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RELATIVE GROWTH OF DIRICHLET SERIES

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Let F and G be entire functions given by Dirichlet series with exponents increasing to $+\infty$. In the term of a generalized order it is introduced a concept of their relative growth and its connection with the growth of F and the growth of G is shown.

1. Introduction. For an entire transcendental function $f(z) = \sum_{n=0}^{\infty} f_n z^n$ let $M_f(r) = \max\{|f(z)| : |z| = r\}$. The classical order of f is defined as follows $\varrho[f] = \inf\{\varrho > 0 : \ln M_f(r) < r^{\varrho}$ for all large $r\}$, i. e. $\varrho[f] = \varlimsup\limits_{r\to+\infty} \frac{\ln\ln M_f(r)}{\ln r}$. Similarly $\lambda[f] = \varliminf\limits_{r\to+\infty} \frac{\ln\ln M_f(r)}{\ln r}$ is the lower order. We say that the function f has a *regular growth* if $\lambda[f] = \varrho[f]$.

Suppose that f and g are entire transcendental functions. Then the functions $M_f(r)$ and $M_g(r)$ are continuous and increasing to $+\infty$ on $[0, +\infty)$. Therefore, there exists the function $M_g^{-1}(x)$ inverse to $M_g(r)$, which is continuous and increasing to $+\infty$ on $(|g(0)|, +\infty)$. The quantity $\varrho_g[f] = \inf\{\varrho > 0 : M_f(r) < M_g(r^{\varrho})$ for all large $r\}$ is called the order of the function f with respect to the function g. It is clear that $\varrho_g[f] = \varlimsup\limits_{r\to+\infty} \frac{\ln M_g^{-1}(M_f(r))}{\ln r}$. Similarly $\lambda_g[f] = \varliminf\limits_{r\to+\infty} \frac{\ln M_g^{-1}(M_f(r))}{\ln r}$ is called the lower order of f with respect to g. If $\lambda_g[f] = \varrho_g[f]$ then we say that f has a regular growth with respect to g. We remark that if $g(z) = e^z$ then $\varrho_g[f] = \varrho[f]$ and $\lambda_g[f] = \lambda[f]$.

Ch. Roy [1] tried to prove the following two theorems.

Theorem A. *If $\varrho[f] < +\infty$ and $\varrho[g] > 0$ then $\varrho_g[f] \geq \varrho[f]/\varrho[g]$, and if, moreover, g has the regular growth then $\varrho_g[f] = \varrho[f]/\varrho[g]$.*

Theorem B. *If the function f has the order $\varrho[f] < +\infty$ and the function $g(z) = \sum\limits_{n=0}^{\infty} g_n z^n$ has the order $\varrho[g] > 0$ and the regular growth then*

$$\varrho_g[f] = \varlimsup_{n\to\infty} \frac{\ln |g_n|}{\ln |f_n|}. \tag{1}$$

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In the proof of Theorem A in [1] it is used the estimate $\ln M_g(r) < r^{\varrho[g]+\varepsilon}$ for each $\varepsilon > 0$ and all $r \geq r_0(\varepsilon)$. To obtain this estimate it is necessary that $\varrho[g] < +\infty$, but not $\varrho[g] > 0$. Therefore, the conclusion of Theorem A is uncertain. In fact, in Theorem A it is necessary to replace the assumptions $\varrho[f] < +\infty$ and $\varrho[g] > 0$ by the following one: neither $\varrho[f] = \varrho[g] = 0$ nor $\varrho[f] = \varrho[g] = +\infty$ holds.

The conclusion of Theorem B is false. Indeed, the functions $f(z) = \sin z$ and $g(z) = \cos z$ have order 1 and the regular growth and by (1) $\varrho_g(f) = 1$. But $\lim\limits_{k\to\infty} \frac{\ln |g_{2k}|}{\ln |f_{2k}|} = +\infty$ and $\lim\limits_{k\to\infty} \frac{\ln |g_{2k+1}|}{\ln |f_{2k+1}|} = 0$. The conclusion of Theorem B is correct under the additional condition $|g_n/g_{n+1}| \nearrow +\infty$ as $n_0 \leq n \to \infty$.

