УДК 515.537.6

A. O. Kuryliak, O. B. Skaskiv, N. Yu. Stasiv

ON THE CONVERGENCE OF RANDOM MULTIPLE DIRICHLET SERIES

A. O. Kuryliak, O. B. Skaskiv, N. Yu. Stasiv. *On convergence of random multiple Dirichlet series*, Mat. Stud. **49** (2018), 122–137.

Let $F_\omega(s)=\sum_{\|n\|=0}^{\infty} f_{(n)}(\omega)\exp\{(\lambda_{(n)},s)\}$, where the exponents $\lambda_{(n)}=(\lambda_{n_1}^{(1)},\ldots,\lambda_{n_p}^{(p)})\in\mathbb{R}^p_+$, $(n)=(n_1,\ldots,n_p)\in\mathbb{Z}^p_+$, $p\in\mathbb{N}$, $\|n\|=n_1+\ldots+n_p$, and the coefficients $f_{(n)}(\omega)$ are pairwise independent random complex variables. In the paper, in particular, we prove the following statements: 1) If $\tau(\lambda)=\overline{\lim}_{\|n\|\to+\infty}\ln\|n\|/\|\lambda_{(n)}\|=0$, then in order that a Dirichlet series be convergent a.s. in the whole space $\mathbb{C}^p$, it is necessary and sufficient that

$$(\forall\Delta>0)\colon\ \sum_{\|n\|=0}^{+\infty}\big(1-F_{(n)}\big(\exp(-\Delta\|\lambda_{(n)}\|)\big)\big)<+\infty.$$

2) If $\tau(\lambda)=0$, then in order that $\sigma\in\partial G_a\cap\big(\mathbb{R}_+\setminus\{0\}\big)^p$ a.s., it is necessary and sufficient that

$$(\forall\varepsilon>0)\colon\ \sum_{\|n\|=0}^{+\infty}\Big(1-F_{(n)}\big(e^{(-1+\varepsilon)(\sigma,\lambda_{(n)})}\big)\Big)<+\infty\ \wedge\ \sum_{\|n\|=0}^{+\infty}\Big(1-F_{(n)}\big(e^{(-1-\varepsilon)(\sigma,\lambda_{(n)})}\big)\Big)=+\infty,$$

where $F_{(n)}(x):=P\{\omega:|f_{(n)}(\omega)|<x\}$, $x\in\mathbb{R}$, $(n)\in\mathbb{Z}^p_+$ is the distribution function of $|f_{(n)}(\omega)|$, ∂G_a is the set of conjugate abscissas of absolute convergence of the random Dirichlet series F_ω.

1. Multiple Dirichlet series with arbitrary real exponents. Let $\Lambda^p=(\lambda_{(n)})_{n\in\mathbb{Z}^p_+}$, where $\lambda_{(n)}=(\lambda_{n_1}^{(1)},\ldots,\lambda_{n_p}^{(p)})\in\mathbb{R}^p_+=[0,+\infty)^p$, $(n)=(n_1,\ldots,n_p)\in\mathbb{Z}^p_+$, $\mathbb{Z}_+:=\mathbb{N}\cup\{0\}$, $p\in\mathbb{N}$. In the case $(\lambda_k^{(j)})_{k\in\mathbb{Z}_+}$, $1\le j\le p$, $0=\lambda_0^{(j)}<\lambda_k^{(j)}<\lambda_{k+1}^{(j)}\uparrow+\infty$ $(1\le k\uparrow+\infty,\ 1\le j\le p)$, we use the notation Λ^p_+. By $\mathcal{D}^p(\Lambda^p)$ we denote the class of formal multiple Dirichlet series of the form

$$F(s)=\sum_{\|n\|=0}^{\infty}a_{(n)}\exp\{(\lambda_{(n)},s)\},\quad s=(s_1,\ldots,s_p)\in\mathbb{C}^p,\ s_j=\sigma_j+it_j\ (j\in\{1,\ldots,p\}),\quad(1)$$

for which there exists $\sigma\in\mathbb{R}^p$ such that

$$a_{(n)}\exp(\lambda_{(n)},\sigma)\to0\quad(\|n\|:=n_1+\ldots+n_p\to+\infty),\qquad(2)$$

where $(a_{(n)})$ is a sequence of complex numbers, and $(\lambda_{(n)},s)=\lambda_{n_1}^{(1)}s_1+\ldots+\lambda_{n_p}^{(p)}s_p$, $(n)=(n_1,n_2,\ldots,n_p)$.

2010 *Mathematics Subject Classification:* 11M32, 11M41, 30B20, 30B50.
Keywords: multiple Dirichlet series; conjugate abscissas of convergence; random exponents.
doi:10.15330/ms.49.2.122-137

For $\sigma \in \overline{\mathbb{R}}^p := [-\infty, +\infty]^p$ we put $\mathbf{\Pi}_\sigma := \{s \in \mathbb{C}^p : \ \mathrm{Re}\, s < \sigma\}$, $\Pi_\sigma := \{x \in \mathbb{R}^p : x < \sigma\}$, $\Pi^0_\sigma := \{x \in \mathbb{R}^p \colon x \geq \sigma\}$, where $\mathrm{Re}\, s = (\mathrm{Re}\, s_1, \ldots, \mathrm{Re}\, s_p)$, $x = (x_1, \ldots, x_p) < b = (b_1, \ldots, b_p) \Longleftrightarrow x_j < b_j$ $(\forall j \in \{1, \ldots, p\})$, and $x \geq b \Longleftrightarrow x_j \geq b_j$ $(\forall j \in \{1, \ldots, p\})$.

The domain of the convergence of the multiple Dirichlet series with exponents Λ^p_+ was investigated in papers [1–11]. We consider the general class $\mathcal{D}^p(\Lambda^p)$ and also classes of Dirichlet series with random coefficients and with random exponents.

Define concepts of the domains of convergence of the series of form (1) with arbitrary exponents Λ^p and the concepts of conjugates abscissa of convergence. We put

$$G_c = \{x \in \mathbb{R}^p : \text{series (1) is convergent in } \mathbf{\Pi}_x\}, \quad \mathbf{G}_c = \{z \in \mathbb{C}^p \colon \mathrm{Re}\, z \in G_c\},$$
$$G_a = \{x \in \mathbb{R}^p : \text{series (1) is absolutely convergent in } \mathbf{\Pi}_x\}, \quad \mathbf{G}_a = \{z \in \mathbb{C}^p \colon \mathrm{Re}\, z \in G_a\}.$$

Here $\mathbf{G}_c$, $\mathbf{G}_a$ are the domains of the convergence and the absolutely convergence of series (1), and, respectively, G_c, G_a are the their traces in $\{x \in \mathbb{R}^p : \ x = \mathrm{Re}\, z, \ z \in \mathbb{C}^p\}$.

