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VISCO-PLASTIC, NEWTONIAN, AND DILATANT FLUIDS: STOKES EQUATIONS WITH VARIABLE EXPONENT OF NONLINEARITY

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Some nonlinear Stokes equations with variable exponent of the nonlinearity are considered. The initial-boundary value problem for these equations is investigated and the existence of the weak and very weak solutions for the problem is proved.

Introduction. Let $n \in \mathbb{N}$ and $T > 0$ be fixed numbers, $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary $\partial\Omega$, $Q_{0,T} := \Omega \times (0,T)$, $\Sigma_{0,T} := \partial\Omega \times (0,T)$, $\Omega_\tau := \{(x,t) \mid x \in \Omega, t = \tau\}$, $\tau \in [0,T]$. We seek a weak solution $\{u, \pi\}$ of the problem

$$u_t - \operatorname{div} \mathcal{N} + \nabla\pi = F(x,t,u) \quad \text{in} \quad Q_{0,T}, \tag{1}$$

$$\operatorname{div}\, u = 0 \quad \text{in} \quad Q_{0,T}, \tag{2}$$

$$\int\limits_\Omega \pi\, dx = 0 \quad \text{in} \quad (0,T), \tag{3}$$

$$u|_{\Sigma_{0,T}} = 0, \tag{4}$$

$$u|_{t=0} = u_0(x), \quad x \in \Omega. \tag{5}$$

In the motion equation (1), $u = (u_1, \dots, u_n)\colon Q_{0,T} \to \mathbb{R}^n$ is the velocity field, $\pi\colon Q_{0,T} \to \mathbb{R}$ is the pressure, F is a given field of the external forces,

$$F(x,t,u) = f(x,t) - \gamma\, G(u), \qquad \gamma > 0, \qquad G(u) = |u|^{q(x)-2}u, \tag{6}$$

$\mathcal{N}$ is the extra stress tensor,

$$\mathcal{N} = \alpha\, D(u) + \beta\, |D_{II}(u)|^{\frac{p-2}{2}} D(u), \qquad p > 1, \tag{7}$$

$D = \frac{1}{2}(\nabla + {}^t\nabla)$ is the symmetric part of the velocity gradient, $D = (D_{ij})$,

$$D(u)_{ij} := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i,j = \overline{1,n}, \tag{8}$$

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$$D_{II}(u) := \frac{1}{8}\sum_{i,j=1}^{n}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big)^2, \tag{9}$$

$\alpha > 0$ is the dynamical viscosity of the fluid, $\beta > 0$ is the additional viscosity of the fluid.

In his paper [1], O.A. Ladyzhenskaya proposes to describe the motion of the viscous incompressible fluids by the initial-boundary value problem for the generalized Navier–Stokes equations

$$u_t - \nu_0\Delta u - \nu_1\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\Big(|\nabla u|^{p-2}u_{x_i}\Big) + \nu_2 H(u,\nabla u) + \nabla\pi = f(x,t), \quad \operatorname{div} u = 0, \tag{10}$$

where $\Delta u = \sum_{i=1}^{3}\frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian of u, $H(u,\nabla u) = \sum_{i=1}^{3} u_i\frac{\partial u}{\partial x_i}$, $\nu_0 > 0$, $\nu_1 > 0$, $\nu_2 = 1$, and $p \geq \frac{5}{2}$. It is well known that the exponent p of the nonlinearity of equations (10) characterizes type of the fluid. In the case $p \in (1,2)$ we have visco-plastic fluid (blood, lava, paint etc.), in the case $p = 2$ we have Newtonian fluid (water), in the case $p > 2$ we have dilatant fluid (clay suspensions, sweet mixture, water-sand systems etc.) (see [2]). Equations (10) with $\nu_2 = 0$ are called the generalized Stokes equations. The Stokes system is a particular case of (10) corresponding to $\nu_1 = \nu_2 = 0$ (see [3], [4]). Equations (1)-(2) are the equations of type (10) with the term $|u|^{q(x)-2}u$ instead of $H(u,\nabla u)$.

We seek a weak and very weak solution to (1)-(5) (see definition below for more details). The weak solution of the Navier–Stokes and Stokes equations with the constant and variable exponents of the nonlinearities is considered in [4]–[11] (see also the references given there). The very weak solution of the Navier–Stokes equations with the constant exponents of the nonlinearities is studied in [2]. Variational inequality for the Stokes equations is considered in [12]. As we know the very weak solution of the Stokes equations with variable exponent of the nonlinearity is not studied yet.

The paper is organized as follows. In Section 1, we formulate the considered problem and main results. The auxiliary statements are given in Section 2. Finally, in Section 3 we prove the main statements.

1. Notation and statement of main result. Let $\|\cdot\|_B \equiv \|\cdot;B\|$ be a norm of some Banach space B, B^* be a dual space, $\langle\cdot,\cdot\rangle_B$ be the scalar product between B^* and B, $B^n := B\times\ldots\times B$ be the nth Cartesian product of the B, $\|z;B^n\| := \|z_1\|_B+\ldots+\|z_n\|_B$ for $z = (z_1,\ldots,z_n) \in B^n$, $(\cdot,\cdot)_H$ be the scalar product in the Hilbert space H, $|\cdot|_H := \sqrt{(\cdot,\cdot)_H}$, $(\cdot,\cdot) := (\cdot,\cdot)_{\mathbb{R}^n}$, and $|\cdot| := |\cdot|_{\mathbb{R}^n}$. For the Banach spaces X and Y the notation $X \circlearrowleft Y$ means the continuous embedding; the notation $X \bar{\circlearrowleft} Y$ means continuous and dense embedding; the notation $X \overset{K}{\subset} Y$ means a compact embedding.

Let us define the scalar product of the tensors $K = (K_{ij})$ and $V = (V_{ij})$ by the rule $K\colon V \equiv \sum_{i,j=1}^{n} K_{ij}V_{ij}$ (see [13, p. 9]). Let $|K| \equiv \sqrt{K\colon K}$. Throughout this paper, for simplicity, we write $\|K;B\|$ instead of $\|\,|K|;B\|$, where B is the Banach space. If $u = (u_1,\ldots,u_n)$ and $K = (K_{ij})$, then $\operatorname{div} u \equiv \frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\ldots+\frac{\partial u_n}{\partial x_n}$, $\nabla u \equiv (n_{ij})$, where $n_{ij} = \frac{\partial u_j}{\partial x_i}$ $(i,j = \overline{1,n})$, and $\operatorname{div} K \equiv \big((\operatorname{div} K)_1,\ldots,(\operatorname{div} K)_n\big)$, where $(\operatorname{div} K)_i = \sum_{j=1}^{n}\frac{\partial K_{ji}}{\partial x_j}$ $(i = \overline{1,n})$ (see [13, p. 9]).

Suppose that $m \in \mathbb{N}$, $\mathfrak{p} \in [1,\infty]$, X is a Banach space, $\mathcal{O} = \Omega$ or $\mathcal{O} = Q_{0,T}$, $\mathcal{M}(\mathcal{O})$ is a set of all measurable functions $v\colon \mathcal{O} \to \mathbb{R}$, $\operatorname{Lip}(\mathcal{O})$ is a set of all Lipschitz-continuous functions $v\colon \mathcal{O} \to \mathbb{R}$, $C^m(\mathcal{O})$, $C_0(\mathcal{O})$, and $D(\mathcal{O})$ are the spaces of the smooth functions (see [14, p. 9, 19]), $C_0^m(\mathcal{O}) := C^m(\mathcal{O}) \cap C_0(\mathcal{O})$, $L^{\mathfrak{p}}(\mathcal{O})$ is the Lebesgue space (see [14, p. 22, 24]), $W^{m,\mathfrak{p}}(\mathcal{O})$ and $W_0^{m,\mathfrak{p}}(\mathcal{O})$ are the Sobolev spaces (see [14, p. 45]), $H^m(\mathcal{O}) := W^{m,2}(\mathcal{O})$, $H_0^m(\mathcal{O}) := W_0^{m,2}(\mathcal{O})$, $C([0,T];X)$ and $C^m([0,T];X)$ are the spaces of the X-valued smooth functions defined on $[0,T]$ (see [15, p. 147]), $L^{\mathfrak{p}}(0,T;X)$ is the Lebesgue-Bochner space (see [15, p. 155]), $W^{m,\mathfrak{p}}(0,T;X)$ is the Sobolev-Bochner space (see [16, p. 286]), and $H^m(0,T;X) := W^{m,2}(0,T;X)$. Also suppose that $\mathcal{V} := \{v \in [D(\Omega)]^n \mid \operatorname{div} v = 0\}$, $V_{\mathfrak{p}} := \{v \in [W_0^{1,\mathfrak{p}}(\Omega)]^n \mid \operatorname{div} v = 0\}$ is the closure of $\mathcal{V}$ in the Sobolev space $[W^{1,\mathfrak{p}}(\Omega)]^n$, H is the closure of $\mathcal{V}$ in the Lebesgue space $[L^2(\Omega)]^n$,

