*Puisque des personnes...*) currently in the process of digitalisation, sheds light on the nature of Leibnizian inassignable infinitesimals.

In a pair of 1695 texts Leibniz made it clear that his incomparable magnitudes violate Euclid's Definition V.4, a.k.a. the Archimedean property, corroborating the non-Archimedean construal of the Leibnizian calculus.

> J'appelle *grandeurs incomparables* dont l'une multipliée par quelque nombre fini que ce soit, ne sçauroit exceder l'autre. –G. W. Leibniz
>
> [C]es touts infinis, et leurs opposés infiniment petits, ne sont de mise que dans le *calcul des géomètres*, tout comme les racines imaginaires de l'algèbre. –*Théophile*

1. Introduction. Figure 130 in l'Hospital's book *Analyse des infiniment petits...* illustrates his *Article 163*. This item concerns the application of what is known today as l'Hôpital's rule to the geometric situation of two curves crossing at a point B on the x-axis.
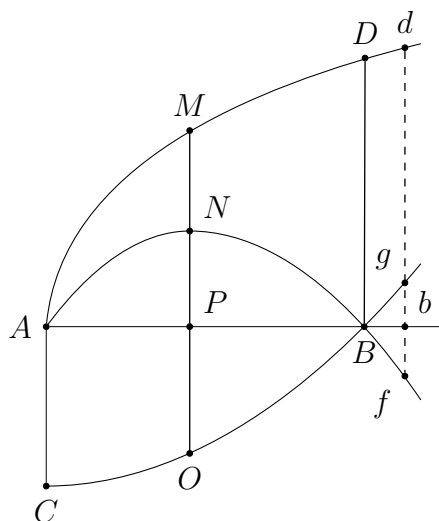
Figure 1: L'Hospital's Figure 130

1.1. Figure 130. L'Hospital's figure places another point b near B on the x-axis, as well as points f and g on the two curves with abscissa (i.e., x-coordinate) equal to b; see Figure 1. The author argues that bf and bg are the differentials corresponding to the two curves. He concludes that the ratio of the ordinates (i.e., y-coordinates) of the two curves equals the ratio of their differentials.

In a 27 July 1705 manuscript *Puisque des personnes...* meant to be sent to Pierre Varignon (but sent to Jacques Lelong on 31 July 1705), Leibniz seeks to address one of the objections raised by Michel Rolle concerning a case where l'Hôpital's rule needs to be applied twice. Here Leibniz writes:

Et ce cas des evanouissants ou naissants est si pres du cas dont il s'agit qu'il n'en diffère d'aucune *grandeur assignable*, comme il est manifeste dans la figure 130 du traité de l'Analyse des infinitesimales, ou [i.e., où] la distance entre B et b est moindre qu'aucune qu'on puisse *assigner*." (Leibniz [82], 1705; emphasis added)

The passage furnishes a succinct summary of Leibniz's take on infinitesimals:

1. the infinitesimal distance Bb is smaller than any assignable magnitude;

2. thus Bb is itself *inassignable*;

3. the points B and b are specific points in an illustration from l'Hospital's book;

4. thus the distance Bb is (not a sequence but) a number.

This apparently straightforward analysis of Leibniz's notion of infinitesimal is resisted by a number of modern scholars. Over three centuries after Leibniz published his first article on infinitesimal calculus in *Acta Eruditorum* ([70], 1684), scholars are still debating the nature of Leibnizian infinitesimals. There are two main interpretations of Leibnizian infinitesimals in the current literature: one associated with historian Henk Bos and the other, with philosopher Hidé Ishiguro.

1.2. Bos on two approaches. Bos writes:

Leibniz considered two different approaches to the foundations of the calculus; one connected with the classical methods of proof by 'exhaustion', the other in connection with a law of continuity. (Bos [27], 1974, p. 55)

Bos goes on to describe the second method as follows: "The chief source for Leibniz's second approach to the justification of the use of 'fictitious' infinitesimals in the calculus is a manuscript..." [27, p. 56]. Here Bos is referring to Leibniz's manuscript *Cum Prodiisset* ([75], 1701).

Leibniz's second approach mentioned by Bos is based on a law of continuity, explained by examples as follows:

[I]n the case of intersecting lines, for instance, arguments involving the intersection could be extended (by introducing an "imaginary" point of intersection and considering the angle between the lines "infinitely small") to the case of parallelism; also arguments about ellipses could be extended to parabolas by introducing a focus infinitely distant from the other, fixed, focus. [27, p. 57]

Bos grants that Leibnizian infinitesimals are fictional:

[Leibniz] had to treat the infinitesimals as 'fictions' which need not correspond to actually existing quantities, but which nevertheless *can be used* in the analysis of problems. ([27, pp. 54–55]; emphasis added)

1.3. Pure fictions and logical fictions. Such usable fictions could be termed *pure fictions*; see Sherry–Katz ([112], 2014) in *Studia Leibnitiana*. Bos' position is largely endorsed by Jesseph ([52], 2015). The question therefore is not *whether* Leibnizian infinitesimals are fictions, but rather *what kind of* fictions: pure fictions or logical fictions; see Section 1.7.

The matter of Appendix 2 in (Bos [27], 1974) was dealt with in Katz–Sherry ([59], 2013, Section 11.3, pp. 606–608) and Bair et al. ([12], 2017, Section 2.7, p. 204); see also Section 3.10 here.

For a comparison of Leibniz's *law of continuity* and the *transfer principle*[1] of Abraham Robinson's framework for analysis with infinitesimals see Katz–Sherry ([58], 2012) as well as Section 2.7. Bos' position is largely endorsed by Paolo Mancosu; see Section 2.4. Marc Parmentier similarly sees two separate techniques in Leibniz's work, including *De Quadratura Arithmetica*:

Sa structure binaire se manifeste également dans l'opposition entre deux types de méthodes utilisables pour réaliser des quadratures. Leibniz présente la première comme un amendement, dûment fondé et démontré, de la méthode des indivisibles.[2] Quant à l'autre méthode, elle est fondée sur les infiniment petits. L'enjeu de la *Quadratura* est dès lors d'établir leur équivalence. (Parmentier [98], 2001, p. 278)

1.4. Extensions and predicates. There is no need necessarily to rely on the idea of *extension* when formalizing Leibniz's procedures exploiting infinitesimals. Indeed, in Edward Nelson's approach to infinitesimal analysis, such entities are found in the ordinary real line. Extensions like $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ (see note 16) are necessary only if one wishes to formalize infinitesimals in the context of Zermelo–Fraenkel set theory based on the language possessing a single relation, namely the membership relation $\in$. If, following Nelson, one allows for a

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[1]See note 18 for a summary concerning transfer.

[2]When Leibniz spoke of the method of indivisibles in DQA, he sometimes had in mind a broader method of traditional geometry originating with Archimedes, namely the technique of exhaustion. Thus, referring to his method *not* exploiting infinitesimals, Leibniz wrote: "Quare methodo indivisibilium quae per spatia gradiformia seu per summas ordinatarum procedit, ut severe demonstrata licebit." (Parmentier's translation: "Voilà ce qui permettra de faire de la méthode des indivisibles et de l'usage des espaces gradiformes soit des sommes des ordonnées qui en sont l'apanage, une méthode et un usage rigoureusement démontrés" [80, p. 63].) This method differs from the "direct method" which exploits infinitesimals. See also note 40.

richer language including also a unary (i.e., one-place) predicate **standard** (together with axioms governing its interaction with the Zermelo–Fraenkel axioms), then infinitesimals (defined as a nonstandard numbers smaller in absolute value than all positive standard ones) can be found within the ordinary real line. (see Section and note 44 for details).

The need for a two-tier number system to account for infinitesimal calculus was felt by philosophers of the Marburg school, Hermann Cohen (1842–1918) and Paul Natorp (1854–1924). They exploited the pair intensive/extensive to describe such a number system, with infinitesimals being *intensive* (Cohen's student Ernst Cassirer, while nominally endorsing the view, in practice went on to analyze other calculi, instead). However, the anti-infinitesimal sentiment fueled by Cantor, Russell, and others at the time was too powerful and the necessary mathematical tools not yet available to enable a convincing formalisation of such ideas; see Mormann–Katz [94] for details. An important link between the Marburg neo-Kantians and Robinson's school is Abraham Fraenkel; see Kanovei et al. ([53], 2018) for details. Felix Klein's take on infinitesimals was more positive than is generally known; see Bair et al. ([13], 2017).

1.5. Infinite sets *vs* infinite numbers in Leibniz. Leibniz held, following Galileo, that infinite aggregates, collections, multitudes, or totalities (what we may refer to as *sets* today) lead to contradiction; see e.g., Knobloch ([66], 2012). Thus, in a letter to Bernoulli dated 22 August 1698, Leibniz wrote:

To be sure, several years ago I have proved that the multitude of all numbers implies a contradiction, if [it is] taken to be a single totality.[3] (Leibniz [74], 1698, p. 535)

On the other hand, Leibniz routinely used fictional infinite (and infinitesimal) *numbers* in his work; see e.g., Section 4.1, Katz–Sherry ([58], 2012), and Blåsjö ([23], 2017).

1.6. Physics, matter, and space. Leibniz occasionally uses the term *syncategorematic* in discussing physics, as described by De Risi:

[Leibniz] endorsed the (quite non-Aristotelian) view that bodies are infinitely divided *in actu*, but in a purely 'syncategorematic' way. This latter notion aimed at expressing the idea that there is no final element in the division of matter (i.e. no point), even though there are more divisions of bodily parts than can possibly be expressed by any finite number" (De Risi [34], 2019; emphasis in the original).

This use of the term "syncategorematic infinity" refers to matter or space, closely related to indefinite divisibility.[4]

1.7. Ishiguro, logical fictions, and alternating quantifiers. An alternative to Bos's interpretation was developed by Ishiguro in ([50], 1990, Chapter 5).[5] Ishiguro argues that a term that seems to express a Leibnizian infinitesimal does not actually designate, denote, or refer; rather, it is a *logical fiction* in the sense of Russell; see e.g., (Russell [107], 1919, p. 45). Such a reading of Leibnizian infinitesimals contrasts with the Bos–Mancosu reading as presented in Section 1.3 in terms of *pure fictions*.

The distinction between pure fiction and logical fiction does not always receive sufficient attention from Leibniz scholars. Thus, Ohad Nachtomy provides the following summary:

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[3]In the original: "Sane ante multos annos demonstravi, numerum seu multitudinem omnium numerorum contradictionem implicare, si ut unum totum sumatur."

[4]See also main text at note 25.

[5]The interpretation in question was developed in the second, 1990 edition of Ishiguro's book, and is not yet found in its first edition (Ishiguro [49], 1972).

Arthur argues that, due to a *syncategorematic* interpretation of the infinitely small, by 1676 Leibniz could use infinitesimals in calculations and avoid the mystery – and indeed the contradictions – that their would-be existence would involve... Thus a continuous whole can be treated as if it consists in an infinity of infinitesimals; but although by such means one can represent truths, *there are not such things in reality as infinite wholes or infinitely small parts.* (Nachtomy [96], 2014; emphasis added)

However, this passage amounts to a banal claim that in "reality" there is no referent for Leibnizian infinitesimals, an assertion agreed to by Bos, Mancosu, and Parmentier. As a summary of the syncategorematic position that it purports to be, Nachtomy's passage is a failure.

Note that Russell himself never applied his concept of *logical fiction* to Leibnizian infinitesimals. According to Ishiguro,

[Leibniz] is treating [infinitely small lines] as convenient theoretical fictions because using signs which looks [sic] as if they stand for quantities sui generis is useful." (Ishiguro [50], 1990, p. 84)

Such "signs which look as if they stand for quantities" turn out to conceal universal and existential quantifiers as follows. Ishiguro contends that Leibniz's continuum is Archimedean, as when she emphasizes "the importance that Leibniz attached to his claim that, strictly speaking, there are only finite numbers and magnitudes" [50, p. 99]. She clarifies the nature of her non-designating claim in the following terms: "we can paraphrase the proposition with a universal proposition with an embedded existential claim" [50, p. 87]. In conclusion,

Fictions [such as Leibnizian infinitesimals] are *not entities* to which we refer. ... They are correlates of ways of speaking which can be reduced to talk about more standard kinds of entities. ([50, p. 100]; emphasis added)

Such fictions, which are *not entities to which we can refer* in Ishiguro's view, are exemplified by Leibnizian infinitesimals:

We saw that Leibniz believed that his language of infinitesimals was rigorous, although there is only a syncategorematic infinitesimal. [50, p. 96]

1.8. Of rabbits and snakes. Ishiguro's reading has been endorsed by a number of Leibniz scholars. While not all of them subscribe to her *not entity* view, Ishiguro's idea that Leibnizian infinitesimals are *not* to be interpreted as non-Archimedean quantities has been endorsed by Arthur ([4], 2013, p. 554), Goldenbaum ([43], 2008, p. 76, note 59), Gray ([45], 2015, p. 10), Levey ([91], 2015, p. 184), Nachtomy ([95], 2009 and [96], 2014), and elsewhere. Rabouin ([100], 2015, note 25, p. 362) describes a Leibnizian infinitesimal as a "'syncategorematic' entity" citing Ishiguro, but points out on the same page that "this arbitrariness [in the choice of ε] does not amount, in modern terms, to a universal quantification (at least in classical first order logic), which would be meaningless to Leibniz."

Knobloch interprets Leibnizian infinitesimals as *variable quantities*; see ([61], 1990), ([62], 1994), ([64], 2002), and ([65], 2008); see also Section 2.3.