The aim of our note is to prove analogues of Theorems A and B for entire Dirichlet series of finite generalized orders.

2. Main results. Let $\Lambda = (\lambda_n)$ be a sequence of nonnegative numbers increasing to $+\infty$ and $S(\Lambda)$ the class of entire Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f_n \exp\{s\lambda_n\}, \quad s = \sigma + it. \tag{2}$$

For $F \in S(\Lambda)$ we denote $M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$.

Let L be the class of continuous increasing functions α such that $\alpha(x) \geq 0$ for $x \geq x_0$, $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$, and on $[x_0, +\infty)$ the function α increases to $+\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha(x(1 + o(1))) = (1 + o(1))\alpha(x)$ as $x \to +\infty$; further, $\alpha \in L_{si}$ if $\alpha \in L$ and for any $c > 0$ $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. It is easy to see that $L_{si} \subset L^0$. Functions from L_{si} are called slowly increasing.

For $\alpha \in L$ and $\beta \in L$

$$\varrho_{\alpha,\beta}[F] = \varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)}, \quad \lambda_{\alpha,\beta}[F] = \varliminf_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)}.$$

are called the *generalized order* and the *lower generalized order* of G, respective. We say that the function F has generalized regular growth if $\varrho_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[F]$. We need the following lemma ([2]).

Lemma 1. *Let $\alpha \in L$ and $\beta \in L$ be continuously differentiable functions, $0 < p < +\infty$ and one of next conditions holds:*

a) $\alpha \in L^0$, $\beta(\ln x) \in L^0$, $\frac{d\beta^{-1}(c\alpha(x))}{d\ln x} \to \frac{1}{p}$ $(x \to +\infty)$ for each $c \in (0, +\infty)$ and $\ln n = o(\lambda_n)$ $(n \to \infty)$;

b) $\alpha \in L_{si}$, $\beta \in L^0$, $\frac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ $(x \to +\infty)$ and $\ln n = o(\lambda_n\beta^{-1}(c\alpha(\lambda_n)))$ $(n \to \infty)$ for each $c \in (0, +\infty)$.

Then

$$\varrho_{\alpha,\beta}[F] = \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n/p)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)},$$

and if, moreover, $\alpha(\lambda_{n+1}/p) = (1 + o(1))\alpha(\lambda_n/p)$ and $\frac{\ln |a_n| - \ln |a_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$\lambda_{\alpha,\beta}[F] = \varliminf_{n\to\infty} \frac{\alpha(\lambda_n/p)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}.$$

We remark that the generalized order can be define as follows $\varrho_{\alpha,\beta}[F] = \inf\{\varrho > 0 : \ln M(\sigma) \le \alpha^{-1}(\varrho\beta(\sigma))$ for all $\sigma \ge \sigma_0(\varrho)\}$. Therefore, by analogy we define the generalized order $\varrho_{\alpha,\beta}[F]_G$ of the function F with respect to a function G, given by an entire Dirichlet series

$$G(s) = \sum_{n=1}^{\infty} g_n \exp\{s\lambda_n\}, \tag{3}$$

as follows $\varrho_{\alpha,\beta}[F]_G = \inf\{\varrho > 0 : \alpha(\ln M_F(\sigma)) \le \alpha(\ln M_G(\beta^{-1}(\varrho\beta(\sigma)))$ for all $\sigma \ge \sigma_0(\varrho)\}$. Since the function $M_G(\sigma)$ is continuous and increasing to $+\infty$ on $(-\infty, +\infty)$, there exists the function $M_G^{-1}(x)$ inverse to $M_G(\sigma)$ which increases to $+\infty$ on $[x_0, +\infty)$. Therefore,

$$\varrho_{\alpha,\beta}[F]_G = \varlimsup_{\sigma\to+\infty} \frac{\beta(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}.$$

Similarly we define the lower generalized order $\lambda_{\alpha,\beta}[F]_G$ of F with respect to G by the formula

$$\lambda_{\alpha,\beta}[F]_G = \varliminf_{\sigma\to+\infty} \frac{\beta(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)},$$

and say that F has generalized regular growth with respect to G if $\varrho_{\alpha,\beta}[F]_G = \lambda_{\alpha,\beta}[F]_G$.