$$(u,v)_H = \sum_{i=1}^n \int_\Omega u_i(x)v_i(x)\,dx, \quad u=(u_1,\dots,u_n), \quad v=(v_1,\dots,v_n), \tag{11}$$

and $\mathcal{B}_+(\mathcal{O}) := \{\mathfrak{q} \in L^\infty(\mathcal{O}) | \operatorname*{ess\,inf}_{y\in\mathcal{O}} \mathfrak{q}(y) > 0\}$. For the sake of convenience we shall write $u(t)$ instead of $u(\cdot,t)$ and $L^p(0,T)$ instead of $L^p((0,T))$ etc. For every $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$ by definition, put

$$\mathfrak{q}_0 := \operatorname*{ess\,inf}_{y\in\mathcal{O}} \mathfrak{q}(y), \qquad \mathfrak{q}^0 := \operatorname*{ess\,sup}_{y\in\mathcal{O}} \mathfrak{q}(y), \tag{12}$$

$$\rho_{\mathfrak{q}}(v;\mathcal{O}) := \int_{\mathcal{O}} |v(y)|^{\mathfrak{q}(y)}\,dy, \quad v \in \mathcal{M}(\mathcal{O}), \tag{13}$$

$$\mathfrak{q}'(y) := \frac{\mathfrak{q}(y)}{\mathfrak{q}(y)-1} \quad \text{for a.e. } y \in \mathcal{O} \tag{14}$$

(note that $\frac{1}{\mathfrak{q}(y)} + \frac{1}{\mathfrak{q}'(y)} = 1$ for a.e. $y \in \mathcal{O}$ and $\mathfrak{q}' \in \mathcal{B}_+(\mathcal{O})$, if $\mathfrak{q}_0 > 1$).

Assume that $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$ and $\mathfrak{q}_0 > 1$. The set $L^{\mathfrak{q}(y)}(\mathcal{O}) := \{v \in \mathcal{M}(\mathcal{O}) | \rho_{\mathfrak{q}}(v;\mathcal{O}) < +\infty\}$ with the Luxemburg norm $\|v; L^{\mathfrak{q}(y)}(\mathcal{O})\| := \inf\{\lambda > 0 | \rho_{\mathfrak{q}}(v/\lambda;\mathcal{O}) \le 1\}$ is called a generalized Lebesgue space. By definition, put

$$C_{weak}([0,T];H) := \{u\colon [0,T] \to H | \forall\, z \in H \quad (u(\cdot),z)_H \in C([0,T])\}.$$

Note that $C([0,T];H) \subsetneq C_{weak}([0,T];H)$. We shall need the following assumptions:

(A): $p \in (1,+\infty)$, $q \in \mathcal{B}_+(\Omega)$, $q_0 > 1$, and $\alpha,\beta,\gamma > 0$.

By definition, put

$$\sigma := \max\{2,p\}, \qquad \sigma' = \frac{\sigma}{\sigma-1}, \qquad p' = \frac{p}{p-1}, \qquad h = \min\Big\{2,\ p',\ \frac{q^0}{q^0-1}\Big\}, \tag{15}$$

$$V := V_\sigma \cap [L^{q(x)}(\Omega)]^n, \qquad U(Q_{0,T}) := L^\sigma(0,T;V_\sigma) \cap [L^{q(x)}(Q_{0,T})]^n. \tag{16}$$

Let us define the function $\mathcal{J}_p\colon [W_0^{1,p}(\Omega)]^n \to \mathbb{R}$ by the rule

$$\mathcal{J}_p(v) := \frac{2}{p}\int_\Omega |D_{II}(v(x))|^{\frac{p}{2}}\,dx, \quad v \in [W_0^{1,p}(\Omega)]^n. \tag{17}$$

By [2, p. 1081] we get that $\mathcal{J}_p$ is convex a Gateaux-differentiable function and

$$\langle \mathcal{J}_p'(u),v\rangle_{[W_0^{1,p}(\Omega)]^n} = \Big\langle -\operatorname{div}\big(|D_{II}(u)|^{\frac{p-2}{2}}D(u)\big),\ v\Big\rangle_{[W_0^{1,p}(\Omega)]^n} =$$

$$= \int_\Omega |D_{II}(u)|^{\frac{p-2}{2}} D(u)\colon D(v)\,dx, \qquad u,v \in [W_0^{1,p}(\Omega)]^n. \tag{18}$$

Definition 1. A pair $\{u,\pi\}$ is called a *weak solution* of problem (1)-(5) if

$$u\in U(Q_{0,T})\cap C([0,T];H),\quad u_t\in L^2(0,T;H),\quad \pi\in L^h(Q_{0,T}),$$

u satisfies initial condition (5), for every $v\in U(Q_{0,T})$ we have

$$\int\limits_{Q_{0,T}}\Big[(u_t,v)+\alpha\,D(u)\colon D(v)+\beta\,|D_{II}(u)|^{\frac{p-2}{2}}D(u)\colon D(v)+(\gamma\,|u|^{q(x)-2}u-f,v)\Big]\,dxdt=0,\quad(19)$$

π satisfies condition (3) and the equality

$$u_t+\alpha\,\mathcal{J}_2'(u)+\beta\,\mathcal{J}_p'(u)+\gamma\,G(u)+\nabla\pi=f\quad\text{in}\quad[D^*(Q_{0,T})]^n.\qquad(20)$$

Note that if u and v are regular enough and if $v(T)=0$, then

$$\int\limits_{Q_{0,T}}(u_t,v-u)\,dxdt=-\int\limits_{Q_{0,T}}(u,v_t)\,dxdt-(u(0),v(0))_H-\frac{1}{2}|u(T)|_H^2+\frac{1}{2}|u(0)|_H^2.\qquad(21)$$

From [2, p. 1089] we have the estimate

$$\mathcal{J}_p(v)-\mathcal{J}_p(u)\geq\langle\mathcal{J}_p'(u),v-u\rangle_{[W_0^{1,p}(\Omega)]^n}.\qquad(22)$$

Let us introduce the function $\mathcal{I}_q\colon[L^{q(x)}(\Omega)]^n\to\mathbb{R}$ by the rule

$$\mathcal{I}_q(v):=\int\limits_{\Omega}\frac{1}{q(x)}\,|v(x)|^{q(x)}\,dx,\quad v\in[L^{q(x)}(\Omega)]^n.\qquad(23)$$

By [17, p. 17] we get that $\mathcal{I}_q$ is convex Gateaux-differentiable, $\mathcal{I}_q'=G$, and so

$$\mathcal{I}_q(v)-\mathcal{I}_q(u)\geq\langle G(u),v-u\rangle_{[L^{q(x)}(\Omega)]^n}.\qquad(24)$$

If we replace v by $v-u$ in (19) and use (5), (21), (22) and (24), we obtain the parabolic variational inequality

$$\int\limits_{Q_{0,T}}(-v_t,u)\,dxdt-(u_0,v(0))_H-\frac{1}{2}|u(T)|_H^2+\frac{1}{2}|u_0|_H^2+\alpha\int\limits_{Q_{0,T}}D(u)\colon\big(D(v)-D(u)\big)\,dxdt+$$
$$+\int\limits_0^T\Big[\beta\big(\mathcal{J}_p(v)-\mathcal{J}_p(u)\big)+\gamma\big(\mathcal{I}_q(v)-\mathcal{I}_q(u)\big)\Big]\,dt-\int\limits_{Q_{0,T}}(f,v-u)\,dxdt\geq0\qquad(25)$$

for every $v\in C^1([0,T];V)$ such that $v(T)=0$. Similarly to [2, p. 1082] we give the following definition.