Breger's view of Leibniz's infinitesimal as a *process* led him to a particularly colorful metaphor:

Whoever is interested in the provability rather than in the art of finding should not stare at the infinitely small magnitude like a rabbit at the snake; he should take a closer look at the process of ever-decreasing divisions. (Breger [28], 2008, p. 188)

Breger's claim that to understand Leibnizian infinitesimals one should "look at the process of ever-decreasing divisions" may not be easy to reconcile with Leibniz's claim that one does *not* reach infinitesimals by a process of ever-decreasing divisions; see e.g., Section 2.4. Breger goes on to offer a sharp criticism of Bos' position in [28, pp. 196–197]. However, Breger's criticism begs the question since it is predicated on the logical fiction hypothesis. Namely, Breger writes:

[A]ccording to Bos there were two strategies by which to justify Leibniz's recourse to infinitesimals: epsilontics and the principle of continuity (Bos, 1974, 55–57). This distinction appears artificial, for the principle of continuity is *of course* also founded on epsilontics: two magnitudes are equal if their difference is smaller than any magnitude that can possibly be expressed. ... In either case the processual nature is the decisive point; it is of no great import whether the process is described by means of *epsilontics* or with reference to the principle of continuity. (Breger [28], 2008, p. 197; emphasis added).

Breger's rabbit-and-snake metaphor contains the germs of a remarkable admission, namely that some historians tend to become paralyzed when faced with *bona fide* infinitesimals.[6]

1.9. Syncategorematic talk.
Some of these authors claim to find support for their view in Proposition 6 from Leibniz's unpublished text *De Quadratura Arithmetica*. Their claims have been challenged in a recent detailed textual study by Blåsjö ([21], 2017); see also Knobloch ([68], 2017), Blåsjö ([22], 2017), and Section 5.

Ishiguro's interpretation of Leibnizian infinitesimals goes under the name *syncategorematic*:[7]

...talk of infinitesimals is, as [Leibniz] says, *syncategorematic* and is actually about 'quantities that one takes... as small as is necessary in order that the error should be smaller than the given error.' (Ishiguro [50], 1990, p. 90; emphasis added)

Note that here Ishiguro applies the term not to physics, matter, or space (see Section 1.6), but rather to an individual magnitude. Used this way, the term is the catchword for the idea that Leibnizian infinitesimals are "signs which look as if they stand for quantities" but in reality signify concealed quantifiers.[8] The passage Ishiguro is alluding to actually provides a piece of evidence *against* her syncategorematic thesis; see Section 3.1.

1.10. Of Leibniz and Weierstrass.
Bowdlerized accounts of Leibniz's position portraying him as a proto-Weierstrassian are ubiquitous in the literature. Thus, commenting on Leibniz's observation that "it is unnecessary to make mathematical analysis depend on or to make sure that there are lines in nature which are infinitely small in a rigorous sense in contrast to our ordinary lines, or as a result, that there are lines infinitely greater than our

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[6]The "epsilontic" theme is continued in Breger's 2017 article: "The fact that the infinitely small (and the incomparably small) magnitudes derived their justification from epsilontics was simply self-evident. Leibniz did not consider it necessary to explain this *in any depth*" (Breger [29], 2017, p. 78; emphasis added). One appreciates Breger's admission that there is no *in-depth* source in Leibniz for the "epsilontic" reading. See further in note 37.

[7]Peter Geach pointed out the inappropriateness of using the term *syncategorematic* in this context: "'Categorematic' and 'syncategorematic'... are words used to describe (uses of) words in a language; an infinite multitude, say, can no more be syncategorematic than it can be pronominal or adverbial. To be sure, the confusion is explicable..." (Geach [39], 1967, p. 41).

[8]On this reading, they are ghosts of departed *quantifiers*; cf. Bair et al. ([11], 2017). Thus, "[i]f first-order differentials have absorbed a logical quantifier, second-order differentials have absorbed two logical quantifiers." (Breger [28], 2008, p. 194).

ordinary ones, etc." (Leibniz [85], 1989, pp. 542–543), *editor* Loemker feels compelled to declare:

If Leibniz had more clearly combined his conception of the infinitesimal as a quantity to be taken at will as less than any assignable quantity whatever with his own analysis of series and his functional conception of the law of continuity, he should have been led to the critical concept of limits upon which the calculus was *at last* theoretically grounded in the nineteenth century by Weierstrass and Cauchy.[9] (Editor Loemker commenting in [85, note 2, p. 546]; emphasis added)

The presentist view of the history of analysis as inexorably progressing toward, and culminating, *at last*, in the Weierstrassian *Epsilontik* was analyzed by Bair et al. ([11], 2017). See also Hacking ([46], 2014) on the distinction between a butterfly and a Latin model for the development of a science.

1.11. Arthur's endorsement of syncategorematic reading. In 2013 Richard Arthur endorses Ishiguro's reading in the following terms:

I take the position here (following Ishiguro 1990) that the idea that Leibniz was committed to infinitesimals as actually infinitely small entities is a misreading: his mature interpretation of the calculus was fully in accord with the Archimedean Axiom. Leibniz's interpretation is (to use the medieval term) *syncategorematic*: Infinitesimals are fictions in the sense that the terms designating them can be treated as if they refer to entities incomparably smaller than finite quantities, but really stand for variable finite quantities that can be taken as small as desired. (Arthur [4], 2013, p. 554)

Here Arthur seeks to apply the qualifier *syncategorematic* to a Leibnizian infinitesimal, rather than either multitude, matter, or space (see Sections 1.5 and 1.6). The Ishiguro–Arthur (IA) syncategorematic thesis concerning Leibniz's infinitesimals has gained wide acceptance in the literature; see Section 1.7. For more details on (Arthur [4]) see Section 3.8.

In 2014 Arthur endorses an allegedly *non-referring* nature of Leibnizian infinitesimals:

Just as the infinite is not an actually existing whole made up of finite parts, so infinitesimals are not existing parts which can be composed into a finite whole. Borrowing a term from the Scholastics, Leibniz called the infinite and the infinitely small *syncategorematic terms*: like 'it' or 'some' in a meaningful sentence, they do not in themselves *refer* to determinate things, but can be used perfectly meaningfully in a specified context. (Arthur [5], 2014, p. 81; emphasis on 'syncategorematic terms' in the original; emphasis on 'refer' added)

In 2015 Arthur renews his endorsement in the following terms:

Ishiguro (1990)... was one of the first to argue that Leibniz can allow for the success of treating the infinite and infinitely small *as if* they are entities (under certain conditions), and that it is this that allows him to claim that mathematical practice is not affected by whether one takes them to be real or not.[10] (Arthur [6], 2015, p. 146, note 16)

Ishiguro attributes such *nonentity* syncategorematic *as-if* infinitesimals to Leibniz without restricting it to any specific period of Leibniz's career. Arthur and Levey acknowledge the presence of infinitesimal *entities* at least in the early Leibniz. Accordingly, they have

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[9]On Cauchy see note 10.

[10]Quantifiers are alluded to on the same page in the following terms: "the justification is in terms that, after *Cauchy*, we would now express in terms of ε and δ" [6, p. 146] (emphasis added). For an analysis of the error of attributing prototypes of (ε, δ) alternating quantifier definitions to Cauchy see Bascelli et al. ([17], 2018).

modified Ishiguro's position to a syncategorematic interpretation starting as early as 1676 (see Arthur [4], 2013, p. 554), recognizing that there is a historical development of Leibniz's mathematical insights and ideas, including the notion of infinitesimals.

In 2018, Arthur's syncategorematic infinitesimal goes *actual*. In a chapter entitled 'Leibniz's syncategorematic actual infinite,' Arthur writes:

[T]o say that a magnitude is *actually* infinitely small in the syncategorematic sense is to say that no matter how small a magnitude one takes, there is a smaller, but there are no *actual* infinitesimals." (Arthur [7], 2018, p. 155; emphasis added)

So is this magnitude actual or not actual? Arthur's desire to incorporate the qualifier "actual" in his title leads him to comical incoherence in his discussion of magnitudes. He goes on to offer yet another endorsement of the *logical fiction* hypothesis in the following terms: "In geometry one may calculate with expressions apparently denoting such entities, on the understanding that they are fictions, standing for variable magnitudes that can be made arbitrarily small..." (ibid., pp. 155–156).

A re-evaluation of Leibniz's contribution to analysis was developed in 2012 (Katz–Sherry [58]), in 2013 (Katz–Sherry [59]), in 2014 (Bascelli et al. [15], Sherry–Katz [112]), in 2016 (Bair et al. [16]), in 2017 (Bair et al. [12], Błaszczyk et al. ([24]), and elsewhere. In an apparent reaction to this work, Arthur wrote in 2018:

Certain scholars of the calculus have denied that the interpretation of infinitesimals as syncategorematic was Leibniz's mature view, and have seen them as *fictions in a different sense*. I shall not mainly be concerned with that line of disagreement here, reserving a detailed critique of such views for another occasion." (Arthur [7], 2018, p. 156; emphasis added).

Meanwhile Arthur's 2019 texts [8], [9] do not contain the reserved critique.

An IA-style reading of Cauchy's infinitesimal as a logical fiction has been challenged in Borovik–Katz ([26], 2012), Bair et al. ([10], 2013), Bascelli et al. ([15], 2014), Bair et al. ([11], 2017), Bascelli et al. ([17], 2018), and elsewhere.

2. Mathematical fictions. We argue that the IA position to the effect that "Fictions [such as Leibnizian infinitesimals] are not entities to which we refer... They are correlates of ways of speaking which can be reduced to talk about more standard kinds of entities" (see Section 1.7) involves equivocation on the meaning of the term *fiction*.

2.1. Entities, nonentities, and referents. To the extent that we have symbolism for mathematical concepts, we can, under suitable conditions, *refer* to them; to the extent that such concepts have no referents, we can also assert that they are "not entities to which we refer", provided we take note of the fact that there is no difference here between Leibnizian infinitesimals on the one hand and e.g., unending decimal strings,[11] on the other. In this sense Ishiguro's attempt at quantifier-assisted transcription of infinitesimals amounts merely to an attempt at long-winded paraphrase of one variety of nonentity by another.[12]

We argue that already in his first publication on the calculus in 1684 and especially starting in the 1690s, Leibniz exploited fictional infinitesimals not reducible to a quantifier paraphrase, and even made it clear that they violate the Archimedean property (see Sections 3.2. and 3.3).

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[11]These were already developed by Simon Stevin at the end of the 16th century; see Katz–Katz ([56], 2012).
[12]Ishiguro uses the techniques of modern mathematics like quantifiers to develop her reading of Leibniz, but overlooks the fact that merely referring to real numbers as *standard*, as she does, does not make them any less lacking in referent than infinitesimals, from the modern viewpoint.

2.2. Smaller than any given quantity. Leibniz repeatedly defined infinitesimals as being smaller than any given quantity. IA read this as "a universal proposition with an embedded existential claim" (Ishiguro [50], 1990, p. 87), namely, as an assertion involving alternating quantifiers (see Section 1.7). However, a more straightforward reading is to interpret *given quantities* as being *assignable* and infinitesimals as *inassignable*, in the terminology Leibniz used both in *Cum Prodiisset* in 1701 (see Section 4.1) and in his manuscript *Puisque des personnes...* in 1705 (see Section 1.1).[13] Sometimes Leibniz also uses the terminology of *incomparables* for infinitesimals (see Section 3.9).

Leibniz famously denied that infinite *totalities* can be viewed as *wholes*, but such a rejection does not necessarily extend, at least in the mathematical realm, to infinite and infinitesimal *magnitudes* and *quantities*, as discussed in Section 1.5.

2.3. Variable quantities from Varignon to Knobloch. Knobloch reads Leibniz' infinitesimals as being variable quantities:

Eventually, Leibniz adhered to 'smaller than any given quantity' or infinitely small that is to a completely consistent fruitful definition of infinitely small. An infinitely small quantity is a variable quantity and can be described in terms of the Weierstrassian epsilon-delta language... It must be a variable quantity that can be described in the Weierstrassian ε-δ-language: smaller than any given quantity. (Knobloch, [69], 2019, pp. 2, 7)

Knobloch's interpretation echoes a related stance expressed by Varignon in 1700–01, when he attempted to defend and clarify Leibniz' infinitesimal calculus in response to attacks by Rolle. In this section, we will compare Knobloch's view with Varignon's.

Rolle subsequently engaged Joseph Saurin in an a polemic that lasted several years. We will analyze the Rolle–Saurin exchange in Section 2.9. For the purposes of this section, we quote the following criticism expressed by Rolle:

On reconnoît d'abord que les effets des méthodes qu'on propose dans la nouvelle Analyse, sont toujours les mêmes quand on substitue des quantités finies à volonté au lieu des Infiniment petits *dx* & *dy*: ce qui prouve que le succès, bon ou mauvais, n'est point attaché à *l'infinie petitesse* qu'on suppose dans le Systême. (Rolle [106], 1703, pp. 324–325; emphasis added)

Here Rolle speaks of arbitrary finite quantities in terms similar to Knobloch's (with the exception of the "Weierstrassian ε-δ"), the difference being that Rolle is *contrasting* such quantities with infinitesimals, whereas Knobloch seeks to *identify* them. Rolle argued that the new *Systême* is superfluous since whatever it can achieve can already be achieved by what he saw as finitist algebraic techniques developed by Fermat and others; see Section 2.8. Clearly both sides in the Rolle–Saurin exchange understood Leibnizian infinitesimals to be characterized by a property Rolle refers to as *l'infinie petitesse*. To pursue his Weierstrassian thesis, Knobloch would be forced to postulate that Leibniz was misunderstood by his contemporaries on account of his infinitesimals (Ishiguro faced a similar dilemma; see Section 4.8).

This series of exchanges was part of a flurry of debate at the Paris Academy of Sciences between 1700 and 1705 focused on the viability of the new Leibnizian calculus; see Blay ([25], 1986) and Mancosu ([92], 1989).

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[13]The distinction assignable *vs* inassignable goes back to the distinction quanta *vs* non-quanta in work of Nicholas of Cusa (1401–1464), which also inspired Galileo's distinction between *quanta* and *non-quanta* according to Knobloch ([63], 1999, p. 89 and [69], 2019). For details on assignable *vs* inassignable see Section 4.1.

