УДК 517.5

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ON BOUNDARY BEHAVIOR OF MAPPINGS WITH TWO NORMALIZED CONDITIONS

E. A. Sevost'yanov, S. A. Skvortsov, N. S. Ilkevych. *On boundary behavior of mappings with two normalized conditions*, Mat. Stud. **49** (2018), 150–157.

The paper is devoted to a study of mappings with finite distortion that have been recently actively investigated last time. We study the boundary behavior of mappings between two fixed domains in metric spaces, which satisfy some moduli estimates. We have proved that families of corresponding inverse mappings with two normalized conditions and integrable majorant are equicontinuous whenever the domain of the mappings has a weakly flat boundary.

1. Introduction. In our recent paper [1], we have obtained equicontinuity of classes of mappings between two domains of metric spaces (X, d) and (X', d'), for which inverse of the mappings satisfy some modulus inequality. The equicontinuity of those classes in $\overline{D'}$, consisting of mappings $g = f^{-1}$ of $D' \subset X'$ onto $D \subset X$, is established under two main conditions, one of which is

$$\operatorname{diam} f(A) := \sup_{x,y\in A} d(x, y) \geqslant \delta > 0\,,$$

where $A \subset D$ is a fixed continuum, and $\delta > 0$ does not depend on f. The second condition is a requirement that $\partial D'$ does not contain any non-degenerate continuum. Remark that, this condition seems us too strong, and we will try to replace it by another assumptions. The main goal of the present paper is to prove some results of [1] under weaker conditions on $\partial D'$. Results given below are new even in Euclidean case, however, they due to general metric spaces. For notions and definitions used below we refer reader to [1], cf. [2] and [3].

Recall, for a given continuous path $\gamma\colon [a, b] \to X$ in a metric space (X, d), that its length is the supremum of the sums

$$\sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1}))$$

over all partitions $a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_k = b$ of the interval $[a, b]$. The path γ is called *rectifiable* if its length is finite. Given a family of paths Γ in X, a Borel function $\rho\colon X \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int_{\gamma} \rho\, ds \;\geqslant\; 1$$

2010 *Mathematics Subject Classification:* 30C65, 30L10, 31C12.
Keywords: metric spaces; quasiconformal mappings; mappings with bounded and finite distortion; equicontinuity; moduli of families of paths.
doi:10.15330/ms.49.2.150-157

for all (locally rectifiable) $\gamma \in \Gamma$. In what follows, $\operatorname{adm}\Gamma$ is the set of all admissible functions for Γ. Everywhere further, for any sets E, F, and G in X, we denote by $\Gamma(E,F,G)$ the family of all continuous paths $\gamma\colon [0,1] \to X$ such that $\gamma(0) \in E$, $\gamma(1) \in F$, and $\gamma(t) \in G$ for all $t \in (0,1)$. Everywhere further (X,d,μ) and (X',d',μ') are metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly. We will assume that μ is a Borel measure such that $0 < \mu(B) < \infty$ for all balls B in X. Given $p \geqslant 1$, the p-modulus of the family Γ is the number

$$M_p(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_X \rho^p(x)\, d\mu(x)\,.$$

Should $\operatorname{adm}\Gamma$ be empty, we set $M_p(\Gamma) = \infty$. Let G and G' be domains with finite Hausdorff dimensions α and $\alpha' \geqslant 1$ in spaces (X,d,μ) and (X',d',μ'), and let $Q\colon G \to [0,\infty]$ be a measurable function. Given $x_0 \in \partial G$, denote $S_i := S(x_0, r_i) = \{x \in G\colon d(x,x_0) = r_i\}$, $i = 1,2$, where $0 < r_1 < r_2 < \infty$. We say that a mapping $f\colon G \to G'$ is a *ring Q-mapping at a point $x_0 \in \overline{G}$*, if the inequality

$$M_{\alpha'}(f(\Gamma(S_1,S_2,A))) \leqslant \int\limits_{A \cap G} Q(x)\eta^{\alpha}(d(x,x_0))\, d\mu(x) \tag{1}$$

holds for any ring

$$A = A(x_0,r_1,r_2) = \{x \in X\colon r_1 < d(x,x_0) < r_2\}, \quad 0 < r_1 < r_2 < \infty\,, \tag{2}$$

and any measurable function $\eta\colon (r_1,r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)dr \geqslant 1 \tag{3}$$

holds. We say that f is a ring Q-homeomorphism in $E \subset \overline{D}$, if and only if f is a ring Q-homeomorphism for every $x_0 \in E$.