Definition 2. A function $u\in U(Q_{0,T})\cap L^\infty(0,T;H)\cap C_{weak}([0,T];H)$ is called a *very weak solution* of problem (1)–(5) if parabolic variational inequality (25) holds for every $v\in C^1([0,T];V)$ such that $v(T)=0$.

Theorem 1. *If condition* **(A)** *holds, $u_0\in H$, and $f\in L^1(0,T;H)$, then problem (1)–(5) has a very weak solution u such that*

$$(u(0),z)_H=(u_0,z)_H\quad\forall\,z\in H.\qquad(26)$$

Theorem 2. *If condition **(A)** holds, $u_0 \in V$, and $f \in L^2(0,T;H)$, then problem (1)–(5) has a weak solution $\{u,\pi\}$ such that*

$$u \in L^\infty(0,T;V), \qquad \nabla\pi \in L^2(0,T;[H^{-1}(\Omega)]^n) + L^{p'}(0,T;[W^{-1,p'}(\Omega)]^n) + [L^{q'(x)}(Q_{0,T})]^n.$$

2. Auxiliary facts. First notice that if D and D_{II} are determined from (8)–(9), then

$$D_{II}(u) = \frac{1}{2}\, D(u)\colon D(u) = \frac{1}{2}\,|D(u)|^2. \tag{27}$$

2.1. Generalized Lebesgue spaces. Properties of the generalized Lebesgue and Sobolev spaces were widely studied in [18], [19], [20], [21], and [22]. In particular, it is well known that if $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$ and $\mathfrak{q}_0 > 1$, then $L^{\mathfrak{q}(y)}(\mathcal{O})$ is the Banach space which is reflexive and separable (see [22, p. 8]). For every $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$, by definition, put

$$S_{\mathfrak{q}}(s) := \max\{s^{\mathfrak{q}_0},\, s^{\mathfrak{q}^0}\}, \quad s \geq 0, \tag{28}$$

Proposition 1. *(Lemma 1 [23, p. 168]). Suppose that $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$, $\mathfrak{q}_0 > 1$, $S_{\mathfrak{q}}$ is defined by (28), and $\rho_{\mathfrak{q}}$ is defined by (13). Then for every $v \in \mathcal{M}(\mathcal{O})$ the following statements hold:*

(i) $\|v; L^{\mathfrak{q}(y)}(\mathcal{O})\| \leq S_{1/\mathfrak{q}}(\rho_{\mathfrak{q}}(v;\mathcal{O}))$ if $\rho_{\mathfrak{q}}(v;\mathcal{O}) < +\infty$;

(ii) $\rho_{\mathfrak{q}}(v;\mathcal{O}) \leq S_{\mathfrak{q}}(\|v; L^{\mathfrak{q}(y)}(\mathcal{O})\|)$ if $\|v; L^{\mathfrak{q}(y)}(\mathcal{O})\| < +\infty$.

Proposition 2. *Suppose that $q \in \mathcal{B}_+(\Omega)$ and $q_0 > 1$. Then the following statements hold:*

(i) (see Lemma 2 [24, p. 46]) we have continuous and dense embeddings

$$L^{q^0}(0,T;L^{q(x)}(\Omega)) \;\bar{\circlearrowright}\; L^{q(x)}(Q_{0,T}) \;\bar{\circlearrowright}\; L^{q_0}(0,T;L^{q(x)}(\Omega)). \tag{29}$$

(ii) (see [19, p. 613]) the Nemyckii operator $G(u) = |u|^{q(x)-2}u$ maps $[L^{q(x)}(\Omega)]^n$ in $[L^{q'(x)}(\Omega)]^n$, it is continuous and bounded, in particular,

$$\|G(z); [L^{q'(x)}(\Omega)]^n\| \leq C_1 \|z; [L^{q(x)}(\Omega)]^n\|, \quad z \in [L^{q(x)}(\Omega)]^n. \tag{30}$$

where the constant $C_1 > 0$ is independent of z.

2.2. Cauchy's problem for system of ordinary differential equations. Take $Q = (0,T)\times\mathbb{R}^\ell$, where $\ell \in \mathbb{N}$. Let us seek a weak solution $\varphi\colon [0,T] \to \mathbb{R}^\ell$ of the problem

$$\varphi'(t) + L(t,\varphi(t)) = M(t), \quad t \in [0,T], \qquad \varphi(0) = \varphi^0, \tag{31}$$

where $M\colon [0,T] \to \mathbb{R}^\ell$ and $L\colon Q \to \mathbb{R}^\ell$ are some functions (for the sake of convenience we have assumed that $L(t,0) = 0$ for every $t \in [0,T]$), and $\varphi^0 = (\varphi_1^0,\ldots,\varphi_\ell^0) \in \mathbb{R}^\ell$.

Proposition 3. *(the Carathéodory-LaSalle theorem, see Theorem 3.24 [25, p. 872]). Suppose that $r \geq 2$, a function $L\colon Q \to \mathbb{R}^\ell$ satisfies L^r-Carathéodory condition, $M \in L^r(0,T;\mathbb{R}^\ell)$, and $\varphi^0 \in \mathbb{R}^\ell$. If there exists nonnegative functions $\alpha_1, \alpha_2 \in L^1(0,T)$ such that for every $\xi \in \mathbb{R}^\ell$ and for a.e. $t \in [0,T]$ the inequality*

$$(L(t,\xi),\xi)_{\mathbb{R}^\ell} \geq -\alpha_1(t)|\xi|^2 - \alpha_2(t) \tag{32}$$

holds, then problem (31) has a global weak solution $\varphi \in W^{1,r}(0,T;\mathbb{R}^\ell)$.

2.3. Special function spaces. Take $s \in \mathbb{N}$. Let us consider the Sobolev space $[H^s(\Omega)]^n$ with the scalar product $((u,v))_s := \sum\limits_{i=1}^{n} (u_i, v_i)_{H^s(\Omega)}$. Suppose that Z_s is the closure of $\mathcal{V}$ in $[H^s(\Omega)]^n$, $Z := Z_1$. From [26, Ch. 1, §6.1], we obtain the embeddings $Z_s \,\overline{\circlearrowleft}\, Z \,\overline{\circlearrowleft}\, H \cong H^* \,\overline{\circlearrowleft}\, Z^* \,\overline{\circlearrowleft}\, Z_s^*$. Let $\{w^\mu\}_{\mu\in\mathbb{N}}$ be a set of all eigenfunctions of the problem

$$((w,v))_s = \lambda\,(w,v)_H \quad \forall\, v \in Z_s, \tag{33}$$

$\{\lambda_\mu\}_{\mu\in\mathbb{N}} \subset \mathbb{R}_{>0}$ is the set of the corresponding eigenvalues. Suppose that $\{w^\mu\}_{\mu\in\mathbb{N}}$ is an orthonormal set in H.

Proposition 4. *(see [26, Ch. 1, §6.3]). If $s \in \mathbb{N}$ and $s \geq \frac{n}{2}$, then the set $\{w^\mu\}_{\mu\in\mathbb{N}}$ of all eigenfunction of problem (33) is a basis for the space Z_s.*

2.4. Compactness theorems. The following facts are needed for the sequel.

Proposition 5. *(the Aubin theorem, see [27] and [28, p. 393]). If $s, h \in (1,\infty)$ are fixed numbers, $\mathcal{W}, \mathcal{L}, \mathcal{B}$ are the Banach spaces, and $\mathcal{W} \overset{K}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$, then*

$$\{u \in L^s(0,T;\mathcal{W}) \;\mid\; u_t \in L^h(0,T;\mathcal{B})\} \overset{K}{\subset} [L^s(0,T;\mathcal{L}) \cap C([0,T];\mathcal{B})].$$

The Aubin's result is extended by the following Proposition 6 and Theorem 3.