In general, the domain G_μ of the existence of the maximal term we define as the interior of the set of the points $\sigma \in \mathbb{R}^p$ such that condition (2) is satisfied.

In the case of a sequence Λ^p_+ the domain G^*_μ of the existence of the maximal term $\mu(\sigma, F)$ is defined ([13]) as the set of the points $\sigma \in \mathbb{R}^p$ such that

$$\mu(\sigma, F) := \mu(\sigma_1, ..., \sigma_p, F) = \max\{|a_{(n)}| \exp(\lambda_{(n)}, \sigma) \colon (n) \in \mathbb{Z}^p_+\} < +\infty. \tag{3}$$

Remark 1. In the case of a sequence Λ^p_+ the interior of the set G^*_μ coincides with the set G_μ. It easy to see that condition (3) follows from condition (2), and the reverse implication, in general, is not true.

The sets ∂G_c, ∂G_a, ∂G_μ are hypersurfaces of *conjugates abscissas's* of the convergence, of the absolutely convergence and of the existence of maximal term of series (1), respectively.

Remark 2. 1. If $\sigma \notin \overline{G}_c$ $(\sigma \notin \overline{G}_a)$, then $\Pi_\sigma \setminus \overline{G}_c \neq \varnothing$ $(\Pi_\sigma \setminus \overline{G}_a \neq \varnothing)$; similarly, if $\sigma \notin \overline{G}_\mu$, then $|a_{(n)}| \exp(\lambda_{(n)}, \sigma) \not\to 0$.
2. If $F \in \mathcal{D}^p(\Lambda^p)$, then $\partial G_\mu \neq \varnothing$ and $G_a \subset G_c \subset G_\mu$, $\partial G_a \subset \overline{G}_c$, $\partial G_c \subset \overline{G}_\mu$.
3. $\sigma \in \partial G_\mu \Longrightarrow \Pi_\sigma \subset G_\mu$; $\sigma \in \partial G_a \Longrightarrow \Pi_\sigma \subset G_a$; $\sigma \in \partial G_c \Longrightarrow \Pi_\sigma \subset G_c$.

The following assertion has, in fact, been proved in paper [3].

Proposition 1 ([3]). *If $F \in \mathcal{D}^p(\Lambda^p_+)$ and $\tau(\Lambda^p_+) := \varlimsup\limits_{\|n\|\to\infty} \dfrac{\ln \|n\|}{\|\lambda_{(n)}\|} = 0$, then*

$$\partial G_a = G^*_1 := \Big\{\alpha \in \mathbb{R}^p \colon \ \varlimsup_{\|n\|\to+\infty} \frac{\ln |a_{(n)}| + (\alpha, \lambda_{(n)})}{\|\lambda_{(n)}\|} = 0\Big\},$$

where $\|\lambda_{(n)}\| = \lambda^{(1)}_{n_1} + ... + \lambda^{(p)}_{n_p}$.

Remark 3. In paper [8, Proposition 1.2] for the case of the sequence Λ^p_+ it has been proved that $\partial G_\mu = G^*_1$, therefore, one immediately has

$$G_\mu = G_1 := \bigcup_{x \in G^*_1} \Pi_x.$$

Remark 4. In the assumptions of each of the two assertions just cited, $G_a = G_1$ and $G_\mu = G_1$, respectively. Therefore, the condition $\tau(\Lambda^p_+) = 0$ implies that $G_a = G_c = G_\mu = G_1$.

In view of Remarks 3 and 4, we have the following statement.

Proposition 2. *If $F \in \mathcal{D}^p(\Lambda^p_+)$ and $\tau(\Lambda^p_+) = 0$, then $G_a = G_c = G_\mu = G_1$ and $\partial G_a = \partial G_c = \partial G_\mu = G_1^*$.*

We now turn to the Dirichlet series $F \in D^p(\Lambda^p)$ with a sequence of exponent Λ^p of general form. We shall first prove the following assertion. In the case of the sequence Λ^p_+, it is proved in [8, Proposition 1.3].

Proposition 3. *Let $F \in \mathcal{D}^p(\Lambda^p)$ be of the form (1) and $\lambda := \varliminf\limits_{\|n\|\to+\infty} \|\lambda_{(n)}\| > 0$. In order that $G_\mu = \mathbb{R}^p$, it is necessary and sufficient that*

$$\lim_{\|n\|\to+\infty} \frac{-\ln|a_{(n)}|}{\|\lambda_{(n)}\|} = +\infty.$$

Proof of Proposition 3. Necessity. Let $G_\mu = \mathbb{R}^p$. Then $|a_{(n)}|\exp(\|\lambda_{(n)}\|\sigma) \to 0$ $(\|n\| \to +\infty)$ for every $\sigma > 0$, thus $-\ln|a_{(n)}| - \|\lambda_{(n)}\|\sigma \to +\infty$ $(\|n\| \to +\infty)$. Hence, $-\ln|a_{(n)}| \geq \|\lambda_{(n)}\|\sigma$ for all $\|n\|$ large enough. Therefore,

$$\varliminf_{\|n\|\to+\infty} \frac{-\ln|a_{(n)}|}{\|\lambda_{(n)}\|} \geq \sigma.$$

It remains to use the arbitrariness of the choice $\sigma > 0$.
Sufficiency. From the assumption it follows that $\ln|a_{(n)}| + \sigma\|\lambda_{(n)}\| \to -\infty$ $(\|n\| \to +\infty)$ for every $\sigma > 0$. Indeed, for any $\sigma_1 > 0$, for each $\sigma > 0$ and for all $\|n\|$ large enough the inequality $-\ln|a_{(n)}| > (\sigma_1 + \sigma)\|\lambda_{(n)}\|$ is fulfilled, thus

$$-\ln|a_{(n)}| - \sigma\|\lambda_{(n)}\| > \sigma_1\|\lambda_{(n)}\| > \sigma_1\lambda/2$$

for all $\|n\|$ sufficiently large. Using the arbitrariness of the choice of $\sigma_1 > 0$, we obtain the required relation.

