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G. Barabash, Ya. Kholyavka, I. Tytar

PERIODIC WORDS CONNECTED WITH THE TRIBONACCI-LUCAS NUMBERS

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We introduce periodic words that are connected with the Tribonacci-Lucas numbers and investigate their properties.

1. Introduction. The Tribonacci numbers T_n are defined by the recurrence relation $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, for all integers $n > 2$, and with initial values $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$ (see, e.g., [1]-[4]). Similar to the Tribonacci numbers, the Tribonacci-Lucas numbers L_n are defined by the recurrence relation $L_n = L_{n-1} + L_{n-1} + L_{n-2}$, for all integers $n > 2$, and with initial values $L_0 = 2$, $L_1 = 1$ and $L_1 = 3$.

Let $L_n(m)$, $0 \le L_n(m) < m$, denote the n-th member of the sequence of integers $L_n = L_{n-1} + L_{n-2} + L_{n-3} \pmod m$, for all integers $n > 2$, and with initial values $L_0 = 2$, $L_1 = 1$ and $L_2 = 3$ ($L_0(m) = 2 \pmod m$, $L_1(m) = 1$ and $L_2(m) = 3 \pmod m$). We reduce L_n modulo m taking the least nonnegative residues. It is referenced A020992 in the On-line Encyclopedia of Integer Sequences [5].

The sequence $L_n(m)$ is periodic and repeats by returning to its starting values because there are only a finite number m^3 of triples of terms possible, and the recurrence of a triple results in recurrence of all the following terms. Let $k(m)$ denote the length of the period of the repeating sequence $L_n(m)$. Few properties of the $k(m)$ are in the following theorem.

Theorem 1. *Let p be an odd prime, $p \neq 11$, and $\binom{p}{11}$ denote the Legendre symbol. In $\mathbb{Z}_p$ the following hold:*

1) *If $\binom{p}{11} = 1$, then $k(p)|(p^2 + p + 1)$ if $x^3 - x^2 - x - 1$ is irreducible $\bmod p$, otherwise $k(p)|(p - 1)$.*
2) *If $\binom{p}{11} = -1$, then $k(p)|(p^2 - 1)$.*

Proof. The statement follows from Theorem 4 [6] and the formula $L_n = T_{n+1} + 2T_n + 2T_{n-1}$, which shows that $k(m)$ is a divisor of period $T_n \pmod m$. □

Using Tribonacci-Lucas numbers, in the present article we shall introduce new kind of the infinite periodic words, TLLP words, and investigate some of their properties. These words are similar to the TLP words [7].

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The letter p is reserved to designate a prime, $p > 3$, and m may be arbitrary integer, $m > 1$. For any notation not explicitly defined in this article we refer to [1], [8].

2. Tribonacci-Lucas words. In analogy to the definition of Tribonacci-Lucas numbers, one defines the Tribonacci-Lucas finite words as the contatenation of the three previous terms $l_n = l_{n-1}l_{n-2}l_{n-3}$, $n > 2$, with initial values $l_0 = 02$, $l_1 = 1$ and $l_2 = 102$ and defines the infinite Tribonacci-Lucas word l, $l = \lim_{n\to\infty} l_n$. The successive initial finite Tribonacci-Lucas words are: $l_0 = 02$, $l_1 = 1$, $l_2 = 102$, $l_3 = 102102$, $l_4 = 1021021021$, $l_5 = 1021021021102102102, \ldots$.

We denote as usual by $|l_n|$ the length (the number of symbols) of l_n. The following proposition summarizes basic properties of Tribonacci-Lucas words.

Theorem 2. *The infinite Tribonacci-Lucas word and the finite Tribonacci-Lucas words satisfy the following properties:*

1) *The words* 111, 22 *and* 00 *are not subwords of the infinite Tribonacci-Lucas word.*
2) *For all $n \geq 0$ let a_n be the last symbol of l_n. Then we have $a_n = 1$ if $n = 1 \pmod 3$ and $a_n = 2$ otherwise.*
3) *For all $n \geq 0$ $|l_n| = L_n$.*

We consider the finite Tribonacci-Lucas words l_n as numbers written in the ternary numeral system and denote them by b_n. Denote by d_n the value of the number b_n in usual decimal numeration system. We write $d_n = b_n$ meaning that d_n and b_n are writing of the same number in different numeral systems.