Proposition 6. *(the Simon theorem, see [6, p. 1097]). If $\mathcal{W}, \mathcal{L}, \mathcal{B}$ are the Banach spaces, and $\mathcal{W} \overset{K}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$, then*

(i) $\{u \in L^s(0,T;\mathcal{W}) \;\mid\; u_t \in L^1(0,T;\mathcal{B})\} \overset{K}{\subset} L^s(0,T;\mathcal{L})$ if $1 \leq s < \infty$;

(ii) $\{u \in L^\infty(0,T;\mathcal{W}) \;\mid\; u_t \in L^h(0,T;\mathcal{B})\} \overset{K}{\subset} C([0,T];\mathcal{L})$ if $1 < h \leq \infty$.

Theorem 3. *Suppose that $\mathcal{W}, \mathcal{L}, \mathcal{B}$ are the Banach spaces, $\mathcal{W} \overset{K}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$, and $h \in (1,+\infty)$. If the sequence $\{\varphi^m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(0,T;\mathcal{W})$, $K \in L^1(0,T)$, $K \geq 0$, the nonnegative sequence $\{c^m\}_{m\in\mathbb{N}}$ is bounded in $L^h(0,T)$, and the following estimate is satisfied*

$$\|\varphi_t^m(t)\|_{\mathcal{B}} \leq K(t) + c^m(t), \quad t \in (0,T), \quad m \in \mathbb{N}, \tag{34}$$

then there exists a subsequence $\{\varphi^{m_j}\}_{j\in\mathbb{N}} \subset \{\varphi^m\}_{m\in\mathbb{N}}$ such that $\varphi^{m_j} \underset{j\to\infty}{\longrightarrow} \varphi$ in $C([0,T];\mathcal{L})$.

Proof. We use the method [6, p. 1097-1098] (see also [2, p. 1084]). Take a point $\delta \in (0,T)$. Denote by $\tau_\delta\varphi$ the translated function of φ, that is $(\tau_\delta\varphi)(t) = \varphi(t+\delta)$. Integrating by parts (see Theorem 2 [16, p. 286]) and using estimate (34), we obtain

$$\|\varphi^m(t+\delta) - \varphi^m(t)\|_{\mathcal{B}} = \Big\| \int\limits_t^{t+\delta} \varphi_t^m(s)\,ds \Big\|_{\mathcal{B}} \leq \int\limits_t^{t+\delta} \|\varphi_t^m(s)\|_{\mathcal{B}}\,ds \leq$$

$$\leq \int\limits_t^{t+\delta} K(s)\,ds + \Big(\int\limits_t^{t+\delta} ds\Big)^{\frac{h-1}{h}} \Big(\int\limits_t^{t+\delta} |c^m(s)|^h\,ds\Big)^{\frac{1}{h}} = \int\limits_t^{t+\delta} K(s)\,ds + \delta^{\frac{h-1}{h}} \Big(\int\limits_0^{T} |c^m(s)|^h\,ds\Big)^{\frac{1}{h}}.$$

Then

$$\|\tau_\delta\varphi^m - \varphi^m; L^\infty(0,T-\delta;\mathcal{B})\| \leq \operatorname*{ess\,sup}_{t\in[0,T-\delta]} \int\limits_t^{t+\delta} K(s)\,ds + C_2\delta^{\frac{h-1}{h}}. \tag{35}$$

The right-hand side of (35) tends to zero in $\delta \to +0$, uniformly for $m \in \mathbb{N}$ (see [6, p. 1098]). Moreover, since the set $\{\varphi^m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(0,T;\mathcal{W})$, by Theorem 5 [29, p. 84], we have that the set $\{\varphi^m\}_{m\in\mathbb{N}}$ is relatively compact in $C([0,T];\mathcal{L})$. □

2.4. Generalized De Rham's theorem. Let $\mathbb{Z}_{\geq -1} := \{s \in \mathbb{Z} | s \geq -1\}$. The following Proposition is needed for the sequel.

Proposition 7. *(the generalized De Rham's theorem, see Theorem 4.1 [7], Remark 4.3 [7], and Lemma 2 [6]). Suppose that* Ω *be a open bounded connected and Lipschitz subset of* $\mathbb{R}^n$, $T > 0$, $s_1, s_2 \in \mathbb{Z}_{\geq -1}$, $h_1, h_2 \in [1, \infty]$, *and* $\mathcal{F} \in W^{s_1,h_1}(0, T; [W^{s_2,h_2}(\Omega)]^n)$. *Then if*

$$\langle \mathcal{F}(\cdot), v \rangle_{[D(\Omega)]^n} = 0 \quad in \quad D^*(0, T)$$

for all $v \in \mathcal{V} = \{v \in [D(\Omega)]^n \mid \operatorname{div} v = 0\}$, *then there exists a unique*

$$\pi \in W^{s_1,h_1}(0, T; W^{s_2+1,h_2}(\Omega)) \tag{36}$$

such that

$$\nabla \pi = \mathcal{F} \quad in \quad [D^*(Q_{0,T})]^n, \tag{37}$$

$$\int\limits_{\Omega} \pi(\cdot)\, dx = 0 \quad in \quad D^*(0, T). \tag{38}$$

Moreover, there exists a positive number C_3 *(independent of* $\mathcal{F}, \pi$*) such that*

$$\|\pi; W^{s_1,h_1}(0, T; W^{s_2+1,h_2}(\Omega))\| \leq C_3 \|\mathcal{F}; W^{s_1,h_1}(0, T; [W^{s_2,h_2}(\Omega)]^n)\|. \tag{39}$$

To detail Proposition 7, let us introduce additional notation.

First let us consider the vector-valued distributions, used in (36) (see [7, Sect. 3.5]). We now define the distribution space because the space, used in (36), will be defined as a subspace of such a space.

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^N$, and let $\mathcal{Y}$ be a complete LCSTVS, that is a locally convex separated topological vector space (the case where it is not the Banach space is used in (42) below). The space of $\mathcal{Y}$-valued distributions on $\mathcal{O}$ is defined by $D^*(\mathcal{O}; \mathcal{Y}) := \mathcal{L}_c(D(\mathcal{O}); \mathcal{Y})$, where $\mathcal{L}_c$ stands for linear continuous (here, it is equivalent to sequentially continuous) and $D(\mathcal{O})$ is the space of the indefinitely differentiable functions with a compact support included in $\mathcal{O}$. Given $f \in D^*(\mathcal{O}; \mathcal{Y})$ and $\varphi \in D(\mathcal{O})$, we frequently denote $\langle f, \varphi \rangle_{D(\mathcal{O})} := f(\varphi) \in \mathcal{Y}$. As usually, we denote $D^*(\mathcal{O}) := D^*(\mathcal{O}; \mathbb{R})$, $D^*(0, T; \mathcal{Y}) := D^*((0, T); \mathcal{Y})$, $D(0, T) := D((0, T))$, and $D^*(0, T) := D^*((0, T))$.

Given $f \in C(\mathcal{O}; \mathcal{Y})$ we identify it to the distribution $\dot{f} \in D^*(\mathcal{O}; \mathcal{Y})$ defined by

$$\langle \dot{f}, \varphi \rangle_{D(\mathcal{O})} := \int\limits_{\mathcal{O}} f(y) \varphi(y)\, dy, \quad \varphi \in D(\mathcal{O}). \tag{40}$$

This provides a topological imbedding $C(\mathcal{O}; \mathcal{Y}) \subset D^*(\mathcal{O}; \mathcal{Y})$.