Putting $\sigma = \max\big\{1,\ \max\{\sigma_j^0 \colon j \in \{1,\dots,p\}\}\big\}$ for any $\sigma^0 = (\sigma_1^0,\dots,\sigma_p^0) \in \mathbb{R}^p$, we have

$$|a_{(n)}|\exp\{(\sigma^0, \lambda_{(n)})\} \leq \exp\{\ln|a_{(n)}| + \sigma\|\lambda_{(n)}\|\} \to 0 \quad (\|n\| \to \infty),$$

i.e. $G_\mu = \mathbb{R}^p$. The proof of Proposition 3 is complete. □

The following assertion generalizes Proposition 3 on the case $\mathbb{R}^l \times \Pi^{p-l}_{\sigma^0} \subset G_\mu$, where $\Pi^{p-l}_{\sigma^0} := \Pi_{\sigma^0_{l+1}} \times \ldots \times \Pi_{\sigma^0_p}$, $\sigma^0 = (\sigma^0_1,\dots,\sigma^0_p) \in \mathbb{R}^p$, $0 \leq l \leq p-1$.

Proposition 4. *Let $F \in \mathcal{D}^p(\Lambda^p)$ be of the form (1) and for some l, $1 \leq l \leq p-1$, $\lambda^l := \varliminf\limits_{\|n\|\to+\infty} \|\lambda_{(n)}\|_l > 0$, where $\|\lambda_{(n)}\|_l := \sum\limits_{k=1}^{l} \lambda^{(k)}_{n_k}$. In order that $\mathbb{R}^l \times \Pi^{p-l}_{\sigma^0} \subset G_\mu$ it is necessary and sufficient that for every $\sigma = (\sigma_1,\dots,\sigma_p) \in \mathbb{R}^l \times \Pi^{p-l}_{\sigma^0}$*

$$\lim_{\|n\|\to+\infty} \frac{\ln(1/|a_{(n)}|) - \sum\limits_{j=l+1}^{p} \sigma_j\lambda^{(j)}_{n_j}}{\|\lambda_{(n)}\|_l} = +\infty. \tag{4}$$

Proof of Proposition 4. Necessity. Let $\beta > 0$ be arbitrary, $\beta_l = (\beta, \dots, \beta) \in \mathbb{R}^l$. From the condition $\sigma = (\beta, \dots, \beta, \sigma_{l+1}, \dots, \sigma_p) \in G_\mu$ it follows that $a_{(n)}e^{(\sigma,\lambda_{(n)})} \to 0$ ($\|n\| \to +\infty$). Therefore, for all $\|n\| \geq k_0$ we have $|a_{(n)}|e^{(\sigma,\lambda_{(n)})} \leq 1$, hence,

$$\frac{-\ln|a_{(n)}| - \sum\limits_{j=l+1}^{p} \sigma_j \lambda_{n_j}^{(j)}}{\sum\limits_{k=1}^{l} \lambda_{n_k}^{(k)}} \geq \beta. \tag{5}$$

It remains to pass to the lower limits as $\|n\| \to +\infty$, and then as $\beta \to +\infty$.
Sufficiency. Let $\widetilde{\sigma} = (\widetilde{\sigma}_{l+1}, \dots, \widetilde{\sigma}_p) < \sigma^* = (\sigma_{l+1}, \dots, \sigma_p) \in \Pi_{\sigma_0}^{p-l}$. Condition (4) implies that for any $\beta > 0$ and for $\|n\| \geq k_0$ inequality (5) is fulfilled. Hence, for arbitrary fixed $(\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_l) \in \mathbb{R}^l$ and $\widehat{\sigma} = (\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_l, \widetilde{\sigma}_{l+1}, \dots, \widetilde{\sigma}_p) \in \mathbb{R}^p$ we obtain

$$-\ln|a_{(n)}| - (\widehat{\sigma}, \lambda_{(n)}) \geq \sum_{k=1}^{l} (\beta - \widetilde{\sigma}_k)\lambda_{n_k}^{(k)} + \sum_{k=1}^{l} (\sigma_j - \widetilde{\sigma}_j)\lambda_{n_j}^{(j)} \geq \sum_{k=1}^{l} (\beta - \widetilde{\sigma}_k)\lambda_{n_k}^{(k)}.$$

We assume that $\beta > 2\max\{\widetilde{\sigma}_k \colon 1 \leq k \leq l\}$. Then,

$$-\ln|a_{(n)}| - (\widehat{\sigma}, \lambda_{(n)}) \geq \frac{\beta}{2}\|\lambda_{(n)}\|_l.$$

It remains to recall the condition $\lambda^l := \varliminf\limits_{\|n\| \to +\infty} \|\lambda_{(n)}\|_l > 0$ and use the arbitrariness of $\beta > 0$. Therefore, $|a_{(n)}|e^{(\widehat{\sigma},\lambda_{(n)})} \to 0$ ($\|n\| \to +\infty$), hence, $\widehat{\sigma} \in G_\mu$ for any $\widehat{\sigma} \in \mathbb{R}^l \times \Pi_{\sigma_0}^{p-l}$. Thus, $\mathbb{R}^l \times \Pi_{\sigma_0}^{p-l} \subset G_\mu$. □

We denote

$$G_0^* := \Big\{\alpha \in \mathbb{R}^p \colon \varliminf_{\|n\| \to +\infty} \frac{-\ln|a_{(n)}|}{(\alpha, \lambda_{(n)})} = 1\Big\}, \quad G_0 := \bigcup_{x \in G_0^*} \Pi_x,$$
$$\widehat{G}_0^* := \Big\{\alpha \in \mathbb{R}^p \colon \varlimsup_{\|n\| \to +\infty} \frac{-\ln|a_{(n)}|}{(\alpha, \lambda_{(n)})} = 1\Big\}, \quad \widehat{G}_0 := \bigcup_{x \in \widehat{G}_0^*} \Pi_x..$$

We prove the following two statements. In the case of the sequence Λ_+^p they are proved in the article [11].