Theorem 3. *For any finite Tribonacci-Lucas word l_n we have*

$$d_0 = 2, \quad d_1 = 1, \quad d_2 = 11, \quad d_n = d_{n-1}3^{L_{n-2}+L_{n-3}} + d_{n-2}3^{L_{n-3}} + d_{n-3}\, (n > 2). \qquad (1)$$

Proof. One can easily verify (1) for the first few n : $d_3 = b_3 = 102102 = (102000+100+2)_3 = (11\cdot 3^3+1\cdot 3^2+2)_{10} = d_2 3^{L_1+L_0}+d_1 3^{L_0}+d_0$, $d_4 = b_4 = 1021021021 = (1021020000+1020+1)_3 = (308\cdot 3^4 + 11\cdot 3^1 + 1)_{10} = d_3 3^{L_2+L_1} + d_2 3^{L_1} + d_1$. Equality (1) follows from Theorem 2 (statement 3)) and the equality

$$d_n = b_n = b_{n-1}\underbrace{0\ldots0}_{L_{n-2}+L_{n-3}} + b_{n-2}\underbrace{0\ldots0}_{L_{n-3}} + b_{n-3} = d_{n-1}3^{L_{n-2}+L_{n-3}} + d_{n-2}3^{L_{n-3}} + d_{n-3}. \qquad \square$$

Let $w_0(m) = 2$ and for arbitrary integer n, $n \geq 1$, $d_n(m) = d_n \pmod m$, $0 \leq d_n(m) < m$, $b_n(m)$ be $d_n(m)$ in the ternary numeration system, $w_n(m) = w_{n-1}(m)l_n(m)$ be the contatenation of the $w_{n-1}(m)$ and $l_n(m)$, where symbols of word $l_n(m)$ it's the same $b_n(m)$. Denote by $w(m)$ the limit $w(m) = \lim_{n\to\infty} w_n(m)$.

Example 1.

$$b_0 = 2, b_1 = 1, b_2 = 102, b_3 = 102102, b_4 = 1021021021, b_5 = 1021021021102102102, \ldots;$$
$$d_0 = 2,\ d_1 = 1,\ d_2 = 11,\ d_3 = 308,\ d_4 = 24982,\ d_5 = 491729033, \ldots;$$
$$m = 5; \quad d_0(5) = 2,\ d_1(5) = 1,\ d_2(5) = 1,\ d_3(5) = 3,\ d_4(5) = 2,\ d_5 = 3, \ldots;$$
$$b_0(5) = 2,\ b_1(5) = 1,\ b_2(5) = 1,\ b_3(5) = 10,\ b_4(5) = 2,\ b_5(5) = 10, \ldots;$$
$$w_0(5) = 2,\ w_1(5) = 21,\ w_2(5) = 211,\ w_3(5) = 21110,\ w_4(5) = 211102,$$
$$w_5(5) = 21110210, \ldots; \qquad w(5) = 2111021011101011010210111101 1\ldots.$$

Definition 1. We say that

1) $w_n(m)$ is a *finite* TLLP word of type 1 modulo m;
2) $w(m)$ is an infinite TLLP word of type 1 modulo m.

Let us give the classical definition of periodicity of words over arbitrary alphabet $\mathcal{A}$ (see [9]).

Definition 2. Let $w = a_0a_1a_2\dots$, $a_i \in \mathcal{A}$, be an infinite word. We say that w is a *periodic word* if there exists a positive integer t such that $a_i = a_{i+t}$ for all $i \geq 0$. The smallest t satisfying the previous condition is called the *period* of w.

Theorem 4. *The infinite TLLP word of type 1 $w(p)$ is a periodic word.*

Proof. The statement follows from (1) because there is only a finite number of $d_n(p)$ and $3^{L_n(p)}$ possible, and the recurrence of the first few terms sequence $d_n(p)$ and $3^{L_n(p)}$ gives recurrence of all subsequent terms. □

Problem 1. *What are properties of the period $w(p)$?*

Using Tribonacci-Lucas words we define a periodic TLLP word $w^*(m)$ (infinite TLLP word type 2 by modulo m). As usual, we denote by ϵ the empty word. Let word $v_n(m)$ be the last $L_n(m)$ symbols of the word l_n if $L_n(m) \neq 0$, otherwise $v_n(m) = \epsilon$.