Rolle attacked the idea of infinitesimals in several ways, including protesting that they are sometimes treated as nonzero quantities and sometimes as absolute zeroes, and in particular that from the equation $dx + x = x$, one must conclude $dx = 0$; see [92].

In his defense of Leibniz' infinitesimals against these arguments, Varignon appealed to Newton's *Principia Mathematica* as the source of imagery and rigor, extensively quoting the Scholium to Lemma XI in Book I. According to Varignon, Rolle had failed to understand that infinitesimals are actually variable quantities, not fixed ones. They decrease continually until they reach zero, but are "considered only in the moment of their evanescence" [92, p. 231].

Varignon states (in his correspondence with Bernoulli) that differentials consist "in being infinitely small and infinitely changing until zero, in being nothing but *quantitates evanescentes, evanescentia divisibilia*, they will always be smaller than any arbitrary given quantity" (Bernoulli [20], 1988, p. 357).

Varignon goes on to explain how this can always be expressed instead with an exhaustion argument, "in the way of the ancients":

Indeed, whatever difference can be assigned between two magnitudes which differ only by a differential it will always be possible, on account of the continual and indefinite variability of this infinitely small differential, and as on the verge of being zero, to find a differential less than the given difference. Which shows, in the way of the ancients, that notwithstanding their difference these two quantities can be taken to be equal. (In Bernoulli [20], 1988, p. 357)

It is suggestive that Varignon had to look to Newton for an appropriate account of infinitesimals as being changing quantities.

2.4. Leibniz's response to Varignon. In a letter dated 28 November 1701, Varignon asked Leibniz to make a precise statement on what should be undestood by *infinitesimal quantity*. Leibniz's letter [76] dated 2 february 1702 contains a response to Varignon. Mancosu summarizes Leibniz's response in the following three points:

(a) There is no need to base mathematical analysis on metaphysical assumptions. (b) We can nonetheless admit infinitesimal quantities, if not as real, [then] as well-founded fictitious entities, as one does in algebra with square roots of negative numbers. Arguments for this position depended on a form of the metaphysical principle of continuity. Or (c) one could organize the proofs so that the error will be always less than any assigned error. (Mancosu [93], 1996, p. 172)

What is conspicuously absent is the Newton–Varignon definition of infinitesimal as a variable quantity (see Section 2.3). Mancosu notes further:

In his letter [Leibniz] merged *two different foundational approaches*. The first was related to the classical methods of proof by exhaustion; the second was based on a metaphysical principle of continuity. (ibid.; emphasis added)

Thus, Mancosu follows Bos in seeing both A-track and B-track methods (see Section 5.1) in Leibniz. The B-track method (mentioned in item (b) of in Mancosu's summary) relies on what is referred to by Bos [27, p. 55] and Katz–Sherry [59] as the *law of continuity*.

As Leibniz wrote in the 2 February 1702 letter to Varignon (GM, IV, 91–95), this is a heuristic law to the effect that the rules of the finite are found to succeed in the infinite, and conversely the rules of the infinite apply to the finite:

Yet one can say in general that though continuity is something ideal and there is never

anything in nature with perfectly uniform parts, the real, in turn, never ceases to be governed perfectly by the ideal and the abstract... (Leibniz as translated in [85, p. 544])

Having formulated the basic distinction between the real and the ideal, Leibniz proceeds to formulate his heuristic law:

... and that *the rules of the finite are found to succeed in the infinite,* ...[14] *And conversely the rules of the infinite apply to the finite,* as if there were infinitely small metaphysical beings, although we have no need of them, and the division of matter never does proceed to infinitely small particles. (ibid.; emphasis added)

We will analyze some reactions to Leibniz's heuristic law in Section 2.7.

2.5. *Fixe et determinée.* The pair of qualifiers *fixe et determinée* occurs in several letters in the Leibniz–Varignon exchange. The first letter in the series is a 28 November 1701 letter from Varignon to Leibniz, complaining about Jean Galloys (Gallois) in the following terms:

M. l'Abbé Galloys... repand ici que vous avez declaré n'entendre par differentielle ou Infinement [sic] petit, qu'une grandeur à la verité tres petite, mais cependant toujours *fixe et determinée*, telle qu'est la Terre par raport au firmament, ou un grain de sable par raport à la Terre... (Varignon [116], 1701, p. 89)

Thus, Galloys (and Varignon following him) use the pair *fixe et determinée* to refer to a specific "small" assignable magnitude.

In a 2 February 1702 response (quoted in Section 5.1), Leibniz speaks of *common incomparables* which are still ordinary assignable numbers. Leibniz describes the latter as not being fixed and determined, by which he means that they need to be made arbitrarily small in an exhaustion-type argument.

In a subsequent letter dated 22 March 1702 from Varignon to Bernoulli (that reached Leibniz in april), Varignon mentions that he showed Leibniz's 2 February 1702 letter to P. Gouye, and describes the latter's "choleric" reaction to a perceived change in Leibniz's stance (GM IV 97). Leibniz responds on 14 April 1702 as follows:

Je reconnois d'avoir dit quelque chose de plus dans ma lettre, aussi estoit-il necessaire, car il s'agissoit d'éclaircir le memoire, mais je ne crois pas qu'il y ait de l'opposition. Si ce Pere [Gouye] en trouve et me la fait connoistre, je tacheray de la lever. Au moins n'y avoit il pas la moindre chose qui dût faire juger que j'entendois une quantité tres petite à la verité, mais tousjours *fixe et determinée.* (Leibniz [77], 1702; emphasis added)

Here Leibniz uses the pair of qualifiers *fixe et determinée* in the sense used by the opponents of the calculus (Galloys and Gouye), namely to refer to a specific "small" assignable number. Leibniz denies that his incomparable (as opposed to *common* incomparables mentioned in the 2 February 1702 letter) is such a number.

The qualifier *common* is similarly used in "Tentamen de motuum coelestium causis" (Leibniz [71], 1689; see also [41], V.VI, 144–161) in reference to the type of incomparables used to justify infinitesimals by means of assignable numbers:

I have assumed in the demonstrations incomparably small quantities, for example the difference between two *common* quantities which is incomparable with the quantities themselves. Such matters as these, if I am not mistaken, can be set forth most lucidly in what follows. And then if someone does not want to employ infinitely small quantities, he can take them to be as small as he judges sufficient to be incomparable, so that they produce an error of

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[14]The passage omitted at this point is analyzed in Section 2.6.

no importance and even smaller than any given [error]. (Leibniz as translated by Jesseph in [51], 2008, p. 227; emphasis added)

Leibniz proceeds to give practical examples of common incomparables:

Just as the Earth is taken for a point, or the diameter of the Earth for a line infinitely small with respect to the heavens, so it can be demonstrated that if the sides of an angle have a base incomparably less than them, the comprehended angle will be incomparably less than a rectilinear angle, and the difference between the sides will be incomparable with the sides themselves; also, the difference between the whole sine, the sine of the complement, and the secant will be incomparable to these differences. (ibid.)

2.6. The passage on atomism. The passage we omitted in Section 2.4 (see note 14) reads as follows: "as if there were atoms, that is, elements of an assignable size in nature, although there are none because matter is actually divisible without limit" (Leibniz as translated by Loemker). Loemker's translation of this passage is imprecise. We reproduce the original:

comme s'il y avait des atomes (c'est à dire des elemens assignables de la nature), quoyqu'il n'y en ait point la matiere estant actuellement sousdivisée sans fin. (Leibniz [76], 1702, p. 93; original spelling retained)

This is the unique mention of *atomes* in Leibniz's letter.

Modern readers may well be puzzled by Leibniz's aside on atomism. Granted Leibniz's consistent opposition to atomism, what need is there to interrupt a discussion of the principles of infinitesimal calculus by an aside concerning physical atomism? Why does Leibniz feel a need, specifically in an infinitesimal context, to distance himself from atomism, a doctrine familiar to schoolchildren today?

A possible answer lies in the 17th century battles – theological and otherwise – over the doctrines of hylomorphism, transubstantiation, and eucharist. Atomism was thought of by the catholic hierarchy at the time as contrary to canon law as codified at the Council of Trent in 1551 (session 13, canon 2) and therefore heretical. The infinitely small, via the language of indivisibles exploited by Cavalieri and others, were thought of as closely related to atomism. Leibniz's aside therefore may have constituted a defensive move. For details see Fouke ([37], 1992), Bair et al. ([14], 2018).

Amir Alexander ([1], 2015) offers a different account of the opposition to indivisibles in the 17th century. Namely, the jesuits saw Euclidean mathematics as an organizing principle that helps man make order out of chaos (and in particular defeat the reformers). They saw indivisibles as introducing a dangerous discordant note in the otherwise (near-)perfect harmony of Euclid, and therefore opposed them as a subversive reform. The jesuits viewed indivisibles as actual errors introduced into the heart of pristine geometry. Indivisibles made geometry paradoxical, unreliable, and chaotic, the very opposite of what they believed it must be. See also Sherry ([111], 2018) and Alexander ([2], 2018). Inspite of their differences, Alexander and Sherry agree on the following: (i) indivisibles were controversial in the 17th century; (ii) the opposition emanated from powerful religious circles; (iii) the opposition was a major factor in the decline of the Italian school of geometry.[15]

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[15]The case of James Gregory is particularly instructive. Gregory studied under the indivisibilist Stefano degli Angeli at Padua. Gregory left Padua in 1668 shortly before degli Angeli's religious order of the jesuats was banned by papal brief in the same year. Gregory's books were subsequently supressed in Italy. The same year also marked an abrupt stop to degli Angeli's output on indivisibles. For details see Bascelli et al. ([18], 2018).

In a similar vein, Leibniz distanced himself from the idea of material *indivisibles* while discussing the fictional nature of infinitesimals in a 20 June 1702 letter to Varignon:

Entre nous je crois que Mons. de Fontenelle, qui a l'esprit galant et beau, en a voulu railler, lorsqu'il a dit qu'il vouloit faire des elemens metaphysiques de nostre calcul. Pour dire le vray, je ne suis pas trop persuadé moy même, qu'il faut considerer nos infinis et infiniment petits autrement que comme des choses ideales ou comme des fictions bien fondées. ... Il est que les substances simples (c'est à dire qui ne sont pas des estres par aggregation) sont veritablement *indivisibles*, mais elles sont immaterielles, et ne sont que principes d'action. (Leibniz [78], 1702, p. 110; emphasis added)

Leibniz expressed similar sentiments in a 1716 letter to Dagincourt; see Section 3.9.

2.7. Leibniz's heuristic law: from Rolle to Robinson. Leibniz addressed a letter to Jean-Paul Bignon in July 1705. Here Leibniz summarizes some objections voiced against infinitesimal calculus by Rolle at the *Academie* in the following terms:

[Ces objections] reviennent à dire en effect qu'en maniant ce nouveau Calcul des infinitesi-males, on ne doit point avoir la liberté d'y joindre les *axiomes et operations* de la Geometrie et de l'Analyse ancienne; qu'on ne doit point substituer *aequalibus aequalia*, qu'on ne doit point dire que de deux quantités égales les quarrés sont egaux aussi, et choses semblables; ... (Leibniz [81], 1705, p. 838; emphasis in the original; emphasis on "axiomes et operations" added)

Leibniz speaks dismissively of such objections (going as far as describing them as *des chicanes* [81, p. 839]). Nonetheless, Rolle's objections are valid. Why do the newly introduced numbers obey the same axioms and operations as those governing the *ancienne* geometry and analysis? Why does the squaring operation extend as expected? We will deal with these objections in more detail in Section 2.9.

2.8. Rolle on Descartes and Fermat. The thrust of Rolle's critique of infinitesimal calculus was that the latter was both unnecessary and plagued by error. Rolle felt that infinitesimal calculus was unnecessary because the problems it solves are solved more easily with what he claimed were ordinary algebraic techniques already available. Rolle was specifically referring to the work of Fermat:

Cependant M. Descartes luy-même dans une autre Lettre à M. Hardy explique & perfectionne la Methode de M. de Fermat. II designe la *difference* des abscisses par un segment de ligne dans la figure, & il la designe encore par la lettre *e* dans le calcul, comme l'avoit déja fait M. de Fermat luy-même. Outre cela il suppose une droite qui rencontre la courbe en deux points, & qui doit devenir Tangente lorsque la *difference* indéterminée des abscisses est prise pour un *zero absolu*. Il poursuit selon les Regles ordinaires de la Geometrie & de l'Algebre, & selon les idées de l'Auteur dont il explique la Methode. (Rolle [105], 1703, p. 2)

Fermat actually used a capital letter E (rather than the lower-case e). The clause "qui doit devenir Tangente lorsque la *difference* indéterminée des abscisses est prise pour un *zero absolu*" is open to interpretation; see e.g., note 28, (Katz et al. [57], 2013), (Bair et al. [14], 2018).