Proposition 5. *If $F \in \mathcal{D}^p(\Lambda^p)$ and $\varliminf\limits_{\|n\| \to +\infty} \|\lambda_{(n)}\| = \lambda > 0$, then*

$$G_0^* \cap \big(\mathbb{R}_+^{p-1} \times (0, +\infty)\big) \subset G_1^* \cap \big(\mathbb{R}_+^{p-1} \times (0, +\infty)\big), \tag{6}$$
$$G_1^* \cap \big(\mathbb{R}_+ \setminus \{0\}\big)^p \subset G_0^* \cap \big(\mathbb{R}_+ \setminus \{0\}\big)^p. \tag{7}$$

Proposition 6. *If $F \in \mathcal{D}^p(\Lambda^p)$ and $\varliminf\limits_{\|n\| \to +\infty} \|\lambda_{(n)}\| = \lambda > 0$, then*

$$G_1^* \cap (-\infty, 0)^p \subset \widehat{G}_0^* \cap (-\infty, 0)^p, \tag{8}$$
$$\widehat{G}_0^* \cap \big((-\infty, 0]^{p-1} \times (-\infty, 0)\big) \subset G_1^* \cap \big((-\infty, 0]^{p-1} \times (-\infty, 0)\big). \tag{9}$$

Proof of Proposition 5. We first prove (6). Let $\sigma \in G_0^* \cap \big(\mathbb{R}_+^{p-1} \times (0,+\infty)\big)$. Then the equality $\varliminf\limits_{\|n\|\to\infty} \ln(1/|a_{(n)}|)/(\sigma,\lambda_{(n)}) = 1$ is holds. Therefore,

$$\frac{\ln|a_{(n)}| + (\sigma,\lambda_{(n)})}{\|\lambda_{(n)}\|} = -\varepsilon_{(n)} \cdot \frac{(\sigma,\lambda_{(n)})}{\|\lambda_{(n)}\|}, \tag{10}$$

where $\varliminf\limits_{\|n\|\to\infty} \varepsilon_{(n)} = 0$. From the condition $\varliminf\limits_{\|n\|\to\infty} \|\lambda_{(n)}\| > 0$ it follows that $\|\lambda_{(n)}\| > 0$ and $(\sigma,\lambda_{(n)}) \geq 0$ for $\|n\|$ large enough, thus $0 \leq \frac{(\sigma,\lambda_{(n)})}{\|\lambda_{(n)}\|} \leq \max\{|\sigma_j|\colon 1\leq j\leq p\} < +\infty$ for such $\|n\|$. Hence we have

$$\varliminf_{\|n\|\to\infty} \varepsilon_{(n)} \cdot \frac{(\sigma,\lambda_{(n)})}{\|\lambda_{(n)}\|} = 0.$$

Therefore,

$$\varlimsup_{\|n\|\to\infty} \frac{\ln|a_{(n)}| + (\sigma,\lambda_{(n)})}{\|\lambda_{(n)}\|} = 0, \tag{11}$$

i.e. $\sigma \in G_1^*$ and $G_0^* \cap \big(\mathbb{R}_+^{p-1} \times (0,+\infty)\big) \subset G_1^* \cap \big(\mathbb{R}_+^{p-1} \times (0,+\infty)\big)$.

We assume now that $\sigma \in G_1^* \cap \big(\mathbb{R}_+ \setminus \{0\}\big)^p$, i.e. $\min\{\sigma_j\colon 1\leq j\leq p\} > 0$. Then, from (11) for any $\varepsilon > 0$ and all $\|n\|$ large enough we have $\ln(1/|a_{(n)}|) > (\sigma,\lambda_{(n)}) - \varepsilon\|\lambda_{(n)}\|$, thus

$$\frac{\ln(1/|a_{(n)}|)}{(\sigma,\lambda_{(n)})} > 1 - \frac{\varepsilon\|\lambda_{(n)}\|}{(\sigma,\lambda_{(n)})} \geq 1 - \frac{\varepsilon}{\min\{\sigma_j\colon j\in\{1,\ldots,p\}\}}.$$

Similarly, for some sequence $(\widehat{n})$ such that $\|\widehat{n}\| \to +\infty$, we obtain

$$\frac{\ln(1/|a_{\widehat{n}}|)}{(\sigma,\lambda_{(\widehat{n})})} < 1 + \frac{\varepsilon\|\lambda_{(\widehat{n})}\|}{(\sigma,\lambda_{(\widehat{n})})} \leq 1 + \frac{\varepsilon}{\min\{\sigma_j\colon j\in\{1,\ldots,p\}\}}.$$

In view of arbitrariness of $\varepsilon > 0$ we get $\sigma \in G_0^* \cap \big(\mathbb{R}_+ \setminus \{0\}\big)^p$, i.e. $G_1^* \cap \big(\mathbb{R}_+ \setminus \{0\}\big)^p \subset G_0^* \cap \big(\mathbb{R}_+ \setminus \{0\}\big)^p$. □

Proof of Proposition 6. We first prove (8). Assume that $\sigma = (\sigma_1,\ldots,\sigma_p) \in G_1^* \cap (-\infty,0)^p$, hence, $\max\{\sigma_j\colon 1\leq j\leq p\} < 0$, i.e. $\min\{-\sigma_j\colon 1\leq j\leq p\} > 0$. Similarly as above, from equality (11), in view of $(\sigma,\lambda_{(n)}) < 0$, we get

$$(\exists k_0)(\forall \|n\|\geq k_0)\colon\ \frac{\ln(1/|a_{(n)}|)}{(\sigma,\lambda_{(n)})} < 1 - \frac{\varepsilon\|\lambda_{(n)}\|}{(\sigma,\lambda_{(n)})} \leq 1 + \frac{\varepsilon}{\min\{-\sigma_j\colon j\in\{1,\ldots,p\}\}},$$
$$(\exists(\widehat{n}))\colon\ \|\widehat{n}\|\to\infty\ \wedge\ \frac{\ln(1/|a_{\widehat{n}}|)}{(\sigma,\lambda_{(\widehat{n})})} > 1 + \frac{\varepsilon\|\lambda_{(\widehat{n})}\|}{(\sigma,\lambda_{(\widehat{n})})} \geq 1 - \frac{\varepsilon}{\min\{-\sigma_j\colon j\in\{1,\ldots,p\}\}}.$$

The latter relationships imply (8).

We now assume that $\sigma \in \widehat{G}_0^* \cap \big((-\infty,0]^{p-1} \times (-\infty,0)\big)$, i.e. $(\sigma,\lambda_{(n)}) \leq 0$ and the equality $\varlimsup\limits_{\|n\|\to\infty} \ln(1/|a_{(n)}|)/(\sigma,\lambda_{(n)}) = 1$ is fulfilled. Then

$$\frac{\ln(1/|a_{(n)}|) - (\sigma,\lambda_{(n)})}{\|\lambda_{(n)}\|} = \varepsilon_{(n)} \cdot \frac{(\sigma,\lambda_{(n)})}{\|\lambda_{(n)}\|}, \tag{12}$$

where $\overline{\lim\limits_{\|n\|\to\infty}}\,\varepsilon_{(n)}=0$. But, $\underline{\lim\limits_{\|n\|\to\infty}}\,\|\lambda_{(n)}\|>0$, thus $\frac{|(\sigma,\lambda_{(n)})|}{\|\lambda_{(n)}\|}\le\max\{|\sigma_j|\colon 1\le j\le p\}<+\infty$ and $(\sigma,\lambda_{(n)})\le 0$ for $\|n\|$ large enough. Then,

$$\underline{\lim_{\|n\|\to\infty}}\,\varepsilon_{(n)}\cdot\frac{(\sigma,\lambda_{(n)})}{\|\lambda_{(n)}\|}=0.$$

Therefore, from (12) we obtain that equality (11) is fulfilled. Thus $\sigma\in G_1^*$, and we get $\widehat{G}_0^*\cap\big((-\infty,0]^{p-1}\times(-\infty,0)\big)\subset G_1^*\cap\big((-\infty,0]^{p-1}\times(-\infty,0)\big)$.