Theorem 5. *The word length $|v_n(m)|$ coincides with $L_n(m)$.*

Proof. This is clear by the construction of $v_n(m)$. □

Since $L_n(m)$ is a periodic sequence with period $k(m)$, the sequence $|v_n(m)|$ is periodic with the same period.

Let $w_0^*(m) = 02$ and for arbitrary integer n, $n \geq 1$, $w_n^*(m) = w_{n-1}^*(m)v_n(m)$. Denote by $w^*(m)$ the limit $w^*(m) = \lim_{n\to\infty} w_n^*(m)$.

Definition 3. We say that

1) $w_n^*(m)$ is a *finite TLLP word of type 2 modulo* m;
2) $w^*(m)$ is an *infinite TLLP word of type 2 modulo* m.

Theorem 6. *The infinite TLLP word of type 2 $w^*(m)$ is a periodic word.*

Proof. Since the sequence $L_n(m)$ is periodic, so the Theorem 6 is a direct corollary of the construction of $w^*(m)$ and Theorem 5. □

3. Polyadic series with TLLP coefficients. We consider the arithmetic properties of polyadic numbers connected with an infinite TLLP words $w(p)$ or $w^*(m)$. For any notation and properties of the polyadic numbers we refer to [10], [11].

Definition 4. We say that

$$\alpha = \sum_{n=0}^{\infty} a_n\, n!, \quad a_n \in \{0, \dots, n\}, \tag{2}$$

is the *canonical representation* of a polyadic number α.

The series (2) converges in all fields $\mathbb{Q}_p$ of p-adic numbers. The ring of polyadic integers is the direct product of the rings $\mathbb{Z}_p$ of p-adic integers for all prime numbers p. In any field $\mathbb{Q}_p$, this series converges to a p-adic integer $\alpha^{(p)}$, which equals the coordinate $\alpha^{(p)}$ in the direct product mentioned above. For any polynomial $P(x)$ with integer coefficients, the polyadic number $P(\alpha)$ has coordinate $P(\alpha^{(p)})$ in the field $\mathbb{Q}_p$.

Definition 5. We say that a polyadic number α is *algebraic* if there exists a polynomial $P(x)$ with integer coefficients such that this polynomial is not identically zero and for any prime p $P(\alpha^{(p)}) = 0$.

A polyadic number not being algebraic is said to be *transcendental*. The transcendality of a polyadic number α means that, for any polynomial $P(x)$ with integer coefficients which is not identically zero, there exists at least one prime p such that $P(\alpha^{(p)}) \neq 0$ in the field $\mathbb{Q}_p$.

Definition 6. We say that a polyadic number α is *infinitely transcendental* if, for any polynomial $P(x)$ with integer coefficients which is not identically zero, there exist infinitely many primes p such that $P(\alpha^{(p)}) \neq 0$.

Definition 7. We say that a polyadic number α is *globally transcendental* if, for any polynomial $P(x)$ with integer coefficients which is not identically zero and any prime p, we have $P(\alpha^{(p)}) \neq 0$ in the field $\mathbb{Q}_p$.

Problems of studying the arithmetic nature of polyadic numbers are very difficult. Few properties of these numbers are in [11].

Definition 8. We say that a polyadic number α is *TLLP polyadic number* if $w = a_0a_1a_2\ldots$ is a TLLP word $w(p)$ or $w^*(m)$, where $a_0, a_1, a_2, \ldots$ are coefficients of the canonical representation α in form (2).

Example 2.

$$p = 5,\ w(5) = 21110210111\ldots,\ \alpha = 2{\cdot}0!+1{\cdot}1!+1{\cdot}2!+1{\cdot}3!+2{\cdot}5!+1{\cdot}6!+1{\cdot}8!+1{\cdot}9!+1{\cdot}10!+\ldots.$$

Theorem 7. *Each TLLP polyadic number is infinitely transcendental.*

Proof. Since the infinite TLLP words $w(p)$ and $w^*(m)$ are periodic, so the Theorem 7 is a corollary of Theorem 1 [11]. □

Problem 2. *Is each TLLP polyadic number globally transcendental number?*

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Ivan Franko National University of Lviv,
galynabarabash71@gmail.com
ya_khol@ukr.net
iratytar1217@gmail.com

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