We say that the boundary of D is *weakly flat* at a point $x_0 \in \partial D$ if for every number $P > 0$ and every neighborhood U of the point x_0 there is a neighborhood $V \subset U$ such that $M_\alpha(\Gamma(E,F,D)) \geqslant P$ for all continua E and F in D intersecting ∂U and ∂V. We say that the boundary ∂D is weakly flat if the corresponding property holds at every point of the boundary.

In what follows,

$$|\gamma| := \{x \in X\colon \exists\, t \in [a,b]\colon \gamma(t) = x\}$$

is a *locus* of a path $\gamma\colon [a,b] \to X$. Two paths $C_1\colon [a,b] \to X$ and $C_2\colon [a,b] \to X$ are called *disjoint* if and only if $|C_1| \cap |C_2| = \varnothing$. In what follows, we consider the following condition **A** :

A $\Leftrightarrow$ *any pair of points $a \in D, b \in \overline{D}$, and $c \in D, d \in \overline{D}$ can be joined by disjoint paths C_1 and C_2 in D.*

We have proved that domains in $\mathbb{R}^n$, $n \geqslant 2$, which are locally connected on the boundary, satisfy the condition **A**, see [1, Proposition].

Proposition 1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, which is locally connected on the boundary. Then any two pairs of points $a \in D$, $b \in \overline{D}$, and $c \in D, d \in \overline{D}$ can by joined by disjoint paths $\gamma_1\colon [0,1] \to \overline{D}$ and $\gamma_2\colon [0,1] \to \overline{D}$, such that $\gamma_i(t) \in D$ for all $t \in (0,1)$, $i = 1,2$, $\gamma_1(0) = a$, $\gamma_1(1) = b$, $\gamma_2(0) = c$, $\gamma_2(1) = d$.*

A domain D is said to be *locally (path) connected* at $x_0 \in \overline{D}$, if for every neighborhood U of x_0 there exists a neighborhood $V \subset U$ of x_0 such that $V \cap D$ is (path) connected. A domain D is *locally (path) connected* in $E \subset \overline{D}$, if the corresponding property holds for every $x_0 \in E$.

Given domains $D \subset X$, $D' \subset X'$, $w_1, w_2 \in D$, $w_1', w_2' \in D'$, and a measurable function $Q\colon X \to [0,\infty]$, $Q(x) \equiv 0$ for $x \notin D$, denote by $\mathfrak{R}_Q^{w_1,w_2,w_1',w_2'}(D,D')$ a family of all homeomorphisms g of D' onto D such that $f = g^{-1}$, $f\colon D \to D'$, is a ring Q-homeomorphism in $\overline{D}$ with $f(w_1) = w_1'$, $f(w_2) = w_2'$. The following result holds.

Theorem 1. *Let $D \subset X$ and $D' \subset X'$ be domains with Hausdorff dimensions $\alpha \geqslant 2$ and $\alpha' \geqslant 2$, correspondingly. Suppose that:*

1) D is locally path connected on $\overline{D}$;

2) $\overline{D}$ and $\overline{D'}$ are compacts in X and X', correspondingly;

3) D' has a weakly flat boundary;

4) the condition $\mathbf{A}$ holds;

5) $Q \in L^1(D)$.

Now, every $g \in \mathfrak{R}_Q^{w_1,w_2,w_1',w_2'}(D,D')$ has a continuous extension on $\partial D'$ such that $g(\overline{D'}) = \overline{D}$, and the family $\mathfrak{R}_Q^{w_1,w_2,w_1',w_2'}(\overline{D},\overline{D'})$, consisting of all extended mappings $\overline{g}\colon \overline{D'} \to \overline{D}$, is equicontinuous on $\partial D'$.