Assume now that $\sigma\in\widehat{G}_0^*\cap\big((-\infty,0]^{p-1}\times(-\infty,0)\big)$, i.e. $(\sigma,\lambda_{(n)})\le 0$ and the equality $\overline{\lim\limits_{\|n\|\to\infty}}\,\ln(1/|a_{(n)}|)/(\sigma,\lambda_{(n)})=1$ is fulfilled. Then, for any $\varepsilon>0$ and all $\|n\|$ large enough we have $\frac{\ln(1/|a_{(n)}|)}{(\sigma,\lambda_{(n)})}<1+\varepsilon$. Therefore,

$$\frac{\ln(1/|a_{(n)}|)-(\sigma,\lambda_{(n)})}{\|\lambda_{(n)}\|}>\frac{\varepsilon(\sigma,\lambda_{(n)})}{\|\lambda_{(n)}\|}\ge\varepsilon\cdot\min\{\sigma_j\colon j\in\{1,\dots,p\}\},$$

and similarly, for some subsequence $(\widehat{n})$ such that $\|\widehat{n}\|\to+\infty$, we get

$$\frac{\ln(1/|a_{\widehat{n}}|)-(\sigma,\lambda_{(\widehat{n})})}{\|\lambda_{(\widehat{n})}\|}<-\varepsilon\cdot\frac{(\sigma,\lambda_{(\widehat{n})})}{\|\lambda_{(\widehat{n})}\|}\le-\varepsilon\cdot\min\{\sigma_j\colon j\in\{1,\dots,p\}\}.$$

Hence, we obtain that equality (11) is fulfilled again. Thus, $\sigma\in G_1^*$. □

Proposition 7. *If $F\in\mathcal{D}^p(\Lambda^p)$ and $\underline{\lim\limits_{\|n\|\to+\infty}}\,\|\lambda_{(n)}\|=\lambda>0$, then $G_\mu\cap\mathbb{R}^p=G_1\cap\mathbb{R}^p$, $\mathbb{R}^p\cap\partial G_\mu=\mathbb{R}^p\cap G_1^*$.*

Proof of Proposition 7. Let $\alpha=(\alpha_1,\dots,\alpha_p)\in\partial G_\mu\cap\mathbb{R}^p$. Assume first that $\min\{\alpha_j\colon 1\le j\le p\}>0$. Then, $\pi_\alpha:=\{t\alpha\colon t\le 1\}\subset\Pi_\alpha$. We consider the Dirichlet series

$$g(t)=\sum a_{(n)}e^{t\lambda^*_{(n)}},\quad \lambda^*_{(n)}=(\alpha,\lambda_{(n)}).$$

The abscissa of existence of the maximal term of this series $\sigma_\mu(g)=1$. Indeed $\sigma_\mu(g)\ge 1$, because $\pi_\alpha\subset\Pi_\alpha$. If inequality $\sigma_\mu(g)>1$ holds, then there exists $t_0>1$ such that

$$a_{(n)}e^{t_0(\alpha,\lambda_{(n)})}=a_{(n)}e^{t_0\lambda^*_{(n)}}\to 0\quad(\|n\|\to+\infty).$$

But then $t_0\alpha\in G_\mu$. By Remark 2.3, $t\alpha\notin G_\mu$ for each $t>1$, and we get a contradiction.

By Proposition 2 from [15] we obtain $\alpha_0(g):=\underline{\lim\limits_{\|n\|\to+\infty}}\,\frac{-\ln|a_{(n)}|}{\lambda^*_{(n)}}=\sigma_\mu(g)$. Then

$$\alpha_0(g)=1,\tag{13}$$

it follows that

$$\underline{\lim_{\|n\|\to+\infty}}\,\frac{-\ln|a_{(n)}|}{(\alpha,\lambda_{(n)})}=1.\tag{14}$$

By Proposition 5 one can deduce that $\alpha\in G_1^*$. So, we have proved that $\partial G_\mu\subset G_1^*$ under some additional assumptions described earlier in the proof.

On the contrary if $\alpha \in G_1^*$, then by Proposition 5 we get (14). But then (13) holds and by Proposition 2 from [15] we obtain $\sigma_\mu(g) = \alpha_0 = 1$. So, for each $t < 1$ one has

$$a_{(n)}e^{t(\alpha,\lambda_{(n)})} = a_{(n)}e^{t\lambda^*_{(n)}} \to 0 \quad (\|n\| \to +\infty).$$

Therefore, $t\alpha \in G_\mu$. Finally we get that $\alpha \in \overline{G}_\mu$ that implies $G_1^* \subset \partial G_\mu$, $G_1 \subset G_\mu$.

Put $\alpha^* := \min\{\alpha_j : 1 \le j \le p\} \le 0$. Then, $\alpha^0 := \min\{\alpha_j + 1 + |\alpha^*| : 1 \le j \le p\} > 0$. We consider the Dirichlet series

$$F_1(z) = \sum a^0_{(n)}e^{(z,\lambda_{(n)})}, \quad a^0_{(n)} := a_{(n)}e^{-(1+|\alpha^*|)(e_1,\lambda_{(n)})}.$$

Remark that $\alpha \in \partial G_\mu(F) \iff \big(\alpha + (1+|\alpha^*|)e_1\big) \in \partial G_\mu(F_1)$, because

$$a^0_{(n)} \exp\big(z + (1+|\alpha^*|)e_1, \lambda_{(n)}\big) = a_{(n)}e^{(z,\lambda_{(n)})}.$$

Also $\alpha \in G_1^*(F) \iff \big(\alpha + (1+|\alpha^*|)e_1\big) \in G_1^*(F_1)$ becouse

$$\varlimsup_{\|n\|\to+\infty} \frac{\ln|a^{(0)}_{(n)}| + (\alpha + (1+|\alpha^*|)e_1, \lambda_{(n)})}{\|\lambda_{(n)}\|} = \varlimsup_{\|n\|\to+\infty} \frac{\ln|a_{(n)}| + (\alpha, \lambda_{(n)})}{\|\lambda_{(n)}\|}.$$