2.9. Rolle, Saurin, and *chicanes***.** The issue of handling the squaring operation in the extended domain, mentioned in Section 2.7, is a subtler problem than might appear at first sight. Leibniz repeatedly emphasizes that he is working with a generalized notion of equality "up to" a negligible term; see e.g., Leibniz's sentences [1] and [2] quoted in Section 3.2.. It is possible to interpret Leibniz's comment as asserting an equality between, say, a pair of

infinitely close *infinite* numbers H and $H + \varepsilon$, where ε is infinitesimal, e.g., $\varepsilon = \frac{1}{H}$. If so, computing the squares of the two numbers we obtain a difference of

$$(H + \varepsilon)^2 - H^2 = H^2 + 2H\varepsilon + \varepsilon^2 - H^2 = 2H\varepsilon + \varepsilon^2 = 2 + \varepsilon^2.$$

Thus, the difference between the squares in this case is an appreciable (non-infinitesimal) amount $2 + \varepsilon^2$, and one can reasonably ask whether Leibniz would consider the squares still equal and under what circumstances. The generalized equality is used, for example, in the proof of Leibniz's rule $d(xy) = xdy + ydx$ which involves dropping the negligible term $dxdy$. This procedure was pertinently criticized by Rolle in [106, p. 327].[16]

Leibniz's comment on squaring quoted in Section 2.7 was prompted by the following comment by Rolle:

On y voit un second dégagement, et une troisiéme substitution; on y quarre les deux membres de la formule $S = \frac{xdy}{dx}$ Ce qui n'a encore esté pratiqué dans la Géometrie transcendante, ni indiqué par aucune regle dans cette Géometrie. (Rolle [105], 1703, p. 33)

Leibniz's comment on extending "axioms and operations" was prompted by Rolle's objection to what he felt was a dubious procedure that consists in

citer des regles ordinaires qui ayent quelque rapport aux operations; supposer qu'elles sont particulieres à l'Analyse des Inf. petits. [105, p. 33–34]

Rolle's objections were quoted by Joseph Saurin in his response; see (Saurin [108], 1705, p. 248). It is instructive to examine Saurin's reaction to Rolle's objections:

mais nous serions tombés-là, dans une extraction de racines, non moins inoüie dans toute la Geometrie transcendante, que la permission que nous nous sommes donnée d'élever la formule au quarré; tant il nous étoit impossible de resoudre le cas proposé par M. Rolle, sans faire des supplémens à nos défectueuses Méthodes. (Saurin [108], 1703, pp. 253–254)

Saurin's sarcasm is palpable, but what about a response to Rolle's objection? Alas, none is forthcoming. Instead, sarcasm turns to *ad hominem*:

Aprés cette vaine & puerile discussion, où m'ont jetté les difficultez de M. Rolle; je suis obligé pour mon honneur de déclarer icy aux Géometres que je sens toute la honte qu'il y a à s'arrêter à des objections de cette nature. Si je le fais, c'est parce qu'elles servent à faire connoître de plus en plus quel est l'esprit de l'Auteur que je refute, etc. (ibid., p. 254)

Saurin proceeds next to Rolle's objection regarding the extension of rules:

citer des regles ordinaires qui ayent quelque rapport aux operations; supposer qu'elles sont particulieres à l'Analyse des Inf. Pet. ... Tous cela paroles jettée en l'air, & qui ne prouve autre chose, sinon que les manieres de l'Auteur sont toûjours les mêmes. (ibid.)

Saurin was unable to appreciate Rolle's objection but in fact, Rolle's objection was more poignant than those formulated three decades later by George Berkeley (see Katz–Sherry [59], 2013). We will analyze Rolle's objection further in Section 2.11.

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[16]This issue can be routinely clarified in Robinson's framework in terms of the standard part function, in the context of the hyperreal extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$. The subring ${}^\flat\mathbb{R} \subseteq {}^*\mathbb{R}$ consisting of the finite elements of ${}^*\mathbb{R}$ admits a map **st** to $\mathbb{R}$, known as *standard part*. The map $\mathbf{st} \colon {}^\flat\mathbb{R} \to \mathbb{R}$ rounds off each finite hyperreal number to its nearest real number. This enables one, for instance, to define the derivative of $t = f(s)$ as $f'(s) = \mathbf{st}\left(\frac{\Delta t}{\Delta s}\right)$ (for infinitesimal $\Delta s \neq 0$). For details see e.g., Keisler ([60], 1986), Katz–Sherry ([58], 2012). See also notes 18 and 28.

2.10. *La Réforme.* As in Leibniz's comments on atomism and indivisibles (see Section 2.6), religious tensions seem just below the surface in the Rolle–Saurin exchange. Rolle appears to feel free to exploit phrases like *selon la réforme* when referring to Leibnizian calculus, even though its practitioners never used the term to describe the new techniques. The term *réforme* occurs at least ten times in (Rolle [105]).

Meanwhile, Saurin had converted to catholicism barely a decade earlier, and would not necessarily have appreciated Rolle's choice of terminology, containing an allusion to the Reformation and the Counter-Reformation. These were developments of a recent past at the time.

In one of his responses, Saurin alludes to Rolle's terminology, and asks rhetorically: where is the reform in all this? (with "reform" italicized): "Y a-t-il là quelque *déguisement*, quelque *supplément*, quelque *reforme*?" (Saurin [109], 1706, p. 12). And again: "D'abord on remarquera que cette solution à laquelle M. Rolle s'est principalement attaché, est non *un nouveau supplément, une nouvelle reforme*, ainsi qu'il l'appelle, mais une solution de surcroît; ..." (ibid.)

Rolle's move of imputing ideologically impure motives to the pro-infinitesimal opposition is not without modern imitators.[17]

2.11. Transfer. As noted in Section 2.9, Rolle was asking for a justification of Leibniz's heuristic law allowing one to extend rules from the ordinary domain to the extended domain of the infinitesimal calculus. In modern terminology such justification is provided by the *transfer principle*. Robinson analyzed Leibniz's heuristic law as follows:

Leibniz did say, in one of the passages quoted above, that what succeeds for the finite numbers succeeds also for the infinite numbers and vice versa, and this is remarkably close to our transfer[18] of statements from $\mathbb{R}$ to $^*\mathbb{R}$ and in the opposite direction. (Robinson [101], 1966, p. 266)

Rolle was unwilling to accept the validity of such transfer. There is little in the responses he received that could have satisfied him, given his rejection of infinitesimal quantities.

Leibniz suggests that infinitesimals ought to be treated as fictitious entities, as one does in algebra with square roots of negative numbers. Thus the difference is not merely a distinct approach to infinitesimals from Varignon's, but a broader difference in their stances on mathematical formalism.

According to Mancosu, "by considering the infinitesimals as well-founded fictions, [Leibniz] was introducing a gap between the formal apparatus and the referents" (Mancosu [93], 1996, p. 173).

3. Evidence: incomparables. We argue that Leibnizian texts tend to support the interpretation of Bos and Mancosu over that of IA.

As a general comment, note that someone seeking to contest the IA interpretation of Leibnizian infinitesimals as logical *fictions* might be tempted to assert that Leibnizian infinitesimals are not fictional but *real.* Such a formulation may unwittingly entail ontological

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[17]See Section 4.3 at note 30.

[18]The *transfer principle* is a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system, still apply (i.e., are "transfered") to an extended number system. Thus, the familiar extension $\mathbb{Q} \hookrightarrow \mathbb{R}$ preserves the property of being an ordered field. To give a negative example, the extension $\mathbb{R} \hookrightarrow \mathbb{R} \cup \{\pm\infty\}$ of the real numbers to the so-called *extended reals* does not preserve the property of being an ordered field. The hyperreal extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ (see note 16) preserves *all* first-order properties, e.g., the formula $\sin^2 x + \cos^2 x = 1$ which remains valid for all hyperreal x, including infinitesimal and infinite values of $x \in {}^*\mathbb{R}$.

commitments as to the *reality* of infinitesimals. However, one can reject Ishiguro's interpretation and still retain the fictionalist interpretation of Leibnizian infinitesimals.

To borrow Moigno's and Connes' terminology, one might say that an infinitesimal is *chimerical*.[19] It does not however follow that they are *logical* chimeras in the IA sense.

3.1. Leibniz's syncategorematic passage. Leibniz wrote in 1702:

Cependant il ne faut point s'imaginer que la science de l'infini est degradée par cette explication, & reduite à des fictions; car il reste toujours un infini syncategorematique, comme parle l'Ecole & il demeure vray par exemple, que 2 est autant que 1/1+1/2+1/4+1/8+ 1/16+1/32+&c. Ce qui est une *series* infinie dans laquelle toutes les fractions, dont les Numerateurs sont l'unité, & les denominateurs de progression Geometrique double, sont comprises à la fois; quoy qu'on n'y employe toujours que des nombres ordinaires, (Leibniz [76], 1702; emphasis in the original)

Having made his remark concerning what he refers to as syncategorematic infinity that involves only ordinary numbers, Leibniz goes on to conclude:

& quoy qu'on n[']y fasse point entrer aucune fraction *infiniment petite*, ou dont le denominateur soit un nombre infini. (ibid.; emphasis added)

The plain meaning of the passage is that an infinitely small fraction (or a fraction whose denominator is an infinite number) is *not* involved in syncategorematic infinity. Thus Leibniz takes what he refers to as *infinitely small fractions* to be *bona fide* infinitesimals of track B type, in the terminology of Section 5.1.

3.2. Euclid, Definition V.4, and incomparables. Leibniz repeatedly made it clear that his system of magnitudes involves a violation of the Archimedean property, viz., Euclid's *Elements*, Definition V.4; see e.g., the passage in Leibniz ([73], 1695, p. 322) as quoted by Bos ([27], 1974, p. 14). This definition is a technical expression of Leibniz's distinction between assignable and inassignable quantities; see Sections 1.1 and 4.1.

The violation of V.4 appears directly to contradict the IA claim that Leibniz was working with an Archimedean continuum. Leibniz frequently writes that his infinitesimals are *useful fictions*; but it is best not to understand them as logical fictions but rather as *pure fictions*; see Section 1.2.

Let us consider in more detail Leibniz's comment in his article *Responsio ad nonnullas difficultates a Dn. Bernardo Niewentiit. . .* on Euclid V.5 (numbered V.4 in modern editions), which is a version of the Archimedean axiom:[20]

[1] Furthermore I think that not only those things are equal whose difference is absolutely zero, but also those whose difference is incomparably small. [2] And although this [difference] need not absolutely be called Nothing, neither is it a quantity comparable to those whose difference it is. [3] It is so when you add a point of a line to another line or a line to a surface, then you do not increase the quantity. [4] The same is when you add to a line a certain line that is incomparably smaller. [5] Such a construction entails no increase. [6] Now I think, in accordance with Euclid Book V def. 5, that only those homogeneous quantities one of which, being multiplied by a finite number, can exceed the other, are comparable. [7] And those

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[19]See Schubring ([110], 2005, p. 456) and Bascelli et al. ([17], 2018, Section 4.1) on Moigno; see Connes ([31], 2004, p. 14) and Kanovei et al. ([54], 2013, Section 8.2, p. 287) on the views of Connes.
[20]See De Risi [33], 2016, Section II.3 for more details on Euclid's Definition V.4.

that do not differ by such a quantity are equal, which was accepted by Archimedes and his followers.[21] (translated from Leibniz [73], 1695, p. 322; numerals [1] through [7] added)

Here Leibniz employs the term *line* in the sense of what we would today call a *segment*. In clause [3], Leibniz exploits the classical example with indivisibles (adding a point to a line doesn't change its length) so as to motivate a similar phenomenon for incomparables in clause [4] (adding an incomparably small line to a finite line does not increase its quantity), namely his *transcendental law of homogeneity* (Leibniz [83], 1710) summarized in clause [5].[22]

In clause [6], Leibniz refers to *homogeneous* quantities satisfying Euclid's definition V.5, i.e., the Archimedean axiom. In a follow-up clause [7], Leibniz goes on to refer to 'those [quantities],' say Q and Q', that 'do not differ by such a quantity,' namely they do *not* differ by a *homogeneous* quantity of the type mentioned in clause [6] (that would *satisfy* Euclid V.5 relative to Q or Q'). Rather, Q and Q' differ by a quantity *not satisfying* Euclid V.5, i.e., a quantity which violates V.5 relative to Q and Q'. Leibniz views such quantities as equal in the sense of a generalized relation of equality governed by his *law of homogeneity*; see Katz–Sherry ([58], 2012).

Leibniz referred to differences as in clause [1] as *incomparably small*. Thus Leibniz is clearly dealing with an *incomparably small* difference $Q - Q'$ which *violates* Euclid V.5 relative to Q or Q'.

3.3. Letter to l'Hospital refutes reading by Arthur. Leibniz is even more explicit about the fact that his *incomparables* violate Euclid V.5 in his letter to l'Hospital dated 14/24 June 1695:

J'appelle *grandeurs incomparables* dont l'une multipliée par quelque nombre fini que ce soit, ne sçauroit exceder l'autre, de la même façon qu'Euclide la pris dans sa cinquieme definition du cinquieme livre.[23] (Leibniz [72], 1695, p. 288; original spelling preserved; emphasis in the original)

In formulas, what Leibniz is saying is that magnitude ε is incomparable with a magnitude r when the following formula is satisfied: $(\forall n)[n\varepsilon < r]$ (for finite n). Thus Leibniz makes it clear that his incomparable magnitudes violate the Archimedean property (a.k.a. Euclid V.4) relative to r. We will analyze Breger's discussion of this passage in Section 3.4.

Arthur's claim in [4, p. 562] based on the very passage from *Responsio* quoted in Section 3.2 that allegedly "Leibniz was quite explicit about *this Archimedean foundation* for his differentials as 'incomparables'" (emphasis added) is therefore surprising. Arthur does

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[21]In the original Latin: "[1] Caeterum aequalia esse puto, non tantum quorum differentia est omnino nulla, sed et quorum differentia est incomparabiliter parva; [2] et licet ea Nihil omnino dici non debeat, non tamen est quantitas comparabilis cum ipsis, quorum est differentia. [3] Quemadmodum si lineae punctum alterius lineae addas, vel superficiei lineam, quantitatem non auges. [4] Idem est, si lineam quidem lineae addas, sed incomparabiliter minorem. [5] Nec ulla constructione tale augmentum exhiberi potest. [6] Scilicet eas tantum homogeneas quantitates comparabiles esse, cum Euclide lib. 5 defin. 5 censeo, quarum una numero, sed finito multiplicata, alteram superare potest. [7] Et quae tali quantitate non differunt, aequalia esse statuo, quod etiam Archimedes sumsit, aliique post ipsum omnes." (Leibniz [73], 1695, p. 322; numerals [1] through [7] added)

[22]In reference to this passage, Breger claims that "the unassignable magnitudes are fictitious, they cannot be determined by any construction" (Breger [28], 2008, p. 196), but fails to deal with Leibniz's very next sentence concerning Euclid V.5 (V.4 in modern editions).