The following theorem establishes connections between $\varrho_{\alpha,\beta}[F]$, $\varrho_{\alpha,\beta}[G]$ and $\varrho_{\alpha,\beta}[F]_G$ and between $\lambda_{\alpha,\beta}[F]$, $\lambda_{\alpha,\beta}[G]$ and $\lambda_{\alpha,\beta}[F]_G$.

Theorem 1. *Let $\alpha \in L$ and $\beta \in L$. Except for cases, when $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = 0$ or $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = +\infty$, the inequality $\varrho_{\alpha,\beta}[F]_G \ge \varrho_{\alpha,\beta}[F]/\varrho_{\alpha,\beta}[G]$ is true and subject to the condition of the generalized regular growth of G this inequality converts into an equality.*

Except for cases, when $\lambda_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[G] = 0$ or $\lambda_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[G] = +\infty$, the inequality $\lambda_{\alpha,\beta}[F]_G \le \lambda_{\alpha,\beta}[F]/\lambda_{\alpha,\beta}[G]$ is true and subject to the condition of the generalized regular growth of G this inequality converts into an equality.

Proof. Indeed,

$$\varrho_{\alpha,\beta}[F]_G = \varlimsup_{x\to+\infty} \frac{\beta(M_G^{-1}(x)}{\beta(M_F^{-1}(x))} = \varlimsup_{x\to+\infty} \frac{\alpha(\ln x)}{\beta(M_F^{-1}(x))}\frac{\beta(M_G^{-1}(x))}{\alpha(\ln x)} \ge$$

$$\ge \varlimsup_{x\to+\infty} \frac{\alpha(\ln x)}{\beta(M_F^{-1}(x))} \varliminf_{x\to+\infty} \frac{\beta(M_G^{-1}(x))}{\alpha(\ln x)} = \varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)} \varliminf_{\sigma\to+\infty} \frac{\beta(\sigma)}{\alpha(\ln M_G(\sigma))} = \frac{\varrho_{\alpha,\beta}[F]}{\varrho_{\alpha,\beta}[G]}$$

and if there exists $\lim\limits_{\sigma\to+\infty} \alpha(\ln M_G(\sigma))/\beta(\sigma) = \lambda_{\alpha,\beta}[G] = \varrho_{\alpha,\beta}[G]$ then

$$\varrho_{\alpha,\beta}[F]_G = \varlimsup_{x\to+\infty} \frac{\alpha(\ln x)}{\beta(M_F^{-1}(x))}\frac{\beta(M_G^{-1}(x))}{\alpha(\ln x)} = \varlimsup_{x\to+\infty} \frac{\alpha(\ln x)}{\beta(M_F^{-1}(x))} \lim_{x\to+\infty} \frac{\beta(M_G^{-1}(x))}{\alpha(\ln x)} = \frac{\varrho_{\alpha,\beta}[F]}{\varrho_{\alpha,\beta}[G]}.$$

The first part of Theorem 1 is proved.

The proof of second part is similar. Indeed,

$$\lambda_{\alpha,\beta}[F]_G = \varliminf_{x\to+\infty} \frac{\alpha(\ln x)}{\beta(M_F^{-1}(x))}\frac{\beta(M_G^{-1}(x))}{\alpha(\ln x)} \le \varliminf_{x\to+\infty} \frac{\alpha(\ln x)}{\beta(M_F^{-1}(x))} \varlimsup_{x\to+\infty} \frac{\beta(M_G^{-1}(x))}{\alpha(\ln x)} = \frac{\lambda_{\alpha,\beta}[F]}{\lambda_{\alpha,\beta}[G]}$$

and if there exists $\lim\limits_{\sigma\to+\infty} \alpha(\ln M_F(\sigma))/\beta(\sigma) = \lambda_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[F]$ then

$$\lambda_{\alpha,\beta}[F]_G = \varlimsup_{x\to+\infty} \frac{\alpha(\ln x)}{\beta(M_F^{-1}(x))} \lim_{x\to+\infty} \frac{\beta(M_G^{-1}(x))}{\alpha(\ln x)} = \frac{\lambda_{\alpha,\beta}[F]}{\lambda_{\alpha,\beta}[G]}.$$