Given domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, $w_1, w_2 \in D$, $w_1', w_2' \in D'$, and a Lebesgue measurable function $Q\colon \mathbb{R}^n \to [0,\infty]$, $Q(x) \equiv 0$ for $x \notin D$, denote by $\mathfrak{E}_Q^{w_1,w_2,w_1',w_2'}(D,D')$ a family of all homeomorphisms $g\colon D' \to D$ of D' onto D such that $f = g^{-1}$, $f\colon D \to D'$, is a ring Q-homeomorphism in $\overline{D}$ with $f(w_1) = w_1'$, $f(w_2) = w_2'$. Since the condition $\mathbf{A}$ holds in $\mathbb{R}^n$ for corresponding domains, see Proposition 1, we obtain the following consequence from Theorem 1.

Corollary 1. *Let $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, be domains with compact closure in $\mathbb{R}^n$. Assume that D is locally path connected on $\overline{D}$, D' has a weakly flat boundary and $Q \in L^1(D)$.*

Now, every $g \in \mathfrak{E}_Q^{w_1,w_2,w_1',w_2'}(D,D')$ has a continuous extension on $\partial D'$ such that $g(\overline{D'}) = \overline{D}$, and the family $\mathfrak{E}_Q^{w_1,w_2,w_1',w_2'}(\overline{D},\overline{D'})$, consisting of all extended mappings $\overline{g}\colon \overline{D'} \to \overline{D}$, is equicontinuous on $\partial D'$.

2. Proof of Theorem 1.

Proof of Theorem 1. Let $g \in \mathfrak{R}_Q^{w_1,w_2,w_1',w_2'}(D,D')$. Since D' has a weakly flat boundary, g extends to a continuous mapping $\overline{g}\colon \overline{D'} \to \overline{D}$ (see [4, Theorem 3], cf. [2, Theorem 4.6] and [3, Theorem 6.1]).

Let us to verify the equality $\overline{g}(\overline{D'}) = \overline{D}$. Indeed, by definition, $\overline{g}(\overline{D'}) \subset \overline{D}$. It remains to show the converse inclusion $\overline{D} \subset \overline{g}(\overline{D'})$. Let $x_0 \in \overline{D}$. Now, we show that $x_0 \in \overline{g}(\overline{D'})$. If

$x_0 \in \overline{D}$, then either $x_0 \in D$, or $x_0 \in \partial D$. If $x_0 \in D$, then there is nothing to prove, since by hypothesis $\overline{g}(D') = D$. Let $x_0 \in \partial D$. Now, there exist $x_k \in D$ and $y_k \in D'$ such that $x_k = \overline{g}(y_k)$ and $x_k \to x_0$ as $k \to \infty$. Since $\overline{D'}$ is compact, we may assume that $y_k \to y_0 \in \overline{D'}$ as $k \to \infty$. Since $f = g^{-1}$ is a homeomorphism, $y_0 \in \partial D'$. Since $\overline{g}^{-1}$ is continuous in $\overline{D'}$, $\overline{g}(y_k) \to \overline{g}(y_0)$. However, in this case, $\overline{g}(y_0) = x_0$, because $\overline{g}(y_k) = x_k$ and $x_k \to x_0$ as $k \to \infty$. Thus, $x_0 \in \overline{g}(\overline{D'})$. The inclusion $\overline{D} \subset \overline{g}(\overline{D'})$ is proved. Therefore, $\overline{D} = \overline{g}(\overline{D'})$, as required.

It remains to show that $\mathfrak{R}_Q^{w_1,w_2,w_1',w_2'}(\overline{D},\overline{D'})$ is equicontinuous at $\partial D'$. We give the proof by contradiction. Now, we can find a point $z_0 \in \partial D'$, a number $\varepsilon_0 > 0$ and sequences $z_m \in \overline{D'}$, $z_m \to z_0$ as $m \to \infty$ and $\overline{g}_m \in \mathfrak{R}_Q^{w_1,w_2,w_1',w_2'}(\overline{D},\overline{D'})$ such that

$$d(\overline{g}_m(z_m), \overline{g}_m(z_0)) \geqslant \varepsilon_0, \quad m = 1, 2, \ldots . \tag{4}$$