[23]Translation: "I use the term *incomparable magnitudes* to refer to [magnitudes] of which one multiplied by any finite number whatsoever, will be unable to exceed the other, in the same way [adopted by] Euclid in the fifth definition of the fifth book [of the *Elements*]" (V.4 in modern editions).

not provide any explanation for his claim but rather merely reproduces the passage we analyzed in Section 3.2, goes on to cite additional passages from Leibniz, and then gets into a discussion of Leibniz's 1684 article and other texts.

Arthur thus fails to explain his inference of an allegedly Archimedean nature of the Leibnizian continuum from this passage. Therefore we can only surmise the nature of Arthur's inference, apparently based on the reference to Archimedes himself in the passage quoted in Section 3.2. However, the term *Archimedean axiom* for Euclid V.4 was not coined until the 1880s (see Stolz [114], 1883), about two centuries after Leibniz.

Thus, Leibniz's mention of Archimedes in [73] could not refer to what is known today as the Archimedean *property* or *axiom*. Rather, Leibniz mentions an ancient authority merely to reassure the reader of the soundness of his methods, or alludes to the method of exhaustion. Arthur's cryptic claim concerning the passage mentioning Archimedes (i.e., that it is indicative of an allegedly Archimedean foundation for the Leibnizian differentials) is misleading and anachronistic.

Leibniz's 1695 letter to l'Hospital (involving a violation of Euclid Definition V.4 by Leibniz's incomparables) is absent from Arthur's bibliography. We will analyze Ishiguro's comments on the letter in Section 3.5.

3.4. Breger on letter to l'Hospital. In Section 3.3 we presented our analysis of the allusion to the violation of Euclid V.4 in Leibniz's 14/24 June 1695 letter to l'Hospital. Breger gives a similar interpretation of this passage:

In a letter to L'Hôpital of 1695, Leibniz gives an explicit definition of incomparable magnitudes: two magnitudes are called incomparable if the one cannot exceed the other by means of multiplication with [sic] an arbitrary (finite) number, and he expressly points to Definition 5 of the fifth book of Euclid quoted above. (Breger [29], 2017, p. 73–74).

Here Breger acknowledges that Leibnizian incomparable magnitudes violate Euclid V.5 (V.4 in modern editions), i.e., that they are non-Archimedean relative to ordinary ones. Breger's position is especially significant. This is because he otherwise pursues an interpretation close to the logical fiction hypothesis (see e.g., Section 1.8), leading Arthur to write:

When Breger says that there is no actual infinite in Leibniz's mathematics, he is primarily concerned to deny the reading of the actual infinite in Leibnizian mathematics as categorematic (as in *non-Archimedean construals* of the continuum and infinitesimals), and I have no quarrel with him... about this." (Arthur [7], 2018, p. 157; emphasis added)

Contrary to Arthur's claim (and to Breger's position expressed elsewhere), Breger does not deny but rather endorses such a "non-Archimedean construal" in [29, pp. 73–74].

We briefly consider the possibility that what Breger might have meant here is an interpretation of Leibnizian incomparable magnitudes as functions or sequences tending to zero. However, such an interpretation is untenable for the following reason. If a violation of V.4 is attributable to e.g., a sequence tending to zero, then it becomes nearly impossible for a system of magnitudes to avoid being in violation of Euclid V.4. Namely, as soon as one incorporates magnitudes corresponding to, say, the ordinary rational numbers, by density one will be able to choose a sequence tending to zero, and thus detect a violation of Euclid V.4 in this sense. The only systems *not* violating Euclid V.4 would be discrete systems like $\mathbb{N}$. It seems clear that this is not the meaning Euclid had in mind when he formulated Definition 4 of his book V. Since Leibniz refers explicitly to Euclid it seems also clear that such a discrete

system of magnitudes is not what Leibniz had in mind, for otherwise he would have likely mentioned such a significant departure from Euclid's intention.

In other words, the ordinary system of magnitudes in Euclid is clearly meant to obey V.4 for otherwise Euclid would not have stated V.4 as a definition. On the other hand, it is clear that such a system cannot be as restrictive as a discrete system like $\mathbb{N}$. With regard to Leibniz, it is particularly clear that his system of ordinary (assignable) magnitudes necessarily incorporates arbitrarily small ones, since Leibniz believed in indefinite divisibility of matter (see e.g., Section 2.6). Such a system of ordinary magnitudes cannot satisfy V.4 if Definition V.4 is interpreted in terms of sequences. Thus the interpretation of magnitudes in terms of sequences is at tension with both Euclid's and Leibniz's intention. See also end of Section 3.5.

3.5. Ishiguro on l'Hospital. The 1695 letter [72] from Leibniz to l'Hospital responding to Nieuwentijt's criticism is cited in Ishiguro's bibliography but Ishiguro erroneously describes Leibniz's criticism of Nieuwentijt here as criticism of... Leibniz's ally, l'Hospital:

It is important to realise however that in this letter Leibniz is using de l'Hospital's own criterion to refute him. De l'Hospital had asserted both that higher differentials are not magnitudes and, if after being multiplied by an infinite number the assumed quantity does not become an ordinary magnitude, then it is not a magnitude at all. It is a nothing. Leibniz responded that if that is what de l'Hospital believes, then he cannot at the same time claim that ddx and $dxdx$ are not magnitudes, since they would, if multiplied by an infinite number ("*per numerum infinitum sed altiorem seu infinites infinitum*") become ordinary magnitudes. This is, however, *not* Leibniz's *own* criterion, as he does not believe that there is such a thing as an infinite number.[24] He is on the contrary trying to explain what differentials and quadratures are by spelling out the thought that leads to them in terms of finite quantities, finite numbers, and Leibniz's concept of 'infinitely many' and of 'incomparable.' (He points out for example that de l'Hospital is wrong to think that if dy is equal, dx would also be equal.). (Ishiguro [50], 1990, p. 89)

Ishiguro repeatedly attributes to l'Hospital what Leibniz describes as Nieuwentijt's errors. Nieuwentijt is not mentioned at all on Ishiguro's page 89.

What are we to make of Ishiguro's command of the details of the alignment of forces among Leibniz's contemporaries with regard to infinitesimal calculus? The shoddiness of her command of such details undermines the credibility of her sweeping claims to the effect that Leibniz was allegedly misunderstood by his contemporaries like Bernoulli and l'Hospital (see Section 4.8), particularly in view of the fact that Leibniz specifically endorses l'Hospital's approach; see Section 1.1.

No scholar of ancient Greece has yet stepped forward to give a syncategorematic reading of Euclid's Definition V.4. It seems reasonable to assume that Leibniz's understanding of Euclid's Definition V.4 and its negation was similar to that of modern scholars; see e.g., De Risi ([33], 2016). Therefore, to account for Leibniz's 1695 texts analyzed in Sections 3.2. and 3.3, advocates of the logical fiction approach would have to extend Ishiguro's hypothesis that Leibniz was misunderstood by his contemporary scholars to apply to modern scholars of Euclid, as well.

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[24]Ishiguro provides no sources to justify her claim that "Leibniz does not believe that there is such a thing as an infinite number." Nor does she pay attention to the distinction between multitude and number; see Section 1.5.

3.6. *Nouveaux Essais sur l'Entendement Humain.* In his 2014 book, Arthur makes the following claim:

Having reached this conclusion in 1676, [Leibniz] holds it from then on: 'there is no infinite number, nor infinite line or other infinite quantity, if these are taken to be genuine wholes.' (NE 157) There is an actual infinite, but it must be understood *syncategorematically*, etc. (Arthur [5], 2014, p. 88)

The same syncategorematic claim, based on the same Leibnizian passage, is reproduced four years later in (Arthur [7], 2018, p. 161). However, a careful examination of the evidence leads one to the opposite conclusion from Arthur's. Arthur's reference (NE 157) is an English translation of Leibniz's treatise *Nouveaux Essais sur l'Entendement Humain.* Here a fictional character named Théophile argues as follows:

Théophile: A proprement parler, il est vrai qu'il y a une infinité de choses, c'est-à-dire qu'il y en a toujours plus qu'on puisse assigner. Mais il n'y a point de nombre infini ni de ligne ou autre quantité infinie, si on les prend pour des véritables touts, comme il est aisé de démontrer. Les écoles ont voulu ou dû dire cela, en admettant un infini syncatégorématique, comme elles parlent, et non pas l'infini catégorématique. (Leibniz [79], 1704, p. 113)

This passage occurs in Chapter 17 (of Book II) entitled *De l'Infinité.* Arthur reproduces the first two sentences of the above passage, but fails to report the outcome of the discussion between Théophile and another fictional character, Philalèthe. The above preliminary comment by Théophile is in response to the opening comment by Philalèthe:

Philalèthe: **1** Une notion des plus importantes est celle du *fini* et de l'*infini*, qui sont regardées comme des modes de la quantité. (ibid.)

A disagreement soon emerges between the two characters. While Philalèthe views the finite and infinite as "des modifications de l'étendue et de la durée," Théophile insists that "la considération du fini et infini a lieu partout où il y a de la grandeur et de la multitude." On the latter view, the infinite is an attribute of magnitude and multitude. It is not *not* an attribute of extension (i.e., continuum or space) and time, as Philalèthe argues. When Philalèthe again attempts to connect the infinite to *space*, Théophile provides a detailed rebuttal,[25] and concludes as follows:

Mais on se trompe en voulant s'imaginer un espace absolu qui soit un tout infini composé de parties, il n'y a rien de tel, c'est une notion qui implique contradiction, et ces touts infinis, et leurs opposés infiniment petits, ne sont de mise que dans le *calcul des géomètres*, tout comme les racines imaginaires de l'algèbre. (Leibniz [79], 1704, p. 114; emphasis added)

Thus according to Théophile, in space, *bona fide* infinitesimals are impossible; but in the *calculations of geometers*, they do have a place. Théophile's suggestion that infinitesimals are possible in calculation on par with imaginary numbers is cogent; see also Section 4.6. Arthur fails to reproduce this crucial passage, which constitutes a piece of evidence against the IA *logical fiction* hypothesis, since there does not exist a syncategorematic paraphrase of imaginary numbers as such logical fictions. Thus the very chapter 17 of *Nouveaux Essais* from which Arthur quotes in support of the IA thesis actually furnishes evidence *against* it.

3.7. Arnauld–Leibniz exchange. In ([6], 2015), Arthur equivocates on the exact meaning of the *syncategorematic* claim with regard to its implications for Leibniz's infinities and

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[25]This connects with Leibniz's syncategorematic views of physical space, as mentioned by de Risi; see Section 1.6.

infinitesimals, but whether or not his reading is identical to Ishiguro's, Arthur does not seek to differentiate his reading from Ishiguro's and on the contrary repeatedly endorses Ishiguro's *logical fiction* reading; see Section 1.11. No less an authority than Knobloch read the text (Arthur [6], 2015) as creating an impression that the syncategorematic reading of Leibniz's mathematical infinitesimals finds support in the Arnauld–Leibniz exchange in 1687. Thus, in his review of Arthur's text [6] for MathSciNet, Knobloch notes that

[Richard] Arthur essentially bases his [syncategorematic] deductions on Leibniz's correspondence with [Antoine] Arnauld and [Burchard] de Volder. (Knobloch [67], 2015)

What exactly is the basis, allegedly deriving from such correspondence, for the IA thesis? Arnauld being more influential than de Volder, we will focus on the Arnauld–Leibniz exchange. Arthur cites the following passage from Leibniz's letter to Arnauld:

I believe that where there are only beings by aggregation, there will not in fact be any real beings; for any being by aggregation presupposes beings endowed with a true unity, because it derives its reality only from that of its constituents. It will therefore have no reality at all if each constituent being is still a being by aggregation, for whose reality we have to find some further basis, which in the same way, if we have to go on searching for it, we will never find.[26] (Leibniz to Arnauld, 30 April 1687 as translated in [6, p. 152])

The reader may well wonder what, if anything, this has to do with mathematical infinitesimals. Indeed, the context of exchange between Arnauld and Leibniz was the latter's views as detailed in his "Discourse on Metaphysics" dating from 1686. Arnauld addressed a letter to Leibniz on 4 March 1687, and Leibniz replied on 30 April 1687. Arnauld and Leibniz are discussing the *metaphysics* related to the structure of *matter*, rather than anything related to mathematical infinitesimals, as is evident from Leibniz's very next sentence:

J'accorde, Monsieur, que dans toute la nature corporelle il n'y a que des machines (qui souvent sont animées) mais je n'accorde pas qu'il n'y ait que des aggregés de substances, et s'il y a des aggregés des substances, il faut bien qu'il y ait aussi des veritables substances dont tous les aggregés resultent.' [90, p. 92]

Thus, the Arnauld–Leibniz exchange is not concerned with the nature of mathematical infinitesimals, contrary to Arthur's claim as reported by Knobloch. Leibniz's position on mathematical infinitesimals is well known: it is not necessary to make mathematical analysis dependent upon *metaphysical* controversies (cf. Section 1.10):

"my intention was to point out that it is unnecessary to make mathematical analysis depend on *metaphysical* controversies or to make sure that there are lines in *nature* which are infinitely small in a rigorous sense in comparison with our ordinary lines..." (Leibniz as translated in [50, p. 86]; emphasis added)

Seeking an explanation of the nature of infinitesimals in Leibniz's comments on the substantial status of being of aggregates of material things, as Arthur attempted to do, is sheer obfuscation.

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[26]In the original: "[J]e croy que là, où il n'y a que des estres par aggregation, il n'y aura pas même des estres reels; car tout estre par aggregation suppose des estres doués d'une veritable unité, parcequ'il ne tient sa realité que de celle de ceux dont il est composé, de sorte qu'il n'en aura point du tout, si chaque estre dont il est composé est encor un estre par aggregation, ou il faut encor chercher un autre fondement de sa realité, qui de cette maniere s'il faut tousjours continuer de chercher ne se peut trouver jamais" (Leibniz [90], pp. 91–92).