□

We remark that except the cases, when $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = 0$ or $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = +\infty$, the inequality $\varrho_{\alpha,\beta}[G]_F \ge \varrho_{\alpha,\beta}[G]/\varrho_{\alpha,\beta}[F]$ is also true. Therefore, $\varrho_{\alpha,\beta}[F]_G \cdot \varrho_{\alpha,\beta}[G]_F \ge 1$, and if the functions F and G have generalized regular growth then $\varrho_{\alpha,\beta}[F]_G \cdot \varrho_{\alpha,\beta}[G]_F = 1$. A similar conclusion can be done relatively $\lambda_{\alpha,\beta}[F]_G$ and $\lambda_{\alpha,\beta}[G]_F$.

Remark also that if we define a generalized order $\varrho^*_{\alpha,\beta}[F]_G$ of the function F with respect to G as follows $\varrho^*_{\alpha,\beta}[F]_G = \inf\{\varrho > 0 : \alpha(\ln M_F(\sigma)) \le \varrho\alpha(\ln M_G(\sigma))$ for all $\sigma \ge \sigma_0(\varrho)\}$, i. e.

$$\varrho^*_{\alpha,\beta}[F]_G = \varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\alpha(\ln M_G(\sigma))},$$

then, repeating the proof of Theorem 1, we obtain the inequality $\varrho^*_{\alpha,\beta}[F]_G \ge \varrho_{\alpha,\beta}[F]/\varrho_{\alpha,\beta}[G]$ and subject to the condition of the generalized regular growth of G we obtain also the equality $\varrho^*_{\alpha,\beta}[F]_G = \varrho_{\alpha,\beta}[F]/\varrho_{\alpha,\beta}[G]$. Hence it follows that if G has generalized regular growth then $\varrho^*_{\alpha,\beta}[F]_G = \varrho_{\alpha,\beta}[F]_G$. This equality can be also obtain using the definitions of $\varrho^*_{\alpha,\beta}[F]_G$ and $\varrho_{\alpha,\beta}[F]_G$.

Using Lemma 1 now we prove the following theorem.

Theorem 2. *Let entire Dirichlet series* (2) *and* (3) *have generalized orders* $\varrho_{\alpha,\beta}[F]$, $\varrho_{\alpha,\beta}[G]$ *and lower orders* $\lambda_{\alpha,\beta}[F]$, $\lambda_{\alpha,\beta}[G]$. *Suppose that* $0 < p < +\infty$, *the functions* $\alpha \in L$ *and* $\beta \in L$ *are continuously differentiable and satisfy conditions either a) or b) of Lemma 1 and* $\alpha(\lambda_{n+1}/p) = (1 + o(1))\alpha(\lambda_n/p)$ *as* $n \to \infty$.

If the function G has generalized regular growth and $\frac{\ln|g_n| - \ln|g_{n+1}|}{\lambda_{n+1}-\lambda_n} \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *then*

$$\varrho_{\alpha,\beta}[F]_G = \varkappa^* =: \varlimsup_{n\to\infty} \frac{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}$$

except for the cases, when $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = 0$ *or* $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = +\infty$.

Moreover, if F has generalized regular growth and $\frac{\ln|f_n| - \ln|f_{n+1}|}{\lambda_{n+1}-\lambda_n} \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *then*

$$\lambda_{\alpha,\beta}[F]_G = \varkappa_* =: \varliminf_{n\to\infty} \frac{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}$$

except for the cases, when $\lambda_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[G] = 0$ *or* $\lambda_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[G] = +\infty$.