Put $g_m := \overline{g}_m|_{D'}$. Since g_m has a continuous extension on $\partial D'$, we may assume that $z_m \in D'$ and $\overline{g}_m(z_m) = g_m(z_m)$. In addition, there exists $z_m' \in D'$, $z_m' \to z_0$ as $m \to \infty$, such that $d(g_m(z_m'), \overline{g}_m(z_0)) \to 0$ as $m \to \infty$. Since $\overline{D}$ is compact, we may assume that $g_m(z_m)$ and $\overline{g}_m(z_0)$ have limits as $m \to \infty$. Let $g_m(z_m) \to \overline{x_1}$ and $\overline{g}_m(z_0) \to \overline{x_2}$ as $m \to \infty$. By continuity of the metrics in (4), $\overline{x_1} \neq \overline{x_2}$. Since homeomorphisms preserve a boundary, $\overline{x_2} \in \partial D$. Without loss of generality, we may assume that $w_1 \neq \overline{x_1}$. By the condition **A**, we may join points w_1 and $\overline{x_1}$ by the path $\gamma_1\colon [0,1] \to \overline{D}$, and points w_2 and $\overline{x_2}$ by the path $\gamma_2\colon [0,1] \to \overline{D}$ such that $|\gamma_1| \cap |\gamma_2| = \varnothing$, $\gamma_i(t) \in D$ for all $t \in (0,1)$, $i = 1,2$, $\gamma_1(0) = w_1$, $\gamma_1(1) = \overline{x_1}$, $\gamma_2(0) = w_2$ and $\gamma_2(1) = \overline{x_2}$. Since D is locally connected on ∂D, there are neighborhoods U_1 and U_2 of $\overline{x_1}$ and $\overline{x_2}$, whose closures do not intersect, and $W_i := D \cap U_i$ are path-connected sets. In what follows,

$$B(x_0, r) := \{x \in X \colon d(x, x_0) < r\} .$$

Without loss of generality, we may assume that $\overline{U_1} \subset B(\overline{x_1}, \delta_0)$ and

$$\overline{B(\overline{x_1}, \delta_0)} \cap |\gamma_2| = \varnothing = \overline{U_2} \cap |\gamma_1| , \quad \overline{B(\overline{x_1}, \delta_0)} \cap \overline{U_2} = \varnothing , \tag{5}$$

$g_m(z_m) \in W_1$ and $g_m(z_m') \in W_2$ for each $m \in \mathbb{N}$. Let a_1 and a_2 be arbitrary points in $|\gamma_1| \cap W_1$ and $|\gamma_2| \cap W_2$, correspondingly. Let t_1, t_2 be such that $\gamma_1(t_1) = a_1$ and $\gamma_2(t_2) = a_2$. We join a_1 and $g_m(z_m)$ by a path $\alpha_m\colon [t_1, 1] \to W_1$ such that $\alpha_m(t_1) = a_1$ and $\alpha_m(1) = g_m(z_m)$. Similarly, we join a_2 and $g_m(z_m')$ by a path $\beta_m\colon [t_2, 1] \to W_2$, $\beta_m(t_2) = a_2$ and $\beta_m(1) = g_m(z_m')$, see Figure 1.

Set

$$C_m^1(t) = \begin{cases} \gamma_1(t), & t \in [0, t_1], \\ \alpha_m(t), & t \in [t_1, 1] \end{cases} , \quad C_m^2(t) = \begin{cases} \gamma_2(t), & t \in [0, t_2], \\ \beta_m(t), & t \in [t_2, 1] \end{cases} .$$

Given sets $A, B \subset X$, denote

$$\operatorname{dist}(A, B) := \inf_{x \in A, y \in B} d(x, y) .$$

Denote, as usual, $|C_m^1|$ and $|C_m^2|$ are loci of paths C_m^1 and C_m^2, respectively. Setting

$$l_0 = \min\{\operatorname{dist}(|\gamma_1|, |\gamma_2|), \operatorname{dist}(|\gamma_1|, U_2)\} ,$$

we consider the covering $A_0 := \bigcup_{x \in |\gamma_1|} B(x, l_0/4)$. Since $|\gamma_1|$ is a compact, we can choose $1 \leqslant N_0 < \infty$ and points $x_1, \ldots, x_{N_0} \in |\gamma_1|$ such that $|\gamma_1| \subset B_0 := \bigcup_{i=1}^{N_0} B(x_i, l_0/4)$. Now

$$|C_m^1| \subset U_1 \cup |\gamma_1| \subset \overline{B(\overline{x_1}, \delta_0)} \cup \bigcup_{i=1}^{N_0} B(x_i, l_0/4) .$$