It remains to apply that was proved above on Dirichlet series F_1. So, we get

$$\alpha \in \partial G_\mu(F) \iff \big(\alpha + (1+|\alpha^*|)e_1\big) \in \partial G_\mu(F_1) \iff$$
$$\iff \big(\alpha + (1+|\alpha^*|)e_1\big) \in G_1^*(F_1) \iff \alpha \in G_1^*(F).$$

Finally, we deduce $\partial G_\mu(F) = G_1^*(F)$. □

In the paper [12] a certain generalization of Valiron's formula was obtained to find the abscissa of the convergence of Dirichlet series of one variable. Also it was investigated relationship between abscissas of convergence and abscissas of existence maximal term of Dirichlet series (see [13, 14]). In [8] p-dimensional ($p \ge 2$) analogues of statements from [12] was obtained (the case $p = 2$ was considered in [7]).

For $\gamma > 0$, $\delta_0 \in \mathbb{R}$, $\delta = (\delta_1, ..., \delta_p) \in \mathbb{R}^p$ denote

$$h_1(\gamma;\delta_0) := \varlimsup_{\|n\|\to\infty} \frac{(\gamma-1)\ln|a_{(n)}| + \delta_0\|\lambda_{(n)}\|}{\ln\|n\|}, \;\; h_2(\gamma;\delta) := \varlimsup_{\|n\|\to\infty} \frac{(\gamma-1)\ln|a_{(n)}| + (\delta,\lambda_{(n)})}{\ln n_1 + ... + \ln n_p}.$$

In [8, Theorem 1] the following statement was proved.

Proposition 8 ([8]). *Let $F \in \mathcal{D}^p(\Lambda^p_+)$.*

i) If there exist $\gamma > 0$, $\delta_0 \in \mathbb{R}$ such that $h_1(\gamma;\delta_0) > p$, then $\gamma G_c \subset G_a + \delta_0 e_1$ and $\gamma G_\mu \subset G_a + \delta_0 e_1$, where $e_1 = (1, ..., 1) \in \mathbb{R}^p$.

ii) If there exist $\gamma > 0$, $\delta \in \mathbb{R}^p$ such that $h_2(\gamma;\delta) > 1$, then $\gamma G_c \subset G_a + \delta$, $\gamma G_\mu \subset G_a + \delta$.

We prove the following statement.

Proposition 9. *Let $F \in \mathcal{D}^p(\Lambda^p)$ of the form* (1)*. Then $\Pi^0_{\alpha_0 e_1} \subset \big(\mathbb{R}^p \setminus G_\mu\big) \subset \big(\mathbb{R}^p \setminus G_c\big)$, where*

$$\alpha_0 := \varliminf_{\|n\|\to+\infty} \frac{-\ln|a_{(n)}|}{\|\lambda_{(n)}\|}.$$

If there exist $\gamma > 0, \delta \in \mathbb{R}^p$ such that

$$\sum_{\|n\|=0}^{+\infty} |a_{(n)}|^{1-\gamma} e^{-(\delta,\lambda_{(n)})} < +\infty, \tag{15}$$

then $\gamma\Pi_{\alpha_0 e_1} \subset G_a + \delta$ and $\gamma G_c \subset \gamma G_1 = \gamma G_\mu \subset G_a + \delta$.

Remark 5. 1. For $p = 1$ and any sequence Λ^1 Proposition 9 was proved in the paper [18] (see also [19, 20]).

2. In the case of Dirichlet series with a sequence of exponents Λ^1_+ Proposition 9 can be found in [13, 14].
3. Condition (15) follows both from the condition $h_1(\gamma;\delta_0) > p$ or from the condition $h_2(\gamma;\delta_0) > 1$. Also it is easy to make sure that the condition (15) is weaker than each of those two conditions.

Proof of Proposition 9. It is obvious that $\mathbb{R}^p \setminus G_\mu \subset \mathbb{R}^p \setminus G_c$. Firstly, we prove that $\Pi^0_{\alpha_0 e_1} \subset \mathbb{R}^p \setminus G_\mu$. The case $\alpha_0 = +\infty$ is trivial, because $\Pi_{\alpha_0 e_1} = \mathbb{R}^p$ and $\Pi^0_{\alpha_0 e_1} = \varnothing$.

Suppose that $\alpha_0 = -\infty$. Then by the definition of α_0, for each $E > 0$ and some sequence $k_j \to +\infty$ $(j \to +\infty)$

$$\ln|a_{(n)}|/\|\lambda_{(n)}\| > E \quad (\|n\| = k_j,\ j \ge 1).$$

Then

$$|a_{(n)}| \exp\{-(Ee_1, \lambda_{(n)})\} > 1 \ (\|n\| = k_j,\ j \ge 1),$$

i.e., $|a_{(n)}| \exp\{(\sigma, \lambda_{(n)})\} \not\to 0$ at the point $\sigma = -Ee_1$. Using arbitrariness of $E > 0$ we deduce $G_\mu = G_c = \Pi_{\alpha_0 e_1} = \varnothing$.

Now we suppose that $\alpha_0 \ne \infty$ and choose $\alpha_0^* = \alpha_0 + \varepsilon, x_0 = \alpha_0^* e_1$, where $\varepsilon > 0$ is arbitrary. So,

$$|a_{(n)}| e^{(x_0,\lambda_{(n)})} = \exp\{\|\lambda_{(n)}\|(\ln|a_{(n)}|/\|\lambda_{(n)}\| + \alpha_0^*)\}.$$

By the definition of α_0 there exists a sequence $k_j \to +\infty$ $(j \to +\infty)$ such that $\ln|a_{(n)}|/\|\lambda_{(n)}\| > -(\alpha_0 + \varepsilon/2)$ $(\|n\| = k_j,\ j \ge 1)$. Then $\ln|a_{(n)}|/\|\lambda_{(n)}\| + \alpha_0 + \varepsilon > \varepsilon/2$ $\quad(\|n\| = k_j,\ j \ge 1)$. Therefore

$$|a_{(n)}| e^{(x_0,\lambda_{(n)})} \ge e^{\varepsilon\|\lambda_{(n)}\|/2} \ge 1 \quad (\|n\| = k_j.\ j \ge 1),$$

Finally, $\Pi^0_{(\alpha_0+\varepsilon)e_1} \subset \mathbb{R}^p \setminus G_\mu$. It remains to use the arbitrariness of $\varepsilon > 0$.