Further let us consider the Sobolev space, used in (36) if $s_1 = 0$ and $s_1 = -1$ (see [7, Sect. 3.6]). Let again $\mathcal{O}$ be an open subset of $\mathbb{R}^N$, Y be a Banach space, and $h \in [1, \infty]$. Given $f \in L^h(\mathcal{O}; Y)$ we identify it to the distribution $\dot{f}$ again defined by (40). This provides a topological imbedding $L^h(\mathcal{O}; Y) \subset D^*(\mathcal{O}; Y)$ and allows to define the derivatives of f to be $\frac{\partial \dot{f}}{\partial y_j}$. Now, we can define $W^{0,h}(\mathcal{O}; Y) := L^h(\mathcal{O}; Y)$ and

$$W^{-1,h}(\mathcal{O}; Y) := \Big\{ f \in D^*(\mathcal{O}; Y) \mid f = f_0 + \sum_{j=1}^{N} \frac{\partial f_j}{\partial y_j}, \ \ f_j \in L^h(\mathcal{O}; Y) \ \ (0 \leq j \leq N) \Big\}.$$

Now let us consider the linear image of the distribution, used in (38) (see [7, Sect. 3.7]). Let $\mathcal{Y}$ and $\mathcal{X}$ be two complete LCSTVS and let $A \in \mathcal{L}_c(\mathcal{Y};\mathcal{X})$. Given $f \in D^*(\mathcal{O};\mathcal{Y})$, its image $Af \in D^*(\mathcal{O};\mathcal{X})$ is defined by

$$(Af)(\varphi) := A(f(\varphi)) \quad \forall\, \varphi \in D(\mathcal{O}). \tag{41}$$

In the case of the Banach spaces, A maps continuously $W^{s,h}(\mathcal{O};\mathcal{Y})$ into $W^{s,h}(\mathcal{O};\mathcal{X})$, where $s \in \mathbb{Z}_{\geq -1}$, $h \in [1,\infty]$.

Since $s_2 \in \mathbb{Z}_{\geq -1}$, we get $s_2+1 \geq 0$ and (36) implies that π lying in $W^{s_1,h_1}(0,T;L^{h_2}(\Omega))$, its image by the map $\int_\Omega \cdot\, dx \in \mathcal{L}_c(L^{h_2}(\Omega);\mathbb{R})$ is defined by (41), $\int_\Omega \pi(x,\cdot)\,dx \in W^{s_1,h_1}(0,T;\mathbb{R})$.

Finally let us consider the separation of the variables, used in (36) (see [7, Sect. 3.8]). The separation of variable for functions which map $C(Q_{0,T};\mathcal{Y})$ onto $C((0,T);C(\Omega;\mathcal{Y}))$, extends by continuity in a one-to-one bicontinuous map from the space $D^*(Q_{0,T};\mathcal{Y})$ onto $D^*(0,T;D^*(\Omega;\mathcal{Y}))$. Using this map to identify the spaces, we get the topological equality

$$D^*(Q_{0,T};\mathcal{Y}) = D^*(0,T;D^*(\Omega;\mathcal{Y})). \tag{42}$$

This identity allows us to consider π either as distributions on $Q_{0,T}$, as in (37), or as distributions on $(0,T)$ with values in a space of distributions on Ω, as in (36).

2.5. Additional statements. The following facts are needed for the sequel.

Proposition 8. *(the Korn inequality, see [18, p. 462]). Suppose that $p \in [1,+\infty)$. Then there exist positive constants K_1 and K_2 such that for every $u \in [W_0^{1,p}(\Omega)]^n$ we have*

$$K_1\|\nabla u; L^p(\Omega)\| \leq \|D(u); L^p(\Omega)\| \leq K_2\|\nabla u; L^p(\Omega)\|. \tag{43}$$

Proposition 9. *(Lemma 5 [6, p. 1098]). Suppose that $y \in W^{1,1}(0,T)$, $y \geq 0$, $K \in L^1(0,T)$, $K \geq 0$, and $y_0 \in \mathbb{R}_{\geq 0}$. Then if the inequalities*

$$\frac{d}{dt}\,y^2 \leq K\,y, \qquad y(0) \leq y_0, \tag{44}$$

hold, then the following estimate is true

$$y(t) \leq y_0 + \frac{1}{2}\int_0^T K(s)\,ds, \quad t \in [0,T]. \tag{45}$$

Lemma 1. *Suppose that H is the Hilbert space, $f^m \underset{m\to\infty}{\longrightarrow} f$ strongly in $L^1(0,T;H)$, and $u^m \underset{m\to\infty}{\longrightarrow} u$ $*$-weakly in $L^\infty(0,T;H)$. Then $\int_0^T (f^m(t),u^m(t))_H\,dt \underset{m\to\infty}{\longrightarrow} \int_0^T (f(t),u(t))_H\,dt$.*

Proof. We use the method [30, p. 51]. Clearly, $\|u^m; L^\infty(0,T;H)\| \leq C_4$, where the constant $C_4 > 0$ is independent of m. Then

$$\int_0^T \Big[(f^m,u^m)_H - (f,u)_H\Big]\,dt = \int_0^T \Big[(f^m-f,u^m)_H + (f,u^m-u)_H\Big]\,dt \leq \int_0^T |f^m-f|_H\cdot|u^m|_H\,dt +$$

$$+\int_0^T (f,u^m-u)_H\,dt \leq C_4\int_0^T |f^m-f|_H\,dt + \int_0^T (f,u^m-u)_H\,dt \underset{m\to\infty}{\longrightarrow} 0.$$

□

3. Proof of the main theorems.

Proof of Theorem 1. The solution will be constructed via Faedo-Galerkin's method.
Step 1. Since $f \in L^1(0,T;H)$, there exists a sequence $\{f^m\}_{m\in\mathbb{N}} \subset C^1([0,T];H)$ such that

$$f^m \underset{m\to\infty}{\longrightarrow} f \quad \text{strongly in} \quad L^1(0,T;H). \tag{46}$$

Then $|f^m(\cdot)|_H \underset{m\to\infty}{\longrightarrow} |f(\cdot)|_H$ in $L^1(0,T)$. Hence, Theorem 4.9 [31, p. 94] implies that there exists a subsequence (we call it $\{f^m\}_{m\in\mathbb{N}}$ again) and a nonnegative $\widetilde{f} \in L^1(0,T)$ such that

$$|f^m(t)|_H \le \widetilde{f}(t) \quad \text{for a.e.} \quad t \in (0,T). \tag{47}$$

Let $\{w^\mu\}_{\mu\in\mathbb{N}}$ and Z_s are taken from Proposition 4, $s \in \mathbb{N}$, and

$$s \ge \max\Big\{2,\ \frac{n}{2},\ 1+n\Big(\frac{1}{2}-\frac{1}{p}\Big),\ n\Big(\frac{1}{2}-\frac{1}{q^0}\Big)\Big\}.$$

Hence, since $\sigma = \max\{p,2\}$, we have $Z_s \circlearrowleft V_\sigma \cap [L^{q^0}(\Omega)]^n \circlearrowleft V$. Let $m \in \mathbb{N}$ and V^m be the subspace spanned by $w^1,\dots,w^m$. By definition, put

$$u^m(x,t) = \sum_{\mu=1}^{m} \varphi_\mu^m(t) w^\mu(x), \quad (x,t) \in Q_{0,T}, \quad m \in \mathbb{N}, \tag{48}$$

where the functions $\varphi_1^m,\dots,\varphi_m^m$ satisfy the equalities

$$(u_t^m(t),w^\mu)_H + \alpha\,\langle \mathcal{J}_2'(u^m(t)),w^\mu\rangle_{[H_0^1(\Omega)]^n} + \beta\,\langle \mathcal{J}_p'(u^m(t)),w^\mu\rangle_{[W_0^{1,p}(\Omega)]^n} +$$

$$+\gamma\,(G(u^m(t)),w^\mu)_H = (f^m(t),w^\mu)_H, \qquad t \in (0,T), \quad \mu = \overline{1,m}, \tag{49}$$

$$\varphi_1^m(0) = \zeta_1^m, \quad \dots, \quad \varphi_m^m(0) = \zeta_m^m, \tag{50}$$

where $\zeta_1^m,\dots,\zeta_m^m \in \mathbb{R}$ and the function $u_0^m(x) := \sum\limits_{\mu=1}^{m} \zeta_\mu^m w^\mu(x)$, $x \in \Omega$, satisfies the condition

$$u_0^m \underset{m\to\infty}{\longrightarrow} u_0 \quad \text{strongly in} \quad H. \tag{51}$$