Proof. By Theorem 1 and Lemma 1 we have

$$\varrho_{\alpha,\beta}[F]_G = \frac{\varrho_{\alpha,\beta}[F]}{\varrho_{\alpha,\beta}[G]} = \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n/p)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)} \varliminf_{n\to\infty} \frac{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\alpha(\lambda_n/p)} \le$$

$$\le \varlimsup_{n\to\infty} \frac{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)} =: \varkappa^*. \tag{4}$$

On the other hand, let $\varkappa^* > 0$. Then for every $\varepsilon \in (0, \varkappa^*)$ there exists an increasing to ∞ sequence (n_k) of integers such that

$$\beta\left(\frac{1}{p} + \frac{1}{\lambda_{n_k}}\ln\frac{1}{|g_{n_k}|}\right) > (\varkappa^* - \varepsilon)\beta\left(\frac{1}{p} + \frac{1}{\lambda_{n_k}}\ln\frac{1}{|f_{n_k}|}\right).$$

and, thus,

$$\varlimsup_{n\to\infty}\frac{\alpha(\lambda_n/p)}{\beta\left(\frac{1}{p}+\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}>(\varkappa^*-\varepsilon)\varlimsup_{n\to\infty}\frac{\alpha(\lambda_n/p)}{\beta\left(\frac{1}{p}+\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}.$$

Since $\varrho_{\alpha,\beta}[G]=\lambda_{\alpha,\beta}[G]$, by Lemma 1 we obtain the inequality $\varrho_{\alpha,\beta}[F]>(\varkappa^*-\varepsilon)\varrho_{\alpha,\beta}[G]$, that is in view of arbitrariness of ε the inequality $\varrho_{\alpha,\beta}[F]=\varrho_{\alpha,\beta}[F]/\varrho_{\alpha,\beta}[F]\geq\varkappa^*$ is true. For $\varkappa^*=0$ the last inequality is obvious. In view of (4) the first part of Theorem 2 is proved. For the proof of the second part we remark that since the function G has generalized regular growth, by Theorem 1 and Lemma 1

$$\lambda_{\alpha,\beta}[F]_G=\frac{\lambda_{\alpha,\beta}[F]}{\lambda_{\alpha,\beta}[G]}=\varliminf_{n\to\infty}\frac{\alpha(\lambda_n/p)}{\beta\left(\frac{1}{p}+\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}\varlimsup_{n\to\infty}\frac{\beta\left(\frac{1}{p}+\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\alpha(\lambda_n/p)}\geq$$

$$\geq\varliminf_{n\to\infty}\frac{\beta\left(\frac{1}{p}+\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\beta\left(\frac{1}{p}+\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}=:\varkappa_*. \tag{5}$$

On the other hand, if $\varkappa_*<+\infty$ then for every $\varepsilon\in(0,\varkappa^*)$ there exists an increasing to ∞ sequence (n_k) of integers such that

$$\beta\left(\frac{1}{p}+\frac{1}{\lambda_{n_k}}\ln\frac{1}{|g_{n_k}|}\right)<(\varkappa_*+\varepsilon)\beta\left(\frac{1}{p}+\frac{1}{\lambda_{n_k}}\ln\frac{1}{|f_{n_k}|}\right).$$

and, thus,

$$\varliminf_{n\to\infty}\frac{\alpha(\lambda_n/p)}{\beta\left(\frac{1}{p}+\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}<(\varkappa_*+\varepsilon)\varliminf_{n\to\infty}\frac{\alpha(\lambda_n/p)}{\beta\left(\frac{1}{p}+\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)},$$

whence, as above, in view of the generalized regular growth of F we obtain the inequality $\lambda_{\alpha,\beta}[F]=\lambda_{\alpha,\beta}[F]/\lambda_{\alpha,\beta}[F]\leq\varkappa_*$. The last inequality is trivial if $\varkappa^*=+\infty$. By virtue of (5) the proof of Theorem 2 is completed. □

3. Corollaries and addition. For $\beta(x)\equiv x$ $(x\geq x_0)$ and $\alpha\in L_{si}$ by $\varrho_\alpha[F]$, $\lambda_\alpha[F]$, $\varrho_\alpha[F]_G$ and $\lambda_\alpha[F]_G$ we denote $\varrho_{\alpha,\beta}[F]$, $\lambda_{\alpha,\beta}[F]$, $\varrho_{\alpha,\beta}[F]_G$ and $\lambda_{\alpha,\beta}[F]_G$, respective. In particular, if $\alpha(x)\equiv\ln x$ for $x\geq x_0$ then we obtain definitions of R-orders and lower R-orders.