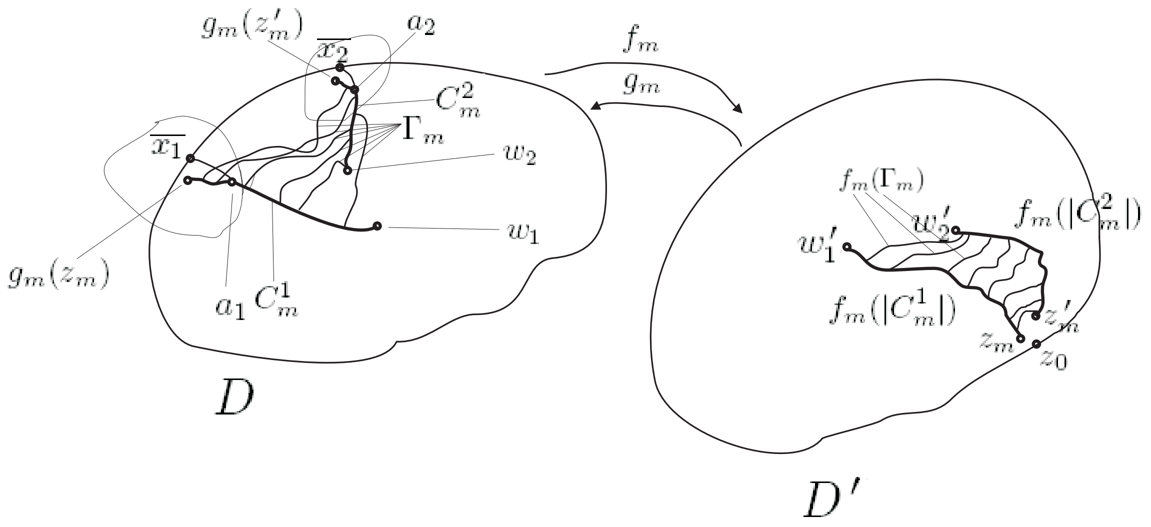

Figure 1. To the proof of Theorem 1

Let Γ_m be a family of paths connecting $|C_m^1|$ and $|C_m^2|$ in D. Now,

$$\Gamma_m = \bigcup_{i=0}^{N_0} \Gamma_{mi}\,, \tag{6}$$

where Γ_{mi} consists of all paths $\gamma\colon [0,1]\to D$ with $\gamma(0)\in B(x_i, l_0/4)\cap|C_m^1|$ and $\gamma(1)\in|C_2^m|$ for $1\leqslant i\leqslant N_0$. Similarly, Γ_{m0} consists of all paths $\gamma\colon [0,1]\to D$ with $\gamma(0)\in B(\overline{x_1},\delta_0)\cap|C_m^1|$ and $\gamma(1)\in|C_2^m|$. By (5) there exists $\sigma_0>\delta_0>0$ such that

$$\overline{B(\overline{x_1},\sigma_0)}\cap|\gamma_2| = \varnothing = \overline{U_2}\cap|\gamma_1|\,,\quad \overline{B(\overline{x_1},\sigma_0)}\cap\overline{U_2}=\varnothing\,. \tag{7}$$

Let $\gamma\in\Gamma_{m0}$. Since $\gamma(0)\in B(\overline{x_1},\delta_0)$ and $\gamma(1)\in|C_2^m|$, by (5) we obtain that $|\gamma|\cap B(\overline{x_1},\delta_0)\neq\varnothing\neq|\gamma|\cap(D\setminus B(\overline{x_1},\delta_0))$. Now, by [5, Theorem 1.I, Ch. 5, § 46], there exists $t_1^*\in(0,1)$ such that $\gamma(t_1^*)\in S(\overline{x_1},\delta_0)$. We may assume that $\gamma(t)\in D\setminus\overline{B(\overline{x_1},\delta_0)}$ for $t>t_1^*$. Similarly, by (7) and [5, Theorem 1.I, Ch. 5, § 46] there exists $t_2^*>t_1^*$ with $\gamma(t_2^*)\in S(\overline{x_1},\sigma_0)$. We may consider that $\gamma(t)\in D\setminus\overline{B(\overline{x_1},\sigma_0)}$ for $t>t_2^*$. Thus,

$$\Gamma_{m0} > \Gamma(S(\overline{x_1},\delta_0), S(\overline{x_1},\sigma_0), A(\overline{x_1},\delta_0,\sigma_0))\,. \tag{8}$$