3.8. Leibniz and Smooth Infinitesimal Analysis. Peckhaus mentions that Arthur compares Leibnizian infinitesimals with nilpotent infinitesimals of Smooth Infinitesimal Analysis (SIA; see e.g., Bell [19]):

In Section 3 Leibniz's conception is compared with... Bell's Smooth Infinitesimal Analysis (SIA) ... which has many points in common with the Leibnizian approach. (Peckhaus [99], 2013).

In fact the original title of Arthur's text was "Leibniz's syncategorematic infinitesimals, Smooth Infinitesimal Analysis, and Newton's Proposition 6."[27] Peckhaus notes moreover that Arthur finds many points in common as well as dissimilarities:

The author comes to the conclusion that despite many points in common, Leibniz's syncategorematic approach to the infinitesimals and Smooth Infinitesimal Analysis "are by no means equivalent". (ibid.)

Note that SIA depends crucially on a category-theoretic foundational framework and on intuitionistic logic to enable nilpotency. While we welcome such a display of foundational pluralism on Arthur's part, we also agree with both John L. Bell and Arthur that SIA is closer to Nieuwentijt's approach to calculus than to Leibniz's.

Note that the nilpotent infinitesimals of SIA are smaller in absolute value than $\frac{1}{n}$, signifying non-Archimedean behavior. The following question therefore arises. Why does Arthur's voluminous output on Leibnizian calculus systematically eschew readings related to Robinson's framework and Nelson's foundational approach (see Section 1.4), based as they are on classical set-theoretic foundations and classical logic, and enabling straightforward transcription of both Leibniz's assignable/inassignable dichotomy and his infinitely many orders of infinitesimal and infinite numbers? The question was essentially posed five years ago in the *Erkenntnis* article [59] and still awaits clarification. Meanwhile, Arthur appears to avoid Robinsonian infinitesimals as zealously as Leibniz avoided atoms and material indivisibles; see Section 2.6.

3.9. Leibniz, Hilbert, and Formalism. The driving force behind the IA interpretation of Leibnizian infinitesimals seems to be a desire to deny them an ontological reality that would smack of Platonism or mathematical realism. Leibniz himself was clear on the matter:

"Quand [nos ami] disputèrent en France avec l'Abbé Gallois [i.e., Galloys], le Père Gouge [i.e., Gouye] & d'autres, je leur témoignai, que je ne croyois point qu'il y eût des grandeurs véritablement infinies ni véritablement infinitésimales, que ce n'étoient que des fictions, mais de fictions utiles pour abréger & pour parler universellement, comme les racines imiginaires dans l'Algèbre, telles que $\sqrt[2]{(-1)}$ ... (Leibniz [84], 1716)

Without choosing sides in the realist/antirealist debate, we note that Leibnizian infinitesimals can be understood as fictions in a sense close to the school of Formalism as developed in the 20th century mathematics by David Hilbert (see e.g., Hilbert [48], 1926, p. 165) and others.

Thus, Abraham Robinson as a formalist distanced himself from platonist and foundationalist views in the following terms:

[M]athematical theories which, allegedly, deal with infinite totalities do not have any detailed ... reference. (Robinson [103], 1975, p. 42)

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[27]A preprint of Arthur's article with this longer title can be viewed at `http://u.math.biu.ac.il/~katzmik/arthur08.pdf`

Robinson explicitly linked the approaches of Leibniz and Hilbert in the following terms: "Leibniz's approach is akin to Hilbert's original formalism, for Leibniz, like Hilbert, regarded infinitary entities as ideal, or fictitious, additions to concrete Mathematics" (Robinson [102], 1967, pp. 39–40).

3.10. A Robinson–Gödel exchange. Bos comments as follows on the connection between the frameworks of Leibniz and Robinson:

The creation o¡f non-standard analysis makes it necessary, according to Robinson, to supplement and redraw the historical picture of the development of analysis . . . it is understandable that for mathematicians who *believe that [the] present-day standards [of mathematical rigor] are final*, nonstandard analysis answers positively the question whether, after all, Leibniz was right. However, I do not think that being "right" in this sense is an important aspect of the appraisal of mathematical theories of the past. (Bos [27], 1974, p. 82; emphasis added)

This criticism of Robinson by Bos is predicated on the assumption that Robinson believed that the present-day standards of mathematical rigor are "final." However, the attribution to Robinson of such naive realist views concerning the finality of this or that piece of mathematics is unsourced and unjustified, as we argue in this section based on the Robinson–Gödel correspondence.

In a 23 August 1973 letter to Kurt Gödel, Robinson refers to his posthumously published paper ([103], 1975, presented at Bristol in 1973) on progress in the philosophy of mathematics. In this paper, Robinson expresses formalist views. The paper was enclosed with the 23 August 1973 letter to Gödel. Robinson's letter is a follow-up on discussions that took place between him and Gödel during Robinson's visit to the Institute for Advanced Study (15–18 August 1973).

In the letter Robinson writes: "I am distressed to think that you consider my emphasis on the model theoretic aspect of Non Standard Analysis wrongheaded" and goes on to describe himself as a "good formalist". Robinson then expresses the sentiment that Gödel is "bound to disagree" with the paper, possibly due to Gödel's realist views; see Gödel ([42], 2003; particularly the introduction by M. Machover) and Dauben ([32], 1995, pp. 268–269).

Without getting into a discussion of the nature and extent of Goedel's realist views, what we wish to highlight is the inaccuracy of attributing such views to Robinson. In his formalist views Robinson was close to both Leibniz and Hilbert.

4. Evidence: inassignables.

4.1. Inassignable dx and assignable $(d)x$. In addition to ratios of inassignable differentials such as $\frac{dy}{dx}$, Leibniz also considered ratios of ordinary values which he denoted $(d)y$ and $(d)x$, so that $\frac{(d)y}{(d)x}$ would be what we call today the derivative. Here dx and $(d)x$ are distinct entities since Leibniz describes them as respectively *inassignable* and *assignable* in *Cum Prodiisset* [75]:

[A]lthough we may be content with the assignable quantities $(d)y$, $(d)v$, $(d)z$, $(d)x$, etc., . . . yet it is plain from what I have said that, at least in our minds, the unassignables [*inassignables* in the original Latin] dx and dy may be substituted for them by a method of supposition even in the case when they are evanescent; . . . (Leibniz as translated in Child [30], 1920, p. 153)

Leibniz used similar terminology in his manuscript *Puisque des personnes. . .* ([82], 1705); see Section 1.1. In Leibniz ([73], 1695), one similarly finds:

...Nous voyons par là que nous pouvons faire comme si le calcul différentiel ne concernait que des quantités ordinaires, même s'il faut en rechercher l'origine dans les *inassignables* pour rendre compte des termes qui sont éliminés ou se détruisent. (Leibniz as translated by Parmentier in [85], 1989, p. 336; emphasis added).

Meanwhile, on the IA reading, dx and $(d)x$ should be identical, both denoting ordinary assignable values (perhaps equipped with a hidden quantifier or placed in a sequence). The distinction between differentials dx and $(d)x$, extensively commented upon by Bos ([27], 1974), is an indication that Leibniz exploits differentials as pure fictions.

This is particularly significant since in *Cum Prodiisset* Leibniz is actually *doing* calculus (thus, he proves the product law for differentiation – Leibniz's rule – relying on the transcendental law of homogeneity; see Katz–Sherry [58], 2012), rather than merely *speculating* about it. Breger wrote:

It has often been noted that Leibniz's *verbal* descriptions of infinitesimal magnitudes vary or even appear *incoherent* ... But in his *use* of them Leibniz is in fact being quite *clear and explicit*; his view of infinitesimals appears not to have altered since the beginning of his Hannover period or a few years later. It is not sufficient to study Leibniz's *verbal* descriptions of infinitesimal magnitudes in isolation; they need to be interpreted in connection with their *mathematical usage*. (Breger [28], 2008, p. 185) (emphases added)

We disagree with Breger's claim of alleged incoherence of Leibniz's verbal descriptions, but we agree concerning the need to focus on mathematical usage.
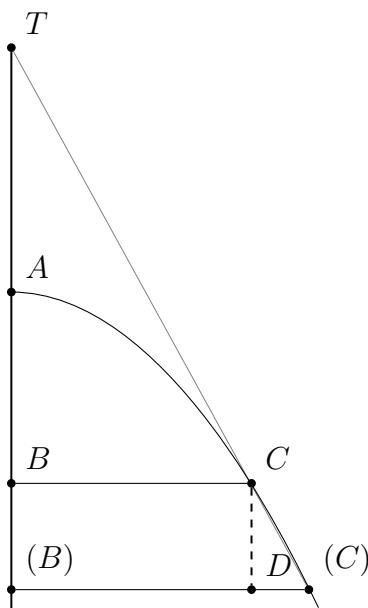
4.2. Characteristic triangle.

Figure 2: Leibniz's tangent line $TC(C)$

For example, consider Leibniz's analysis, recently translated in [89], of the inassignable characteristic triangle $CD(C)$, where D is vertex of the right angle whereas C and (C) are the other two vertices; see Figure 2. This characteristic triangle is taken to be similar to the assignable triangle TBC. Leibniz writes:

[G]râce à ce triangle *inassignable*, c'est-à-dire à l'intervention de la raison entre quantités inassignables CD et $(C)D$ (que notre calcul différentiel donne au moyen des quantités ordinaires ou assignables), on peut trouver la raison entre les quantités *assignables TB* et BC, et donc tracer la tangente TC. (Leibniz [89], 2018, p. 155; emphasis added)

Leibniz's tangent line TC is thought of as passing through both infinitely close points C and (C).[28] In the same text, Leibniz sees incomparables as equivalent to inassignables:

[O]n voit en optique, quand les divers rayons proviennent d'un même point, et que ce point est placé à l'infini ou de façon *inassignable* (ou encore, comme j'ai coutume de dire souvent, "est éloigné de façon *incomparable*"), que les rayons sont parallèles. (ibid.; emphasis added)

Both involve a violation of Euclid's V.4; see Section 3.5.

4.3. Differentials according to Breger and Spalt. Breger writes:

I would now like to turn briefly to Leibniz's first publication of his infinitesimal calculus from 1684. It has been said that Leibniz introduced infinitesimals here as finite magnitudes (Boyer, 1959, 210; Bos, 1974, 19, 62–64). *This is not wrong, but it is misleading.* Leibniz in fact explains that one can choose any dx you like, and he then defines dy as the magnitude that has the same relation to dx as the ordinate to the subtangent. (Breger [28], 2008, p. 188; emphasis added)

Breger's comments are misleading because they misrepresent Bos's position. The first of the pages 62–64 in Bos's article mentioned by Breger is page 62. On this page, Bos is analyzing *Cum Prodiisset* ([75], 1701), rather than Leibniz's 1684 article [70] mentioned in Breger's passage. Here Bos insists on the difference between the Leibnizian differentials dx and $(d)x$, where the former is inassignable whereas the latter is an ordinary assignable quantity. In the passage quoted above, Breger clearly has $(d)x$ in mind, but uses the notation dx. Significantly, Breger fails to mention anything here about this crucial distinction. Elsewhere, Breger acknowledges a violation of Euclid V.4 in Leibniz; see Section 3.4.

Similarly to Breger, Spalt overlooks the implications of the crucial Leibnizian distinction between dx (inassignable) and $(d)x$ (assignable) in his discussion of Leibnizian differentials when he writes:

Leibniz gives a rigorous *geometric* justification of his rules for differentials, and, in so doing, to a significant extent he builds on his law of continuity. Obviously, Leibniz's differential calculus has nothing whatsoever to do with a use of infinitely-small non-variable 'numbers', as are known in the modern theory of nonstandard analysis. Leibniz' differentials aren't 'numbers', but (variable) *geometrical* 'continuous' 'magnitudes'. (Spalt [113], 2015, p. 121; translation ours)

Spalt raises the issue of the differentials being geometric magnitudes rather than numbers. This may be an interesting issue to explore. However, this issue is transverse to the question of the non-Archimedean nature of incomparable magnitudes. Leibniz made it clear in a letter to l'Hospital that such magnitudes are non-Archimedean; see Sections 3.3 and 3.4.

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[28]To provide a modern interpretation, one can normalize the equation of the line $TC(C)$ as $ax + by = c$ where $a^2 + b^2 = 1$. Then one can apply the standard part function (see note 16) to the coefficients of the equation of the line $TC(C)$ to obtain the equation $a_o x + b_o y = c_o$ of the true tangent line, where $r_o = \mathbf{st}(r)$ for each $r = a, b, c$. In this sense, the line $TC(C)$ and the true tangent line coincide *up to negligible terms*. Leibniz often points out that he is working with a generalized notion of equality. It the case of the characteristic triangle, such a notion is applied to secant lines and tangent lines.

Spalt does mention the Leibnizian differentials $(d)x$ in a separate discussion, where he claims that "The variable length dx with limit 0 is called 'infinitely small'" (Spalt [113], 2015, p. 118). However, thinking of the Leibnizian dx as a variable with limit 0 is merely another version of the logical fiction hypothesis.

Spalt goes on to castigate Bos for concluding that

LEIBNIZ had proposed 'two distinct' (Bos 1974, p. 55) respectively, two 'very distinct' (Bos 1980, p. 70) concepts of differentials." (ibid.)

Bos' position was outlined in Section 1.2. Spalt continues:

Advocates of nonstandard analysis routinely refuse to acknowledge this; the allures of Leibniz' reputation, and of the beautiful field of activity 'historiography of analysis', are too irresistible.[29] (ibid.)