Clearly, (49)–(50) coincide with (31) if we put $\varphi := (\varphi_1^m,\dots,\varphi_m^m)$, $\varphi^0 := (\zeta_1^m,\dots,\zeta_m^m)$, $L := (L_1,\dots,L_m)$, $M := (M_1,\dots,M_m)$,

$$L_\mu(t,\varphi(t)) := \alpha\,\langle \mathcal{J}_2'(u^m(t)),w^\mu\rangle_{[H_0^1(\Omega)]^n} + \beta\,\langle \mathcal{J}_p'(u^m(t)),w^\mu\rangle_{[W_0^{1,p}(\Omega)]^n} + \gamma\,(G(u^m(t)),w^\mu)_H,$$

$$M_\mu(t) := (f^m(t),w^\mu)_H, \ \ t \in (0,T), \ \ \mu = \overline{1,m}.$$

Similarly to [25, p. 874], we prove that $L\colon (0,T)\times\mathbb{R}^m \to \mathbb{R}^m$ satisfies L^∞-Carathéodory condition. Moreover, $M \in L^2(0,T;\mathbb{R}^m)$. Using (18), (43), and the Friedrichs inequality (see [15, p. 50]), we obtain

$$\langle \mathcal{J}_2'(u^m(t)),u^m(t)\rangle_{[H_0^1(\Omega)]^n} = \int\limits_{\Omega_t} |D(u^m)|^2\,dx \ge C_5 \int\limits_{\Omega_t} |\nabla u^m|^2\,dx \ge C_6 \int\limits_{\Omega_t} |u^m|^2\,dx, \tag{52}$$

where the constant $C_6 > 0$ is independent of u^m. Since

$$\int\limits_{\Omega} \Big|\sum_{\mu=1}^{m} \varphi_\mu^m(t) w^\mu(x)\Big|^2\,dx = |\varphi^m(t)|^2,$$

from (18) and (52), we get

$$(L(t,\varphi(t)),\varphi(t))_{\mathbb{R}^m} \geq \alpha C_6 \int\limits_{\Omega_t} |u^m|^2\,dx = \alpha C_6 |\varphi^m(t)|^2 \geq 0.$$

Then estimate (32) holds. Thus the Carathéodory-LaSalle theorem (see Proposition 3) yields the existence of the weak global solution $\varphi \in H^1(0,T;\mathbb{R}^m)$ to problem (49)–(50). Since $\{w^\mu\}_{\mu\in\mathbb{N}} \subset Z_s \subset [H^s(\Omega)]^n$, (48) yields that $u^m \in H^1(0,T;Z_s) \subset [H^1(Q_{0,T})]^n$. Clearly,

$$u^m(0) = u_0^m. \tag{53}$$

Step 2. Multiplying both sides of (49) by $\varphi^m_\mu(t)$, summing over $\mu = \overline{1,m}$, and taking into account (18) and (27), we have

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t} |u^m|^2\,dx + \int\limits_{\Omega_t} \Big[\alpha\,|D(u^m)|^2 + 2^{\frac{2-p}{2}}\beta\,|D(u^m)|^p + \gamma\,|u^m|^{q(x)}\Big]\,dx = \int\limits_{\Omega_t} (f^m,u^m)\,dx. \tag{54}$$

Then Cauchy-Bunyakowski-Schwarz's inequality and (53) yield that

$$\frac{d}{dt}\int\limits_{\Omega_t} |u^m|^2\,dx \leq 2\Big(\int\limits_{\Omega_t} |f^m|^2\,dx\Big)^{\frac{1}{2}}\Big(\int\limits_{\Omega_t} |u^m|^2\,dx\Big)^{\frac{1}{2}}, \qquad \Big(\int\limits_{\Omega_t} |u^m|^2\,dx\Big)^{\frac{1}{2}}\Big|_{t=0} = \|u_0^m\|_{[L^2(\Omega)]^n},$$

and from Proposition 9 we get the following estimate

$$\Big(\int\limits_{\Omega} |u^m(x,t)|^2\,dx\Big)^{\frac{1}{2}} \leq \|u_0^m\|_{[L^2(\Omega)]^n} + \int\limits_0^T \Big(\int\limits_{\Omega} |f^m(x,s)|^2\,dx\Big)^{\frac{1}{2}}\,ds, \quad t\in[0,T].$$

Therefore, taking into account (46) and (51), we obtain

$$\|u^m; L^\infty(0,T;H)\| \leq C_7. \tag{55}$$

Integrating both sides of (54) in $t \in [0,T]$ and using (55), we get

$$\frac{1}{2}\int\limits_{\Omega_T} |u^m|^2\,dx + \int\limits_{Q_{0,T}} \Big[\alpha\,|D(u^m)|^2 + 2^{\frac{2-p}{2}}\beta\,|D(u^m)|^p + \gamma\,|u^m|^{q(x)}\Big]\,dxdt \leq \frac{1}{2}\int\limits_{\Omega} |u_0^m|^2\,dx +$$

$$+\int\limits_0^T \Big(\int\limits_{\Omega_t} |f^m|^2\,dx\Big)^{\frac{1}{2}}\Big(\int\limits_{\Omega_t} |u^m|^2\,dx\Big)^{\frac{1}{2}}\,dt \leq C_7\Big\{1 + \int\limits_0^T \Big(\int\limits_{\Omega_t} |f^m|^2\,dx\Big)^{\frac{1}{2}}\,dt\Big\} \leq C_8.$$

Thus, (see the Korn inequality (43) and Proposition 1)

$$\|u^m; L^\sigma(0,T;V_\sigma)\| \leq C_9, \tag{56}$$

$$\|u^m; [L^{q(x)}(Q_{0,T})]^n\| \leq C_{10}. \tag{57}$$

Here the constants $C_7,\dots,C_{10} > 0$ are independent of m.

Estimates (55)–(57) imply that there exists $\{u^{m_j}\}_{j\in\mathbb{N}} \subset \{u^m\}_{m\in\mathbb{N}}$ such that

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad *\text{-weakly in} \quad L^\infty(0,T;H) \quad \text{and weakly in} \quad U(Q_{0,T}), \tag{58}$$

$$u^{m_j}(T) \underset{j\to\infty}{\longrightarrow} \chi_T \quad \text{weakly in} \quad H. \tag{59}$$

Step 3. Take $m, \ell \in \mathbb{N}$, $m \geq \ell$, $v \in V^\ell$. By (55) we get

$$|(u^m(t), v)_H| \leq \|u^m; L^\infty(0,T;H)\| \cdot |v|_H \leq C_{11}, \quad t \in (0,T), \tag{60}$$

where the constant $C_{11} > 0$ is independent of m, t.

On the other hand, from (18) and the Korn inequality (43) we obtain

$$|\langle \mathcal{J}_2'(u^m(t)), v\rangle_{[H_0^1(\Omega)]^n}| = \Big|\int\limits_{\Omega_t} D(u) \colon D(v)\, dx\Big| \leq C_{12}\|u^m(t)\|_{V_2}\|v\|_{V_2}, \quad t \in (0,T). \tag{61}$$

From (18), (27), the Hölder inequality, and the Korn inequality (43) we have

$$|\langle \mathcal{J}_p'(u^m(t)), v\rangle_{[W_0^{1,p}(\Omega)]^n}| = \Big|\int\limits_{\Omega_t} |D_{II}(u^m)|^{\frac{p-2}{2}} D(u) \colon D(v)\, dx\Big| \leq C_{13}\int\limits_{\Omega_t} |D(u^m)|^{p-1}|D(v)|\, dx \leq$$

$$\leq C_{13}\Big(\int\limits_{\Omega_t} |D(u^m)|^p\, dx\Big)^{\frac{p-1}{p}} \Big(\int\limits_{\Omega_t} |D(v)|^p\, dx\Big)^{\frac{1}{p}} \leq C_{14}\|u^m(t)\|_{V_p}^{p-1}\|v\|_{V_p}, \quad t \in (0,T). \tag{62}$$