By the definition of α_0

$$\alpha_0 < \frac{-\ln|a_{(n)}|}{\|\lambda_{(n)}\|} + \varepsilon/2 \quad (\|n\| \ge k_0),$$

hence $|a_{(n)}| e^{\left(\alpha_0^* e_1, \lambda_{(n)}\right)} < \exp\{-\|\lambda_{(n)}\|\varepsilon/2\} \le 1$ $(\|n\| \ge k_0)$, where $\alpha_0^* := \alpha_0 - \varepsilon$ and $\varepsilon > 0$ is arbitrary. Choose $x_0 = \gamma\alpha_0^* e_1 - \delta$. By $\|n\| \ge k_0$ we get

$$|a_{(n)}| e^{(x_0,\lambda_{(n)})} = |a_{(n)}|^{1-\gamma} e^{-(\delta,\lambda_{(n)})} \Big(|a_{(n)}| e^{\left(\alpha_0^* e_1,\lambda_{(n)}\right)}\Big)^\gamma \le |a_{(n)}|^{1-\gamma} e^{-\left(\delta,\lambda_{(n)}\right)}.$$

By the assumptions we obtain that the series (1) converges absolutely at the point $x_0 = \gamma\alpha_0^* e_1 - \delta$. Therefore $\Pi_{\gamma\alpha_0^* e_1} \subset G_a + \delta$. It remains to use $\Pi_{\gamma\alpha_0^* e_1} = \gamma\Pi_{\alpha_0^* e_1}$ and arbitrariness of $\varepsilon > 0$.

Let us suppose that condition (15) holds. Put $\alpha^* = \alpha - e_1\varepsilon$, where $\varepsilon > 0$ and $\alpha \in G_1^*$ are arbitrary. Similar as above, by the definition of an upper limit in G_1^* we get

$$\ln|a_{(n)}| + (\alpha, \lambda_{(n)}) < \|\lambda_{(n)}\|\varepsilon/2 \Longrightarrow |a_{(n)}|\exp(\alpha^*, \lambda_{(n)}) < \exp(-\|\lambda_{(n)}\|\varepsilon/2) \le 1 \quad (\|n\| \ge k_0).$$

Choose $x_0 = \gamma\alpha^* - \delta$. Then

$$|a_{(n)}|e^{(x_0,\lambda_{(n)})} = |a_{(n)}|^{1-\gamma}e^{-(\delta,\lambda_{(n)})}\Big(|a_{(n)}|e^{\left(\alpha^*,\lambda_{(n)}\right)}\Big)^{\gamma} \le |a_{(n)}|^{1-\gamma}e^{-\left(\delta,\lambda_{(n)}\right)} \quad (\|n\| \ge k_0).$$

So we deduce that series (1) converges absolutely at the point $x_0 = \gamma\alpha^* - \delta = \gamma(\alpha - e_1\varepsilon) - \delta$. Then $\Pi_{x_0} = \big(\gamma\Pi_\alpha - \gamma e_1\varepsilon - \delta\big) \subset G_a$ and $\gamma\Pi_\alpha \subset G_a + \gamma e_1\varepsilon + \delta$. It remains to remark arbitrariness of $\varepsilon > 0$ and $\alpha \in G_1^*$. Therefore,

$$\gamma G_1 = \bigcup_{\alpha\in G_1^*} \gamma\Pi_\alpha \subset G_a + \delta.$$

Using Proposition 7 we obtain $G_\mu = G_1$. □

Similar to [15] in the case of Dirichlet series with exponents Λ^1, from Proposition 9 we get such a result.

Proposition 10. *Let $F \in \mathcal{D}^p(\Lambda^p)$ of the form (1). If $\ln\|n\| = o\big(\ln|a_{(n)}|\big)$ $(\|n\| \to +\infty)$, then $\Pi_{\alpha_0 e_1} \subset G_a$ and $G_c = G_1 = G_\mu = G_a$.*

Choosing in Proposition 9 $\delta = 0$, $\gamma = 1 - \gamma_1$, we get the following corollary.

Proposition 11. *Let $F \in \mathcal{D}^p(\Lambda^p)$ be of the form (1), $\underline{\lim}_{\|n\|\to+\infty} \|\lambda_{(n)}\| := \lambda > 0$. For some $\gamma_1 \le 1$*

$$\sum_{\|n\|=0}^{+\infty} |a_{(n)}|^{\gamma_1} < +\infty. \tag{16}$$

Then

$$G_a \supset (1-\gamma_1)G_1 = (1-\gamma_1)G_\mu \supset (1-\gamma_1)G_c. \tag{17}$$

Denote $\|\ln n\| = \ln n_1 + \ldots \ln n_p$,

$$h_1 = \overline{\lim_{\|n\|\to+\infty}} \frac{-\ln|a_{(n)}|}{\|\ln n\|}, \quad h_2 = \underline{\lim}_{\|n\|\to+\infty} \frac{-\ln|a_{(n)}|}{\|\ln n\|}, \quad \gamma_1 = \begin{cases} \frac{1}{h_1}, & \text{if} \quad h_1 \in [-\infty, 0), \\ \frac{1}{h_2}, & \text{if} \quad h_2 \in (1, +\infty]. \end{cases}$$

Proposition 12. *Let $F \in \mathcal{D}^p(\Lambda^p)$ and be of the form (1). If $h_1 \in [-\infty, 0)$ or $h_2 \in (1, +\infty]$, then the relation $\gamma G_c \subset \gamma G_1 = \gamma G_\mu \subset G_a + \delta$ holds with $\gamma = 1 - \gamma_1$. Here we assume that $1/\infty = 0$.*

Proof of Proposition 12. Firstly, suppose $h_1 < 0$. We check that Proposition 11 can be applied with

$$\gamma_1 = h_0 = 1/(h_1(1-\varepsilon/2)), \tag{18}$$

where $0 < \varepsilon < 1$ is arbitrary. Remark that in the case $h_1 = -\infty$, in equality (18) $h_1 < 0$ is arbitrary. By hypothesis for each $\varepsilon \in (0,1)$ there exists k_0 such that for all $\|n\| \geq k_0(\omega)$

$$\frac{\ln |a_{(n)}|}{\| \ln n\|} > -h_1(1-\varepsilon).$$

Then by $h_0 = 1/(h_1(1-\varepsilon/2))$, $0 < \varepsilon < 1$, because $h_0(\omega) < 0$ we have

$$|a_{(n)}|^{h_0} \leq \exp\Big\{ -h_0 h_1(1-\varepsilon)\| \ln n\| \Big\} = \exp\Big\{ -\frac{1-\varepsilon}{1-\varepsilon/2}\| \ln n\| \Big\},$$

but $\frac{1-\varepsilon}{1-\varepsilon/2} > 1$. Therefore, by Proposition 11 we get (17), where $\gamma_1 = 1/(h_1(1-\varepsilon/2))$. It remains to use the arbitrariness of $\varepsilon \in (0,1)$.