Spalt's move of attributing questionable ideological motives to the pro-infinitesimal opposition is not without historical precedent.[30]

4.4. Infinitesimals and contradictions according to Rabouin.
Rabouin's argument for a version of the logical fiction hypothesis runs as follows:

It should then be clear why infinitesimals were called by Leibniz "fictions". [1] In and of itself, there is no such thing as a "quantity smaller than any other quantity". [2] This would amount to the existence of a minimal quantity [3] and one can show that a minimal quantity implies *contradiction* [4] (as a simple consequence of Archimedes axiom). [5] So "infinitesimals" as "infinitely small quantities" are terms without reference. They only have a contextual meaning and should be paraphrased not by terms, but by sentences in which only finite quantities occur. (Rabouin [100], 2015, pp. 362–363; numerals [1] through [5] added; emphasis added)

Now it is correct to assert that there is no quantity smaller than any other quantity, as per Rabouin's sentence [1]. However, this was not Leibniz's characterisation of infinitesimals. Rather, an infinitesimal is inassignable and is smaller than every *assignable* quantity (see e.g., Sections 1.1 and 4.1). This does not imply the existence of a minimal quantity, contrary to Rabouin's [2]. Thus the "contradiction" posited by Rabouin in [3] is not there. The reference to the "Archimedes axiom" in [4] is gratuitous; this axiom is not required to obtain a contradiction starting from Rabouin's incorrect hypotheses. Thus, his conclusion [5] rests on shaky premises.

4.5. Irrationals, imaginaries, infinitesimals.
Leibniz employed the qualifier "impossible" in reference to irrational numbers:

Des irrationnels naissent les quantités impossibles ou *imaginaires*, dont la nature est étrange, mais dont l'utilité ne doit pourtant pas être méprisée. En effet, même si celles-ci signifient en soi quelque chose d'impossible, cependant, non seulement elles montrent la source de l'impossibilité, ainsi que la façon dont la question pourrait être corrigée afin de ne pas être impossible, mais aussi on peut, par leur intervention, exprimer des quantités réelles. (Leibniz [89], 2018, p. 152; emphasis in the original)

Leibniz goes on to discuss a few examples, and concludes:

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[29]In the original: "Fürsprecher der Nichtstandard-Analysis pflegen sich dieser Einsicht zu verschließen: Die Verlockungen des Leibniz'schen Renommees sowie des schönen Betätigungsfeldes "Geschichtsschreibung der Analysis" sind zu unwiderstehlich" (Spalt [113], 2015, p. 121).

[30]See e.g., Section 2.10 on Rolle and *la réforme*.

Ces expressions ont ceci de merveilleux que dans le calcul elles ne recouvrent *rien d'absurde ou de contradictoire* et ne peuvent cependant être montrées dans la *nature*, c'est-à-dire dans les choses concrètes. (Leibniz [89], 2018, p. 153; emphasis added)

Leibniz states clearly that such expressions entail no contradiction. On the other hand, in "nature" there is no referent for such expressions; cf. Bos on infinitesimals as summarized in Section . Similarly with regard to imaginaries,

les valeurs des quantités réelles doivent parfois nécessairement être exprimées par l'intervention des quantités imaginaires et que de là naissent des formules non moins utiles à toute l'étendue de l'analyse que ne le sont les formules communes. Et ces quantités, je les appelle *impossibles en apparence*, car à la vérité elles sont réelles, et je rapporte les préceptes par lesquels ceci peut être reconnu. [89, pp. 107–108]

Thus both irrationals and imaginaries are only *apparently* impossible, according to Leibniz. Leibniz takes the argument a step further and makes it clear that such notions imply no contradiction:

Il y a une grande différence entre les quantités imaginaires, ou impossibles par accident, et celles qui sont absolument impossibles [parce qu'elles] impliquent contradiction. [89, p. 108]

Leibniz has made it clear that imaginaries (as opposed to *absolutely impossible* quantities) in fact *do not* imply a contradiction. Finally, Leibniz extends the argument to the infinitely large and small:

De fait, les imaginaires ou impossibles par accident, qui ne peuvent être exhibées parce que fait défaut ce qui est nécessaire et suffisant pour que se produise une intersection, peuvent être comparées avec les Quantités infinies et infiniment petites, qui naissent de la même façon. (ibid.)

Thus according to Leibniz irrationals, imaginaries, and infinitesimals imply no contradiction; see also Sections 4.2 and 4.6. In modern terminology, they are only impossible in the sense of representing new types of mathematical entities.

4.6. Apparent impossibility as possibility. Leibniz illustrates the *apparent impossibility* of imaginaries discussed in Section 4.5 via an analysis of a geometric configuration involving a circle, say C and a line, say L (see [89], p. 153). If the nearest distance from L to the center of C is greater than than the radius of C, then C and L are disjoint. The usual formulas for points in the intersection $C \cap L$ then contain imaginary terms (in modern language, the intersection points will appear once the curves C and L are complexified). Leibniz closes the discussion somewhat inconclusively, by commenting that to create *real* intersection points, one needs to *change the data of the problem* by either increasing the radius of C or moving L closer to the center of C ([89], p. 154).

Leibniz's comment does not solve the problem of accounting for the use of imaginaries in a situation where one *can't change the data of the problem*. Leibniz's comment does not help in situations where one is forced to make sense of imaginaries and cannot avoid them. Such a situation arises e.g., in the solution of the cubic when imaginaries necessarily arise in an intermediate stage of the calculation, a technique Leibniz was proficient at; see e.g., Sherry–Katz ([112], 2014, p. 169).[31]

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[31]Here Leibniz is quoted to the effect that "For this is the remarkable thing, that, as calculation shows, such an imaginary quantity is only observed to enter those cubic equations that have no imaginary root, all their roots being real or possible."

Thus Leibniz's discussion of imaginaries in the text translated in [89] (unlike other texts) is incomplete (perhaps it was completed in subsequent manuscripts), and amounts to walking away from the problem of the status of imaginaries rather than resolving it.

Leibniz's $C \cap L$ example is therefore not comparable to his example of the characteristic triangle ([89], p. 154–155) discussed above, where Leibniz does not walk away from the problem but rather presents a successful solution for finding the tangent line, *without changing the data of the problem.* The solution is in terms of infinitesimals. Neither example is meant to imply that imaginaries and/or infinitesimals are either absolutely impossible or contradictory; on the contrary. There are many problems treated by Leibniz where both imaginaries and infinitesimals appear in solutions.

Our analysis undermines Rabouin's claim to the effect that

Le parallèle avec les imaginaires est très souvent mentionné par ceux qui défendent une vue des infinitésimaux comme entités idéales qu'on adjoindrait au domaine des objets réels pour la résolution de problème. Or il est frappant que dans notre texte comme dans [3b], les imaginaires soient en fait présentés comme indiquant que le problème n'a pas de solution. Si Leibniz précise que les infiniment petits et les points à l'infini entrent dans des problèmes qui, eux, ont des solutions, on se gardera néanmoins de forcer un parallèle que Leibniz ne place pas là où l'interprétation formaliste le place. (Rabouin in [89], 2018, p. 95, note 2 carrying over to bottom of page 96)

Contrary to Rabouin's claim, the parallel between imaginaries and infinitesimals as noncontradictory new types of entities is valid and is *not* undermined by Leibniz's $C \cap L$ example, as we discussed in the current section.

Leibniz repeatedly likens infinitesimals to imaginaries (see also Sections 3.6 and 3.9), and at least once described the latter as having their *fundamentum in re* (basis in fact; see Leibniz [72], 1695, p. 93), providing evidence against the IA reading that would surely deny them such a basis.

4.7. Hierarchical structure. Further support for the reading by Bos is provided by the *hierarchical* structure on the Leibnizian differentials dx's, dx^2's, ddx's, etc., ubiquitous in Leibniz's texts. One can replace dx by a sequence of finite values ε_n and furnish a concealed quantifier incorporated into a modern limit notion so as to interpret dx as shorthand for the sequence (ε_n). However, one notices that $\lim_{n\to\infty} \varepsilon_n = 0$, as well as $\lim_{n\to\infty} \varepsilon_n^2 = 0$, and also unsurprisingly $\lim_{n\to\infty}(\varepsilon_n + \varepsilon_n^2) = 0$. Thus, the Leibnizian substitution

$dx + dx^2 = dx$ (1)

in accordance with his transcendental law of homogeneity (see Leibniz [83], 1710 and also Katz–Sherry [59], 2013) becomes a meaningless tautology $0 + 0 = 0$. Furthermore, if such identities are to be interpreted in terms of taking *limits*, then an absurd equality $dx = dx^2$ would also be true. To interpret Leibniz's substitution (1) in both a syncategorematic and a meaningful manner, IA would have to introduce additional ad hoc proto-Weierstrassian devices with no shadow of a hint in the original Leibniz.

4.8. "Historically unforgivable sin". Ishiguro mentions "Leibniz's followers like Johann Bernoulli, de l'Hospital, or Euler, who were all brilliant mathematicians rather than philosophers," (Ishiguro [50], 1990, pp. 79–80) but then goes on to yank Leibniz right out of his historical context by claiming that their *modus operandi*

is prima facie a strange thing to ascribe to someone who, like Leibniz, was obsessed with general methodological issues, and with the logical analysis of all statements and the well-foundedness of all explanations. (ibid., p. 80)

Having thus abstracted Leibniz from his late 17th–early 18th century context, Ishiguro proceeds to insert him in a late 19th century Weierstrassian one.[32] On purely mathematical grounds, such a paraphrase is certainly possible.[33] However, such an approach to a historical figure would apparently not escape Unguru's censure:

It is . . . a historically unforgiveable sin . . . to assume wrongly that mathematical equivalence is tantamount to historical equivalence. (Unguru [115], 1976, p. 783)

Ishiguro seems to have been aware of the problem and at the end of her Chapter 5 she tries again to explain "why I believe that Leibniz's views on the contextual definition of infinitesimals is [sic] different from those of other mathematicians of his own time who sought for operationist definitions for certain mathematical notions" (Ishiguro [50], 1990, p. 99), but with limited success. Ishiguro's dubious command of the positions of Leibniz's contemporaries was discussed in Section 3.5.

4.9. Leibniz against exhaustion. Parmentier quotes Leibniz's *De Quadratura Arithmetica* as follows:

J'ai dit jusqu'ici des infinis et des infiniment petits des choses qui paraîtront obscures à certains, comme paraît obscure toute chose nouvelle; rien cependant que chacun ne puisse aisément comprendre en y consacrant un peu de réflexion pour, l'ayant compris, en avouer la fécondité. Peu importe que de telles quantités soient ou non naturelles, on peut se contenter de les introduire par le biais d'une fiction dans la mesure où elles offrent bien des commodités dans les formulations, dans la pensée, et finalement dans l'invention aussi bien que dans la démonstration, en rendant inutiles l'usage des figures inscrites et circonscrites, les raisonnements par l'absurde et la démonstration qu'une erreur est plus petite que toute erreur assignable. (Leibniz as translated in [98, p. 284])

Leibniz's last sentence asserts that infinitesimals make it unnecessary to get involved in *exhaustion*-type proofs involving inscribed and circumscribed polygons, arguments with error smaller than any assignable error, etc. This would be Leibniz' own refutation of Ishiguro's thesis. Namely, Ishiguro claimed that infinitesimals stand for exhaustion proofs, based on the following passage from Leibniz:

these incomparable magnitudes - are not at all fixed or determined but can he taken to be as small as we wish in our geometrical reasoning and so have the effect of the infinitely small in the rigorous sense. If any opponent tries to contradict this proposition, it follows from our calculus that the error will be less than any possible assignable error since it is in our power to make this incomparably small magnitude small enough for this purpose inasmuch as we can always take a magnitude as small as we wish. (Leibniz as translated in [50, p. 87])

However, Leibniz wrote that, on the contrary, infinitesimals make it unnecessary to get involved in exhaustion proofs.[34]

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[32]One of the first occurrences of a modern definition of continuity in the style of the *Epsilontik* can be found in Schwarz's summaries of 1861 lectures by Weierstrass; see Dugac ([35], 1973, p. 64), Yushkevich ([117], 1986, pp. 74–75). This definition is a verbal form of a definition featuring a correct quantifier order (involving alternations of quantifiers).

[33]Thus, Robinson noted: "the method of ultrapowers provides a ready means for translating a non-standard proof into a standard mathematical proof in each particular case. However, in the course of doing so we may complicate the proof considerably, so that frequently the resulting procedure will be less desirable from a heuristic point of view. At the same time there may well exist a shorter mathematical proof which can be obtained independently" (Robinson [101], 1966, p. 19). For a more detailed analysis see Henson–Keisler ([47], 1986).

[34]Note that Ishiguro's sentence "If magnitudes are incomparable, they can be neither bigger nor smaller" [50,

4.10. Leibniz, Des Bosses, and Infinity. Ishiguro quotes a passage from Leibniz's letter to Des Bosses dated 3 March 1706, where Leibniz writes:

Meanwhile I have shown that these expressions are of great use for the abbreviation of thought and thus for discovery as they cannot lead to error, since it is sufficient to substitute for the infinitely small, as small a thing as one may wish, so that the error may be less than any given amount, hence it follows that there can be no error. (Leibniz as translated in [50, p. 85])

Ishiguro concludes: "It seems then that throughout his working life as a mathematician Leibniz did not think of founding the calculus in terms of a *special kind of small magnitude*" [50, p. 86] (emphasis added). But the plain meaning of the Leibnizian passage is that there are two *distinct* methods, one involving infinitesimals and one involving errors "less than any given amount," the former being advantageous over the latter.

Even more significantly, Ishiguro fails to mention that a few months later, on 1 september 1706, Leibniz wrote another letter to Des Bosses that sheds light on the question of mathematical infinity. In this letter, Leibniz responds to a list of propositions banned by soon-to-become General Michelangelo Tamburini in 1705. The list was sent to Leibniz confidentially by jesuit Des Bosses. The fourth of these banned propositions is the following:

4. Our minds, to the extent that they are finite, cannot know anything certain about the infinite; consequently, we should never make it the object of our discussions. (translation from Ariew et al. [3], 1998, p. 258)

Leibniz comments as follows:

Unless I am mistaken, mathematicians have already refuted the fourth proposition, and I myself have published some samples of the science of the infinite. However, I maintain, strictly speaking, that an infinite composed from parts is neither one nor a whole, and it is not conceived as a quantity except through a fiction of the mind. (Leibniz as translated in [88], 2007, p. 53)

Here Leibniz affirms that human mind can indeed conceive of infinity (contrary to proposition 4 rejected by both the jesuits and himself), and moreover that he published "samples of the science of the infinite" to prove this. Here clearly mathematical infinity is not a mere sign for hidden quantifiers, contrary to Ishiguro's position.