Corollary 1. *Let $\alpha\in L_{si}$, $\frac{d\alpha(x)}{d\ln x}=O(1)$ as $x\to+\infty$, $\alpha(\lambda_{n+1})=(1+o(1))\alpha(\lambda_n)$ and $\ln n=o(\lambda_n\alpha(\lambda_n))$ as $n\to\infty$. If the function G has generalized regular growth and $\frac{\ln|g_n|-\ln|g_{n+1}|}{\lambda_{n+1}-\lambda_n}\nearrow+\infty$ as $n_0\leq n\to\infty$ then*

$$\varrho_\alpha[F]_G=\varlimsup_{n\to\infty}\frac{\ln(1/|g_n|)}{\ln(1/|f_n|)} \tag{6}$$

except for the cases, when $\varrho_\alpha[F]=\varrho_\alpha[G]=0$ or $\varrho_\alpha[F]=\varrho_\alpha[G]=+\infty$.

Moreover, if F has generalized regular growth and $\frac{\ln|f_n|-\ln|f_{n+1}|}{\lambda_{n+1}-\lambda_n}\nearrow+\infty$ as $n_0\leq n\to\infty$ then

$$\lambda_\alpha[F]_G=\varliminf_{n\to\infty}\frac{\ln(1/|g_n|)}{\ln(1/|f_n|)} \tag{7}$$

except the cases, when $\lambda_\alpha[F]=\lambda_\alpha[G]=0$ or $\lambda_\alpha[F]=\lambda_\alpha[G]=+\infty$.

In this corollary the function α is absent on the right-hand side of formulas (6) and (7). A question arises: when is it possible to remove this function from the left-hand side of these formulas.

We put $\Phi(\sigma) = \ln M_G(\sigma)$. Then the function Φ is convex and continuously differentiable except the countable set of points, in which there exist one-sided derivatives and besides the left-sided derivative is less than the right-sided derivative. We remark that if $\frac{d\ln\Phi(\sigma)}{d\ln\sigma} \to +\infty$ as $\sigma \to +\infty$ and Φ^{-1} is the inverse function to Φ then $\frac{d\ln\Phi^{-1}(x)}{d\ln x} \to 0$ as $x \to +\infty$, that is Φ^{-1} is slowly increasing. We remark also that $\sigma = o(\ln M_G(\sigma))$ as $\sigma \to +\infty$, i. e. $\Phi^{-1}(x) = o(x)$ as $x \to +\infty$. Finally, $\frac{\Phi^{-1}(\ln M_G(\sigma))}{\sigma} = 1$. Therefore, if we choose the function $\alpha \in L$ such that $\alpha(x) = \Phi^{-1}(x)$ then we get the following corollary.

Corollary 2. *Suppose that Dirichlet series* (2) *and* (3) *are entire. Let* $\Phi(\sigma) = \ln M_G(\sigma)$, $\frac{d\ln\Phi(\sigma)}{d\ln\sigma} \to +\infty$ *as* $\sigma \to +\infty$, $\Phi^{-1}(\lambda_{n+1}) = (1+o(1))\Phi^{-1}(\lambda_n)$ *and* $\ln n = o(\lambda_n\Phi^{-1}(\lambda_n))$ *as* $n \to \infty$. *If* $\frac{\ln|g_n|-\ln|g_{n+1}|}{\lambda_{n+1}-\lambda_n} \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *then*

$$\varlimsup_{\sigma\to+\infty} \frac{M_G^{-1}(M_F(\sigma))}{\sigma} = \varlimsup_{n\to\infty} \frac{\ln(1/|g_n|)}{\ln(1/|f_n|)}.$$

Moreover, if F *has generalized regular growth and* $\frac{\ln|f_n|-\ln|f_{n+1}|}{\lambda_{n+1}-\lambda_n} \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *then*

$$\varliminf_{\sigma\to+\infty} \frac{M_G^{-1}(M_F(\sigma))}{\sigma} = \varliminf_{n\to\infty} \frac{\ln(1/|g_n|)}{\ln(1/|f_n|)}.$$

Now we suppose that F_j $(1 \le j \le m)$ are entire functions given by Dirichlet series

$$F_j(s) = \sum_{n=1}^{\infty} f_{n,j}\exp\{s\lambda_n\}, \tag{8}$$

In [3] the following theorem is proved.