Using generalized Hölder inequality (see [20, p. 431]) and (30), we get

$$\Big|\big(G(u^m(t)), v\big)_H\Big| \leq C_{15}\|G(u^m(t)); [L^{q'(x)}(\Omega)]^n\| \cdot \|v; [L^{q(x)}(\Omega)]^n\| \leq$$

$$\leq C_{16}\|u^m(t); [L^{q(x)}(\Omega)]^n\| \cdot \|v; [L^{q(x)}(\Omega)]^n\|, \quad t \in (0,T). \tag{63}$$

Finally note that (see (47))

$$|(f^m(t), v)_H| \leq |f^m(t)|_H |v|_H \leq \widetilde{f}(t)|v|_H, \quad t \in (0,T). \tag{64}$$

Using (61)–(64), from (49), we obtain the estimate

$$\Big|\frac{d}{dt}\,(u^m(t), v)_H\Big| \leq C_{17}\Big(\widetilde{f}(t) + a_m(t)\Big)\|v\|_V, \quad t \in (0,T), \tag{65}$$

where $\widetilde{f} \in L^1(0,T)$, the constant $C_{17} > 0$ is independent of $t, m, \widetilde{f}, v$,

$$a_m(t) = \|u^m(t)\|_{V_2} + \|u^m(t)\|_{V_p}^{p-1} + \|u^m(t); [L^{q(x)}(\Omega)]^n\|, \quad t \in (0,T). \tag{66}$$

Estimates (56) and (57) (see also (29)) yield that the set $\{a_m\}_{m\in\mathbb{N}}$ is bounded in $L^r(0,T)$, where $r = \min\{2,\ \frac{p}{p-1},\ q_0\} > 1$. Therefore, (60), (65), and Theorem 3 with $\mathcal{W} = \mathcal{L} = \mathcal{B} = \mathbb{R}$ imply that

$$(u^{m_j}(\cdot), v)_H \underset{j\to\infty}{\longrightarrow} (u(\cdot), v)_H \qquad \text{in} \quad C([0,T]) \tag{67}$$

for every fixed $v \in V^\ell$, where $\ell \in \mathbb{N}$.

Step 4. Take arbitrary $z \in H$. Let us prove that

$$(u^{m_j}(\cdot), z)_H \underset{j\to\infty}{\longrightarrow} (u(\cdot), z)_H \quad \text{in} \quad C([0,T]). \tag{68}$$

Clearly, (68) yields that $\chi_T = u(T)$ (see (59)) and (26) holds.

There exists $\{v^\ell\}_{\ell\in\mathbb{N}}$ such that $v^\ell \underset{\ell\to\infty}{\longrightarrow} z$ in H and $v^\ell \in V^\ell$ for all $\ell \in \mathbb{N}$. In particular,

$$\forall\, \varepsilon > 0 \quad \exists\, \ell_0 \in \mathbb{N} \quad \forall\, \ell \in \mathbb{N} \quad (\ell \geq \ell_0): \quad |z - v^\ell|_H < \varepsilon.$$

Moreover, (67) yields that $(u(\cdot), v^\ell)_H \in C([0,T])$, $\ell \in \mathbb{N}$. In particular, if $t_0 \in [0,T]$ and $\{t_k\}_{k\in\mathbb{N}} \subset [0,T]$ such that $t_k \underset{k\to\infty}{\longrightarrow} t_0$, then

$$\forall\, \varepsilon > 0 \quad \exists\, k_0 \in \mathbb{N} \quad \forall\, k \in \mathbb{N} \quad (k \geq k_0): \quad |(u(t_k), v^{\ell_0})_H - (u(t_0), v^{\ell_0})_H| < \varepsilon.$$

First let us prove that

$$\text{the function} \quad [0,T] \ni t \mapsto (u(t), z)_H \in \mathbb{R} \quad \text{is continuous.} \tag{69}$$

Indeed, by (58) we have that $(u(\cdot), z)_H \in L^\infty(0,T)$ and so we may define $(u(t), z)_H \in \mathbb{R}$ for every $t \in [0,T]$. Then

$$|(u(t_k), z)_H - (u(t_0), z)_H| = |(u(t_k) - u(t_0), z)_H| = |(u(t_k) - u(t_0), z - v^{\ell_0})_H +$$

$$+ (u(t_k) - u(t_0), v^{\ell_0})_H| \leq |u(t_k) - u(t_0)|_H |z - v^{\ell_0}|_H + |(u(t_k) - u(t_0), v^{\ell_0})_H| \leq 2C_7\varepsilon + \varepsilon,$$

where the constant $C_7 > 0$ is taken from (55). Thus (69) holds.

Further from (67) we get

$$\forall\, \varepsilon > 0 \quad \exists\, j_0 \in \mathbb{N} \quad \forall\, j \in \mathbb{N} \quad (j \geq j_0): \quad \|(u^{m_j} - u, v^{\ell_0})_H; C([0,T])\| < \varepsilon.$$

Therefore,

$$\|(u^{m_j} - u, z)_H; C([0,T])\| = \|(u^{m_j} - u, v^{\ell_0})_H + (u^{m_j} - u, z - v^{\ell_0})_H; C([0,T])\| \leq$$

$$\leq \|(u^{m_j} - u, v^{\ell_0})_H; C([0,T])\| + \|\, |u^{m_j} - u|_H; C([0,T])\| \cdot |z - v^{\ell_0}|_H \leq \varepsilon + 2C_7\varepsilon$$

and (68) holds.

Step 5. Take $\ell, j \in \mathbb{N}$ such that $m_j \geq \ell$. Then, by (49), similarly to (25), we obtain

$$\int\limits_{Q_{0,T}} (-v_t, u^{m_j})\, dxdt - (u_0^{m_j}, v(0))_H + \frac{1}{2}|u_0^{m_j}|_H^2 + \alpha \int\limits_{Q_{0,T}} D(u^{m_j})\colon D(v)\, dxdt +$$

$$+ \int\limits_0^T \Big[\beta\, \mathcal{J}_p(v) + \gamma\, \mathcal{I}_q(v)\Big]\, dt - \int\limits_{Q_{0,T}} (f^{m_j}, v - u^{m_j})\, dxdt \geq \frac{1}{2}|u^{m_j}(T)|_H^2 + \alpha \int\limits_{Q_{0,T}} |D(u^{m_j})|^2\, dxdt +$$

$$+ \beta \int\limits_0^T \mathcal{J}_p(u^{m_j})\, dt + \gamma \int\limits_0^T \mathcal{I}_q(u^{m_j})\, dt \tag{70}$$

for every $v \in C^1([0,T]; V^\ell)$ such that $v(T) = 0$. Using the Fatou lemma and the lower semi-continuous of $\mathcal{J}_p$ and $\mathcal{I}_q$ (see Lemma 2.1 [32]), from (70) we easily get (25). This completes the proof of Theorem 1. □

Remark 1. Similarly to (65), from (49) and (55) we get

$$\Big|\frac{d}{dt}\int\limits_{\Omega}|u^m(t)|^2\,dx\Big|\le C_{18}\Big(\widetilde{f}(t)+b_m(t)\Big),\quad t\in(0,T),$$

where the constant $C_{18}>0$ is independent of m,t,f,

$$b_m(t)=\|u^m(t)\|^2_{V_2}+\|u^m(t)\|^p_{V_p}+\rho_q(u^m(t);\Omega),\quad t\in(0,T),$$

and ρ_q is defined by (13). Since the set $\{b_m\}_{m\in\mathbb{N}}$ is bounded in $L^h(0,T)$ only for $h=1$, we do not have the statement of Theorem 3 and we can not prove that $|u(\cdot)|_H\in C([0,T])$.