Similarly,

$$\Gamma_{mi} > \Gamma(S(x_i,l_0/4), S(x_i,l_0/2), A(x_i,l_0/4,l_0/2))\,. \tag{9}$$

Putting

$$\eta(t)=\begin{cases}4/l_0, & t\in[l_0/4,l_0/2],\\ 0, & t\notin[l_0/4,l_0/2]\,,\end{cases}\qquad \eta_0(t)=\begin{cases}1/(\sigma_0-\delta_0), & t\in[\delta_0,\sigma_0],\\ 0, & t\notin[\delta_0,\sigma_0]\,,\end{cases}$$

and $f_m:=g_m^{-1}$, we obtain by (1) that

$$\begin{gathered} M_{\alpha'}(f_m(\Gamma(S(\overline{x_1},\delta_0),S(\overline{x_1},\sigma_0),A(\overline{x_1},\delta_0,\sigma_0))))\leqslant(1/(\sigma_0-\delta_0))^{\alpha}\cdot\|Q\|_1<c_1<\infty\,,\\ M_{\alpha'}(f_m(\Gamma(S(x_i,l_0/4),S(x_i,l_0/2),A(x_i,l_0/4,l_0/2))))\leqslant(4/(l_0))^{\alpha}\cdot\|Q\|_1<c_2<\infty\,, \end{gathered}\tag{10}$$

where c_1 and c_1 are some positive constants, not depending on m, and $\|Q\|_1 = \int\limits_D Q(x)\,d\mu(x)$. We conclude from (6), (8), (9), (10) and subadditivity of modulus that

$$M_{\alpha'}(f_m(\Gamma_m)) \leqslant (4N_0/l_0^{\alpha} + (1/(\sigma_0 - \delta_0))^{\alpha})\|Q\|_1 := c < \infty\,. \tag{11}$$

On the other hand, let $\delta_1 := \min\{\mathrm{dist}\,(w_1', \partial D'), \mathrm{dist}\,(w_2', \partial D')\}$. Now, we obtain that

$$d'(f_m(|C_m^1|)) \geqslant d'(z_m, w_1') \geqslant \delta_1/2\,, \quad d'(f_m(|C_m^2|)) \geqslant d'(z_m', w_2') \geqslant \delta_1/2 \tag{12}$$

for some $M_0 \in \mathbb{N}$ and for all $m \geqslant M_0$. Let $U := B(z_0, r_0) = \{x \in X'\colon d'(x, z_0) < r_0\}$, where $0 < r_0 < \delta_1/4$ and δ_1 is from (12). Notice, that $f_m(|C_m^1|) \cap U \neq \varnothing \neq f_m(|C_m^1|) \cap (D' \setminus U)$ for sufficiently large $m \in \mathbb{N}$, because $d'(f_m(|C_m^1|)) \geqslant \delta_1/2$ and $z_m \in f_m(|C_m^1|)$, $z_m \to z_0$ as $m \to \infty$. Similarly, $f_m(|C_m^2|) \cap U \neq \varnothing \neq f_m(|C_m^2|) \cap (D' \setminus U)$. Since $f_m(|C_m^1|)$ and $f_m(|C_m^2|)$ are continua,

$$f_m(|C_m^1|) \cap \partial U \neq \varnothing, \quad f_m(|C_m^2|) \cap \partial U \neq \varnothing\,, \tag{13}$$

see, e.g., [5, Theorem 1.I, Ch. 5, § 46]. Since $\partial D'$ is weakly flat, given $P > 0$, there exists a neighborhood $V \subset U$ of z_0 such that

$$M_{\alpha'}(\Gamma(E, F, D')) > P \tag{14}$$

for any continua $E, F \subset D'$ with $E \cap \partial U \neq \varnothing \neq E \cap \partial V$ and $F \cap \partial U \neq \varnothing \neq F \cap \partial V$. Observe that

$$f_m(|C_m^1|) \cap \partial V \neq \varnothing, \quad f_m(|C_m^2|) \cap \partial V \neq \varnothing \tag{15}$$

for sufficiently large $m \in \mathbb{N}$.