Theorem C. *Let* α *and* β *be continuously differentiable functions, the condition b) of Lemma 1 hold, and all functions* (8) *have generalized regular growth. Suppose that* $\alpha(\lambda_{n+1}) = (1+o(1))\alpha(\lambda_n)$ *and* $\frac{\ln|f_{n,j}|-\ln|f_{n+1,j}|}{\lambda_{n+1}-\lambda_n} \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *for each* j. *If*

$$\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right) = (1+o(1))\prod_{j=1}^{m}\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_{n,j}|}\right)^{\omega_j}, \quad n\to\infty, \tag{9}$$

where $\omega_j > 0$ *and* $\sum_{j=1}^m \omega_j = 1$, *then function* (2) *has generalized regular growth and* $\varrho_{\alpha,\beta}[F] = \prod_{j=1}^m (\varrho_{\alpha,\beta}[F_j])^{\omega_j}$.

Here we prove the following analogue of Theorem C.

Theorem 3. *Let* α *and* β *be continuously differentiable functions, the condition b) of Lemma 1 hold, and all functions* (8) *have generalized regular growth with respect to* G. *Suppose that* $\alpha(\lambda_{n+1}) = (1+o(1))\alpha(\lambda_n)$ *and* $\frac{\ln|a_{n,j}|-\ln|a_{n+1,j}|}{\lambda_{n+1}-\lambda_n} \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *for each* j *and condition* (9) *holds.*

If the function G *has generalized regular growth and* $\frac{\ln|g_n|-\ln|g_{n+1}|}{\lambda_{n+1}-\lambda_n} \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *then* $\varrho_{\alpha,\beta}[F]_G = \prod_{j=1}^m (\varrho_{\alpha,\beta}[F_j]_G)^{\omega_j}$ *except for cases, when* $\varrho_{\alpha,\beta}[F_j] = \varrho_{\alpha,\beta}[G] = 0$ *or* $\varrho_{\alpha,\beta}[F_j] = \varrho_{\alpha,\beta}[G] = +\infty$ *for some* j.

Proof. Since functions (8) have generalized regular growth with respect to G, by Theorem 1

$$\varrho_{\alpha,\beta}[F_j]_G = \lambda_{\alpha,\beta}[F_j] = \lim_{n\to\infty} \frac{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|f_{n,j}|}\right)},$$

i. e. in view of (9) there exists

$$\lim_{n\to\infty} \frac{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)} = \lim_{n\to\infty} \frac{\prod\limits_{j=1}^{m}\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_{n,j}|}\right)^{\omega_j}}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)} =$$

$$= \lim_{n\to\infty}\prod_{j=1}^{m}\left(\frac{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_{n,j}|}\right)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}\right)^{\omega_j} = \prod_{j=1}^{m}\left(\lim_{n\to\infty}\frac{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_{n,j}|}\right)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}\right)^{\omega_j} = \prod_{j=1}^{m}\left(\frac{1}{\varrho_{\alpha,\beta}[F_j]_G}\right)^{\omega_j},$$

whence

$$\lim_{n\to\infty} \frac{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)} = \prod_{j=1}^{m}(\varrho_{\alpha,\beta}[F_j]_G)^{\omega_j}. \tag{10}$$

In view of (9)

$$\lambda_{\alpha,\beta}[F]_G \geq \underline{\lim}_{n\to\infty} \frac{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}.$$

Therefore, from (10) we obtain

$$\prod_{j=1}^{m}(\varrho_{\alpha,\beta}[F_j]_G)^{\omega_j} = \lambda_{\alpha,\beta}[F]_G \leq \varrho_{\alpha,\beta}[F]_G = \prod_{j=1}^{m}(\varrho_{\alpha,\beta}[F_j]_G)^{\omega_j},$$

i. e. the function F has generalized regular growth with respect to G and $\varrho_{\alpha,\beta}[F]_G = \prod_{j=1}^{m}(\varrho_{\alpha,\beta}[F_j]_G)^{\omega_j}$. □

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