Proof of Theorem 2. The solution will be constructed via Faedo-Galerkin's method.
Step 1. Since $f\in L^2(0,T;H)$, we may choose the sequence $\{u^m\}_{m\in\mathbb{N}}$ in the same way as in proof of Theorem 1 but with f instead of f^m in (49) and with the condition $u_0^m\underset{m\to\infty}{\longrightarrow}u_0$ in V instead of (51), i.e. u^m satisfies (48), (53), and

$$(u_t^m(t),w^\mu)_H+\alpha\,\langle\mathcal{J}_2'(u^m(t)),w^\mu\rangle_{[H_0^1(\Omega)]^n}+\beta\,\langle\mathcal{J}_p'(u^m(t)),w^\mu\rangle_{[W_0^{1,p}(\Omega)]^n}+$$
$$+\gamma\,(G(u^m(t)),w^\mu)_H=(f(t),w^\mu)_H,\qquad t\in(0,T),\quad \mu=\overline{1,m},\tag{71}$$

instead of (49). Then (55)–(57) hold. By (57) we have

$$\|G(u^m);[L^{q'(x)}(Q_{0,T})]^n\|\le C_{19},\tag{72}$$

where the constant $C_{19}>0$ is independent of m. Thus, there exists a subsequence $\{u^{m_j}\}_{j\in\mathbb{N}}\subset\{u^m\}_{m\in\mathbb{N}}$ such that

$$u^{m_j}\underset{j\to\infty}{\longrightarrow}u\quad *\text{-weakly in } L^\infty(0,T;H)\ \text{ and weakly in } U(Q_{0,T}),\tag{73}$$

$$G(u^{m_j})\underset{j\to\infty}{\longrightarrow}\chi\quad\text{weakly in}\quad [L^{q'(x)}(Q_{0,T})]^n.\tag{74}$$

Note that the proof of Theorem 1 implies that u is a very weak solution of problem (1)–(5).
Step 2. Multiplying both sides of (71) by $(\varphi_\mu^m(t))'$ and summing over $\mu=\overline{1,m}$, we have

$$|u_t^m(t)|_H^2+\alpha\,\langle\mathcal{J}_2'(u^m(t)),u_t^m(t)\rangle_{[H_0^1(\Omega)]^n}+\beta\,\langle\mathcal{J}_p'(u^m(t)),u_t^m(t)\rangle_{[W_0^{1,p}(\Omega)]^n}+$$
$$+\gamma\,(G(u^m(t)),u_t^m(t))_H=(f(t),u_t^m(t))_H.\tag{75}$$

By [2, p. 1092], we get

$$\langle\mathcal{J}_p'(u^m(t)),u_t^m(t)\rangle_{[W_0^{1,p}(\Omega)]^n}=\frac{d}{dt}\mathcal{J}_p(u^m(t))=\frac{d}{dt}\Big(2^{\frac{2-p}{2}}\|D(u^m(t))\|^p_{L^p(\Omega)}\Big),\tag{76}$$

$$\langle\mathcal{J}_2'(u^m(t)),u_t^m(t)\rangle_{[H_0^1(\Omega)]^n}=\frac{d}{dt}\,\|D(u^m(t))\|^2_{L^2(\Omega)}.\tag{77}$$

Taking into account Theorem 3 [33, p. 73], we prove that

$$(G(u^m(t)),\ u_t^m(t))_H=\frac{d}{dt}\Big(\int\limits_{\Omega}\frac{1}{q(x)}\,|u^m(x,t)|^{q(x)}\,dx\Big).\tag{78}$$

Using (75)–(78) and the estimate $|(f,u_t^m)_H|\le \frac12|f|_H+\frac12|u_t^m|_H$, we obtain

$$\frac12|u_t^m(t)|_H^2+\frac{d}{dt}\Big(\alpha\,\|D(u^m(t))\|^2_{L^2(\Omega)}+2^{\frac{2-p}{2}}\beta\|D(u^m(t))\|^p_{L^p(\Omega)}+$$
$$+\gamma\int\limits_\Omega\frac{1}{q(x)}\,|u^m(x,t)|^{q(x)}\,dx\Big)\le\frac12|f(t)|_H^2. \tag{79}$$

Integrating (79) in $t\in(0,T)$, we have

$$\|u^m;L^\infty(0,T;V)\|\le C_{20}, \tag{80}$$

$$\|u_t^m;L^2(0,T;H)\|\le C_{21}, \tag{81}$$

where the constants $C_{20},C_{21}>0$ are independent of m.

By (80)–(81), we have

$$u^{m_j}\underset{j\to\infty}{\longrightarrow}u\quad *-\text{weakly in}\quad L^\infty(0,T;V),\quad u_t^{m_j}\underset{j\to\infty}{\longrightarrow}u_t\quad\text{weakly in}\quad L^2(0,T;H).$$

Since $V\overset{K}{\subset}H$, the Simon theorem (see Proposition 6) and (80)–(81) imply that

$$u^{m_j}\underset{j\to\infty}{\longrightarrow}u\quad\text{in}\quad C([0,T];H)\quad\text{and in}\quad L^2(0,T;H).$$

Hence (5) holds and $u^{m_j}(x,t)\underset{j\to\infty}{\longrightarrow}u(x,t)$ for a.e. $(x,t)\in Q_{0,T}$. Then $\chi=G(u)$ (see (74)).

Step 5. By definition, put $\mathfrak{M}:=\Big\{\sum_{\mu=1}^m\alpha_\mu^m(t)w^\mu(x)\big|\alpha_\mu^m\in C^1([0,T]),\mu=\overline{1,m},m\in\mathbb{N}\Big\}$. Take $v\in\mathfrak{M}$. Since (22) holds, from (71) we have

$$\int\limits_{Q_{0,T}}\Big[(u_t^{m_j},v-u^{m_j})+\alpha\,D(u^{m_j})\colon D(v)+\gamma\,(G(u^{m_j}),v)-(f,v-u^{m_j})\Big]\,dxdt+\beta\int\limits_0^T\mathcal{J}_p(v(t))\,dt\ge$$
$$\ge\int\limits_{Q_{0,T}}\Big[\alpha\,|D(u^{m_j})|^2+\gamma\,|u^{m_j}|^{q(x)}\Big]\,dxdt+\beta\int\limits_0^T\mathcal{J}_p(u^{m_j}(t))\,dt.$$

Hence, taking lower limits, we get

$$\int\limits_{Q_{0,T}}\Big[(u_t,v-u)+\alpha\,D(u)\colon(D(v)-D(u))+\gamma\,(G(u),v-u)-(f,v-u)\Big]\,dxdt+$$
$$+\int\limits_0^T\beta\,\Big(\mathcal{J}_p(v(t))-\mathcal{J}_p(u(t))\Big)\,dt\ge0 \tag{82}$$

for every $v\in\mathfrak{M}$. Since $U(Q_{0,T})=\overline{\mathfrak{M}}$, (82) holds for every $v\in U(Q_{0,T})$.

Choose $v=u+\lambda\,w$, where $\lambda>0$ and $w\in U(Q_{0,T})$, in (82). Dividing by λ and letting $\lambda\to+0$, we obtain $\langle\mathcal{F},w\rangle_{U(Q_{0,T})}\ge0$, where

$$\mathcal{F}=u_t+\alpha\,\mathcal{J}_2'(u)+\beta\,\mathcal{J}_p'(u)+\gamma\,G(u)-f.$$

We can also take v as before with $\lambda < 0$ and get $\langle \mathcal{F}, w\rangle_{U(Q_{0,T})} \leq 0$. Hence,

$$\langle \mathcal{F}, w\rangle_{U(Q_{0,T})} = 0, \quad w \in U(Q_{0,T}), \tag{83}$$

and (19) holds.
Step 6. Setting $w(x,t) = v(x)\varphi(t)$, $x \in \Omega$, $t \in (0,T)$, from (83) we obtain

$$\int_0^T \langle \mathcal{F}(t), v\rangle_{[D(\Omega)]^n}\, \varphi(t)\, dt = 0, \qquad v \in [D(\Omega)]^n, \quad \varphi \in D(0,T).$$

Using notation (14)–(15), we get

$$\mathcal{F} \in L^2(0,T;[H^{-1}(\Omega)]^n) + L^{p'}(0,T;[W^{-1,p'}(\Omega)]^n) + [L^{q'(x)}(Q_{0,T})]^n \subset W^{0,h}(0,T;[W^{-1,h}(\Omega)]^n).$$

Then the generalized De Rham's theorem (see Proposition 7) yields that there exists a function $\pi \in W^{0,h}(0,T;[W^{0,h}(\Omega)]^n) = L^h(Q_{0,T})$ such that (37)–(38) are satisfied. Thus (3) and (20) hold. This completes the proof of Theorem 2. □

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