Indeed, let $z_m \in f_m(|C_m^1|)$, $z_m' \in f_m(|C_m^2|)$, where $z_m, z_m' \to z_0 \in V$ as $m \to \infty$. Now, $f_m(|C_m^1|) \cap V \neq \varnothing \neq f_m(|C_m^2|) \cap V$ for sufficiently large $m \in \mathbb{N}$. In addition, $d'(V) \leqslant d'(U) \leqslant 2r_0 < \delta_1/2$. Besides, by (12) we obtain that $d'(f_m(|C_m^1|)) > \delta_1/2$. Thus, $f_m(|C_m^1|) \cap (D' \setminus V) \neq \varnothing$ and, consequently, $f_m(|C_m^1|) \cap \partial V \neq \varnothing$ (see [5, Theorem 1.I, Ch. 5, § 46]). Similarly, $d'(V) \leqslant d'(U) \leqslant 2r_0 < \delta_1/2$. By (12) $d'(f_m(|C_m^2|)) > \delta_1$, thus $f_m(|C_m^2|) \cap (D' \setminus V) \neq \varnothing$. By [5, Theorem 1.I, Ch. 5, § 46] we have, that $f_m(|C_m^1|) \cap \partial V \neq \varnothing$. Thus, (15) is proved.

By (13), (14) and (15), we obtain that $M_{\alpha'}(f_m(\Gamma_m)) = M_{\alpha'}(\Gamma(f_m(|C_m^1|), f_m(|C_m^2|), D')) > P$, which contradicts (11). The contradiction obtained above disproves the assumption made in (4). The theorem is proved. □

Example 1. Consider the case $X = X' = \mathbb{R}^n$, $\alpha = \alpha' = n$, $n \geqslant 2$. Let $p \geqslant 1$ be a number, such that $n/p(n-1) < 1$. Put $\alpha \in (0, n/p(n-1))$. Let $\mathbb{B}^n = \{x \in \mathbb{R}^n\colon |x| < 1\}$. We define a sequence of mappings f_m of $\mathbb{B}^n$ onto the $B(0, 2)$ in the following way:

$$f_m(x) \;=\begin{cases} \frac{1+|x|^{\alpha}}{|x|}\cdot x\,, & 1/m \leqslant |x| \leqslant 1,\\ \frac{1+(1/m)^{\alpha}}{(1/m)}\cdot x\,, & 0 < |x| < 1/m, \end{cases}$$

see Figure 2. Notice, that f_m satisfies (1) in $\overline{\mathbb{B}^n}$ for $Q = \left(\frac{1+|x|^{\alpha}}{\alpha|x|^{\alpha}}\right)^{n-1} \in L^1(\mathbb{B}^n)$ at every $x_0 \in \overline{\mathbb{B}^n}$, see [6, proof of Theorem 7]. By [7, Lemma 4.3], $B(0, 2)$ has a weakly flat boundary. Observe that f_m fixes an infinite number of points of the unit ball for all $m \geqslant 2$.

By Theorem 1, the family $\mathfrak{G} = \{g_m\}_{m=1}^{\infty}$, $g_m := f_m^{-1}$, is equicontinuous in $\overline{B(0, 2)}$.