The preponderance of the evidence in the primary sources indicates that Leibniz did indeed found his calculus on a special kind of fictional small magnitude.

5. *De Quadratura Arithmetica.* Leibniz's unpublished text *De Quadratura Arithmetica...* (DQA) was written shortly after he developed the infinitesimal calculus in 1675. Thus the work dates from an early period of his mathematical career. Here Leibniz wrote:

Nec refert an tales quantitates sint in rerum natura, sufficit enim fictione introduci, cum loquendi cogitandique, ac proinde inveniendi pariter ac demonstrandi compendia praebeant, ne semper inscriptis vel circumscriptis uti... necesse sit.[35] (Leibniz [87], p. 69)

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p. 88] involves an equivocation on the term *incomparable*: if *incomparable* is taken to mean the definition from the theory of partially ordered sets, then this is a tautology (roughly "if magnitudes cannot be compared, then they cannot be compared"); if *incomparable* is taken to mean "any positive-integer multiple is still less than any positive real" then Ishiguro's statement is mathematically incorrect, for the hyperreals are a totally ordered field, hence every element can be compared with any other.

[35]Jesseph's translation: " 'Nor does it matter whether there are such quantities in nature, for it suffices that they are introduced as fictions, since they allow the abbreviations of speech and thought in the discovery as

5.1. B-track reading. A straightforward interpretation of this passage from DQA is that there exist two approaches to the calculus:

(A) one involving inscribed and circumscribed figures, called the method of exhaustion; and

(B) a method involving what he referred to elsewhere as *useful fictions*, and enabling abbreviations of speech and thought when compared to method A.

The theme of a pair of distinct approaches occurs often in Leibniz's writing. Thus, in his 2 February 1702 letter to Varignon, Leibniz writes:

Et c'est pour cet effect que j'ay donné un jour des lemmes des incomparables dans les Actes de Leipzic, qu'on peut entendre comme on vent[36] [sic], soit des infinis *à la rigueur*, soit des grandeurs seulement, qui n'entrent point en ligne de compte les unes au prix des autres. Mais il faut considerer en même temps, que ces incomparables communs mêmes n'estant nullement fixes ou determinés, et pouvant estre pris aussi petits qu'on veut dans nos raisonnemens Geometriques, font l'effect des infiniment petits rigoureux... (Leibniz [76], 1702, p. 92; emphasis added)

The passage is analyzed in its context in Section 2.5.

5.2. A-track reading. An alternative interpretation following Ishiguro of this type of passage in Leibniz is that the B-method is merely shorthand for the A-method involving hidden quantifiers, in the spirit of Russell's *logical fictions*. Such an interpretation of the passage seems more forced than the one we gave above.[37]

In DQA Leibniz wrote that infinitely small (say, ε) means smaller than any given quantity, and that infinitely large means larger than any given quantity.

Such descriptions similarly admit two readings: (A) for every given positive quantity there is an $\varepsilon > 0$ smaller than it; or (B) this specific ε is smaller than every given (i.e., assignable) positive quantity. In reading (B) the ε is *inassignable*. The dichotomy of assignable *vs* inassignable quantity (or magnitude) was used frequently by Leibniz, e.g., in *Cum Prodiisset* [75] and *Puisque des personnes...* [82][38]

Leibniz's reference to Archimedes in various texts typically is a reference to the method of exhaustion, and is sometimes accompanied by a claim that infinitesimals violate Euclid Definition V.4 (see e.g., Section 3.3). The latter is called today the Archimedean property (but was not in Leibniz's time).

The term *syncategorematic* itself is incidental to the true issues involved. The real issue is whether Leibniz was relying on hidden quantifiers (as per IA) or not.

5.3. Blåsjö–Knobloch differences over DQA. Knobloch wrote in reference to DQA:

In this treatise, Leibniz laid the rigorous foundation of the theory of infinitely small and infinite quantities or, in other words, of the theory of quantified indivisibles. In modern

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well as demonstration' (Leibniz 1993, p. 69)" (Jesseph [52], 2015, p. 198). A longer passage including this one was quoted in Parmentier's French translation in Section 4.9.

[36]I.e., "veux".

[37]Based on the *second* sentence of the Leibnizian passage quoted in Section 5.1, Breger claims that "[Leibniz] stresses that the incomparably small magnitudes are certainly not 'fixes ou determinés'; they can be chosen as small as one wants" (Breger [29], 2017, p. 77). However, Breger fails to mention the fact that the *first* setence of the passage we quoted makes it clear that Leibniz's comments on incomparables not being "fixes ou determinés" apply to *common* incomparables of track A, rather than the "infinies à la rigueur" of track B.

[38]See also note 13 and Section 4.1 on the dichotomy assignable *vs* inassignable.

terms Leibniz introduced 'Riemannian sums'[39] in order to demonstrate the integrability of continuous functions. (Knobloch [64], 2002, p. 59)

Knobloch then proceeds to describe Leibniz's method, and notes:

While the "common method of indivisibles"[40] considered inscriptions and circumscriptions of mixtilinear figures, the step figure is neither an inscription nor a circumscription, rather something in between. In modern terms: Leibniz demonstrated the integrability of a huge class of functions by means of Riemannian sums which depend on intermediate values of the partial integration intervals. (ibid., p. 63)

Thus, Knobloch argues that Leibniz's technique in DQA represented an advance over earlier inscription and circumscription techniques.

Jesseph argues in [52] that the techniques in DQA were limited by their reliance on the knowledge of the tangent lines to the curve (and/or the corresponding differential). Therefore the applicability of the techniques depended on the availability of such data.[41] Accordingly, the solution of the quadrature problem in DQA depends on a differentiation problem.

Blåsjö analyzes Leibniz's technique in DQA as follows. The so-called general integration theorem in DQA assumes the existence of tangent lines, not only of the "general function" one starts with, say $f(x)$, but also for a secondary function, say $d(x)$ (following the notation in Blåsjö [21], 2017), is constructed from $f(x)$ by means of the tangents to $f(x)$. The theorem relates the areas under f and d to each other in a manner closely related to modern integration by parts. The assumption that $f(x)$ has tangents everywhere (or possibly almost everywhere) is essential since otherwise there wouldn't even be any function $d(x)$ to investigate. The assumption that $d(x)$ has a tangent everywhere is less essential. The *knowledge* of the tangents themselves does not play an essential role in the proof, but it is essential that the curve has no "reversion points", which is a notion that Leibniz has not defined otherwise than in terms of tangents (and hence assumed that tangents must exist, for a non-differentiable curve could reverse directions without having a tangent corresponding to the turning point). This concerns the general condition under which the theorem is valid. Was Leibniz explicating precise and rigorous conditions of validity? Clearly he was not. The conditions he does state are of an intuitive nature and are not intended as rigorous conditions of validity. For the latter purpose, they are clearly insufficient, for example because of the issue of the tangents. Leibniz's result does not provide any general integration theory, and is actually a single specific integration technique, closely related to integration by parts. Just as the modern integration by parts formula, it can be considered to apply in great generality, but it is only useful in cases where we have some geometric information equivalent to the derivatives or antiderivatives involved (otherwise it is just an exercise in expressing some unknown integral in terms of another unknown integral). For more details see Blåsjö ([21], 2017).

Our thesis in the present text is independent of resolving these differences among scholars concerning DQA. Namely, the contention that Leibniz's infinitesimal was inassignable in the sense of violating the Archimedean property when compared to ordinary (assignable)

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[39]The correct technical term for this concept is *Riemann sum*. The adjective *Riemannian* is also used in a technical sense, but in other contexts; e.g., *Riemannian geometry*.
[40]See note 2.
[41]Thus, Leibniz applies his method in DQA to find the quadrature of general cycloidal segments (Edwards [36], 1979, p. 251). Here the calculation exploits the geometric knowledge related to the construction of the curve.

quantities is independent of the Blåsjö–Knobloch differences over DQA. Whatever the foundational significance of DQA may have been (and this is the subject of their disagreement), the fact remains that here Leibniz talks about two separate methods: track A and track B. Leibniz seeks to justify the direct track-B method (exploiting inassignable infinitesimals) in terms of an exhaustion-type track-A method.

5.4. Logarithmic curve. Leibniz also investigated the special case of the logarithmic curve in Proposition 46 in DQA. In the statement of Proposition 46, Leibniz speaks of information that can be obtained from the hyperbola in terms of which the logarithmic curve was defined. Thus the investigation depends on the knowledge that $\log(x)$ is the area under $\frac{1}{x}$ (in other words, the knowledge of the derivative of the logarithm). This can be easily understood in the context of integration by parts. Therefore Leibniz's propostion on the logarithmic curve is not an example of performing quadratures without knowledge of derivatives or tangents.

5.5. Manuscript remained unpublished. Leibniz never published the DQA.[42] Was that because he realized that the A-method, while not contradicting the B-method, was an impediment to the *Ars Inveniendi*? Jesseph notes in [52, p. 200] that Leibniz may have set aside the DQA without publishing it because he had turned his attention to more powerful methods that he would introduce in the 1680s in what he called "our new calculus of differences and sums, which involves the consideration of the infinite", and "extends beyond what the imagination can attain"[43] (GM V, 307).

6. Conclusion. Leibniz did not merely use an infinitesimal approach as an unrigorous way of doing calculus that makes the work easier. Rather, what we are arguing is that historically there have been two separate approaches to the calculus: track (A) and track (B). The historical calculus has often been criticized from an anachronistic modern set-theoretic viewpoint that would make both approaches appear unrigorous to the extent that they did not possess a set-theoretic justification which is considered a *sine-qua-non* of rigor in today's mathematics. From such an anachronistic standpoint, *both* historical approaches were unrigorous by today's standards.

The work of Fermat, Gregory, Leibniz, Euler, Cauchy and others created a body of *procedures* called infinitesimal calculus and/or analysis, in what could be referred to as the pioneering phase of the discipline. Following the pioneering phase, efforts were made to develop set-theoretic justification for this body of procedures. Eventually this effort succeeded both for track (A) and track (B). Assigning names is a matter of debate but in a sense Edward Nelson's syntactic approach ([97], 1977) is particularly fundamental because it shows that one can take an infinitesimal to be a primitive notion within the context of the ordinary real line, in the spirit of what natural philosophers since at least Pascal have envisioned. Nelson's approach exploits a unary (i.e., one-place) predicate **standard**; the formula **standard**(x) reads "x is standard". Thus, mathematical entities can be either *standard* or *nonstandard*. This applies in particular to real numbers. The standard/nonstandard distinction can be seen as a formalisation of Leibniz's assignable/inassignable distinction.[44] For details on the

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[42]The loss of a manuscript version in transit from Paris to Hannover in 1679, signaled by Knobloch [68, p. 282], could have been overcome by writing a new version, Leibniz having been only 33 at the time.

[43]The full sentence in the original reads: "Ainsi il ne faut point s'étonner, si notre nouveau calcul des différences et des sommes, qui enveloppe la considération de l'infini et s'éloigne par conséquent de ce que l'imagination peut atteindre, n'est pas venu d'abord à sa perfection." See Gerhardt's edition ([41], vol. V, p. 307). The passage appears in Leibniz's article "Considerations sur la différence qu'il y a entre l'analyse ordinaire et le nouveau calcul des transcendantes" in *Journal des Sçavans* in 1694.

[44]In Nelson's approach, the violation of the Archimedean property takes the form $(\exists \varepsilon > 0)(\forall^{\mathbf{st}} n \in \mathbb{N})$

systems developed by Nelson, Hrbacek, Kanovei, and others see Fletcher et al. ([38], 2017).

Ishiguro and Arthur have argued that what appears to be a B-track method is in reality a Russellian illusion that is eliminable by careful analysis of the pioneering texts of Leibniz. This is the meaning of their *logical fiction* hypothesis; Arthur goes so far as to speak, in the title of his forthcoming chapter ([8], 2019), of "Archimedean infinitesimals," a word string that would appear as a freshman quantifier-order error to many a mathematically educated scholar. We hold the IA hypothesis to be an error of interpretation and have argued that it is not backed by solid textual evidence in Leibniz.

Leibniz, by arguing in favor of exploiting *inassignable* infinitesimals even though they are fictions, differed from l'Hopital and Bernoulli who were prepared to assign a loftier ontological status to infinitesimals as truly existing entities. In this Leibniz was remarkably modern, and anticipated formalist strategies that fully emerged in the 20th century. As C. H. Edwards, Jr. points out,

It is important to note that the differentials dx are fixed non-zero quantities; they are neither variables approaching zero nor ones that are intended to eventually approach zero. [36, p. 261]

Many mathematicians, historians, and philosophers nowadays are in favor of pluralism as far as foundations of mathematics are concerned. Today we have both A-type and B-type set-theoretic foundations, as well as category-theoretic foundations both of classical and intuitionistic types. The historical studies currently available suggest that in the case of Leibnizian infinitesimal calculus, A-type foundations are insufficiently expressive to capture the spirit of Leibniz's work. Pluralism is a good thing in principle but the A-type logical fiction interpretation is not a viable alternative to the B-type pure fictional one.

The *logical fiction* reading of Leibnizian infinitesimals has become entrenched to an extent that some Leibniz scholars feel compelled to endorse it publicly, while in private correspondence conceding that Leibniz used a dual strategy which we have elaborated in terms of a distinction between Leibnizian methods (A) and (B) (see Section 5.1), the latter involving *pure fictional* infinitesimals, as opposed to the IA *logical fiction* hypothesis. The Ishiguro–Arthur hypothesis must be rejected as having little basis in Leibniz's writings.

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$\left[\varepsilon < \frac{1}{n}\right]$, where $\forall^{\mathbf{st}}$ is universal quantification over *standard* elements only.

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