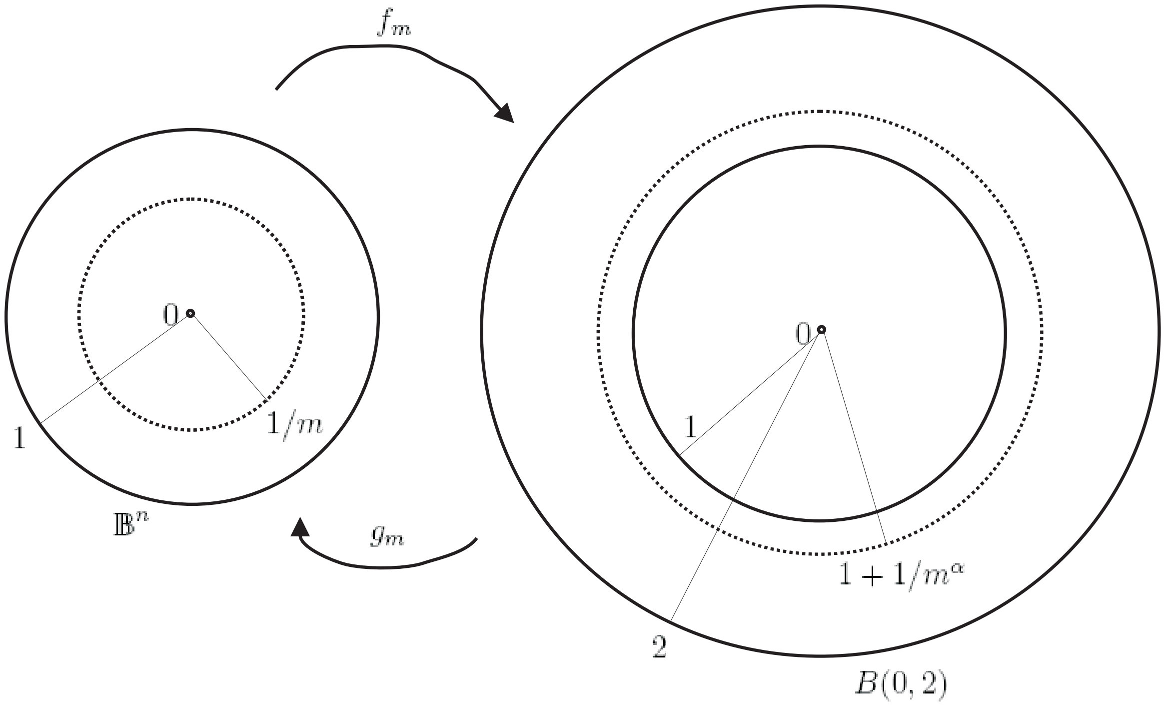

Figure 2. To Example 1

Observe that the “inverse” family $\mathfrak{F} = \{f_m\}_{m=1}^{\infty}$ is not equicontinuous in $\mathbb{B}^n$. Indeed, $|f_m(x_m) - f(0)| = 1 + 1/m^{\alpha} \not\to 0$ as $m \to \infty$, where $|x_m| = 1/m$).

The reason for the last circumstance is that g_m are not ring Q-homeomorphisms with some integrable Q in $B(0, 2)$. Let us show this.

Let $K_I(x, f)$ be a so-called inner dilatation of f at x (see [6, relation (45)]). By direct calculations, we observe that

$$g_m(y) := f_m^{-1}(y) = \begin{cases} \frac{y}{|y|}(|y|-1)^{1/\alpha}, & 1+1/m^{\alpha} \leqslant |y| < 2, \\ \frac{(1/m)}{1+(1/m)^{\alpha}} \cdot y\,, & 0 < |y| < 1+1/m^{\alpha}\,, \end{cases}$$
$$m = 1, 2, \dots, \quad g_m : B(0, 2) \to \mathbb{B}^n.$$

Arguing similarly to the proof of [6, Theorem 7], we conclude that

$$K_I(y, f_m) = \begin{cases} \frac{|y|}{\alpha(|y|-1)}\,, & 1+1/m^{\alpha} \leqslant |y| < 2, \\ 1\,, & 0 < |y| < 1+1/m^{\alpha}\,. \end{cases}$$

Note that $K_I(y, f_m) \notin L^1(B(0, 2)) = L^1(g_m(\mathbb{B}^n))$. Indeed, by the Fubini theorem,

$$\int\limits_{B(0,2)} K_I(y, f_m)\, dm(y) \geqslant \int\limits_{1<|y|<2} K_I(y, f_m)\, dm(y) =$$

$$= \frac{1}{\alpha} \cdot \int\limits_{1<|y|<2} \frac{|y|}{|y|-1}\, dm(y) = \frac{\omega_{n-1}}{\alpha} \cdot \int\limits_1^2 \frac{r^n}{r-1} dr = \infty,$$

where, as usual, ω_{n-1} denotes the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ and $dm(y)$ is the element of Lebesgue measure. By [8, Statement 1.3], $K_I(y, f_m) \leqslant c_n \cdot Q(y)$ for almost every $y \in B(0,2)$, where $c_n > 0$ is some positive constant. Consequently, $Q(y)$ is not integrable, because $K_I(y, f_m)$ is not integrable, as well.

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Received 02.06.2018