In the case $h_2 > 1$ we choose $\gamma_1 = h_0 + 2\varepsilon$, $h_0 = 1/h_2$ ($h_0 = 0$ in the case $h_2 = +\infty$). By the definition of a lower limit with $h_0 + \varepsilon$. Indeed by the assumption, for any $\varepsilon > 0$ with $h_0 + \varepsilon < 1$ there exists k_0 such that for all $\|n\| \geq k_0$

$$\frac{\ln |a_{(n)}|}{\| \ln n\|} < \frac{-1}{h_0+\varepsilon}.$$

Then

$$|a_{(n)}|^{h_0+2\varepsilon} = \exp\Big\{(h_0+2\varepsilon)\ln |a_{(n)}|\Big\} \leq \exp\Big\{ -\frac{h_0+2\varepsilon}{h_0+\varepsilon}\| \ln n\| \Big\}.$$

Therefore, by Proposition 11 we obtain (17), where $\gamma_1 = h_0 + 2\varepsilon$. It remains to use the arbitrariness of $\varepsilon \in (0,1)$. □

2. Dirichlet series with random coefficients. Let $\big(\Omega, \mathcal{A}, P\big)$ be a probability space. In this section we consider the Dirichlet series of the form

$$F(s) = F_\omega(s) = F(s,\omega) = \sum_{\|n\|=0}^{+\infty} f_{(n)}(\omega)\exp\{(\lambda_{(n)}, s)\}, \tag{19}$$

where $\big(f_{(n)}(\omega)\big)$ is a sequence of random variables $f_{(n)} \colon \Omega \to \mathbb{C}$ on the probability space $\big(\Omega, \mathcal{A}, P\big)$, and $\Lambda^p = (\lambda_{(n)})_{n\in\mathbb{Z}^p_+}$ is the same as the above vector-sequence with non-negative components. By $\mathcal{D}^p(\Lambda^p)$ denote the class of Dirichlet series of the form (19) such that $F_\omega \in D^p(\Lambda^p)$ a.s. (almost surely), i.e. for every fixed $\omega \in \Omega \setminus B$, $P(B) = 0$.

Everywhere in this section, we assume that the following condition is fulfilled:

$$\tau(\Lambda^p) := \varlimsup_{\|n\|\to\infty} \frac{\ln \|n\|}{\|\lambda_{(n)}\|} = 0.$$

From this condition it follows that $\varliminf\limits_{\|n\|\to+\infty} \|\lambda_{(n)}\| > 0$. Then by Proposition 2, $\partial G_a = \partial G_c = \partial G_\mu = G_1^*$ a.s., and by Proposition 5, $G_1^* \cap \big(\mathbb{R}_+ \setminus \{0\}\big)^p = G_0^* \cap \big(\mathbb{R}_+ \setminus \{0\}\big)^p$ a.s. Actually, $\sigma \in \partial G_a \cap \big(\mathbb{R}_+ \setminus \{0\}\big)^p$ a.s. if and only if

$$\varliminf_{\|n\|\to+\infty} \frac{-\ln |f_{(n)}(\omega)|}{(\sigma, \lambda_{(n)})} = 1 \quad \text{a.s.},$$

as well, by Proposition 6, $G_1^* \cap (-\infty, 0)^p = \widehat{G}_0^* \cap (-\infty, 0)^p$ a.s., in particular, $\sigma \in \partial G_a \cap (-\infty, 0)^p$ a.s. if and only if

$$\overline{\lim_{\|n\|\to+\infty}} \frac{-\ln|f_{(n)}(\omega)|}{(\sigma, \lambda_{(n)})} = 1 \quad \text{a.s.}$$

Theorem 1. *Let $F \in \mathcal{D}^p(\Lambda^p)$ be of the form (19). We assume that $\big(|f_{(n)}(\omega)|\big)$ is a sequence of pairwise independent random variables with the distribution functions $F_{(n)}(x) := P\{\omega : |f_{(n)}(\omega)| < x\},\ x \in \mathbb{R},\ (n) \in \mathbb{Z}_+^p$. The following assertions hold:*
a) *In order that $\sigma \in \partial G_a \cap \big(\mathbb{R}_+ \setminus \{0\}\big)^p$ a.s., it is necessary and sufficient that*

$$(\forall \varepsilon > 0)\colon \ \sum_{\|n\|=0}^{+\infty} \Big(1 - F_{(n)}\big(e^{(-1+\varepsilon)(\sigma,\lambda_{(n)})}\big)\Big) < +\infty \wedge \sum_{\|n\|=0}^{+\infty} \Big(1 - F_{(n)}\big(e^{(-1-\varepsilon)(\sigma,\lambda_{(n)})} + 0\big)\Big) = +\infty.$$

b) *In order that $\sigma \in \partial G_a \cap \big(-\infty, 0)^p$ a.s., it is necessary and sufficient that*

$$(\forall \varepsilon > 0)\colon \ \sum_{\|n\|=0}^{+\infty} \Big(1 - F_{(n)}\big(e^{(-1-\varepsilon)(\sigma,\lambda_{(n)})}\big)\Big) < +\infty \ \wedge \ \sum_{\|n\|=0}^{+\infty} \Big(1 - F_{(n)}\big(e^{(-1+\varepsilon)(\sigma,\lambda_{(n)})}\big)\Big) = +\infty.$$

c) *In order that $G_a = \mathbb{R}^p$ a.s., it is necessary and sufficient that*

$$(\forall \Delta > 0)\colon \ \sum_{\|n\|=0}^{+\infty} \big(1 - F_{(n)}\big(\exp(-\Delta\|\lambda_{(n)}\|)\big)\big) < +\infty.$$

Proof of Theorem 1. **a)** *Necessity.* By the assumption, $(\exists B \in \mathcal{A}, P(B) = 1)(\forall \omega \in B)\colon$

$$\overline{\lim_{\|n\|\to+\infty}} \frac{\ln|f_{(n)}(\omega)|}{(\sigma, \lambda_{(n)})} = -1.$$

By the definition of $\overline{\lim}$, since $(\sigma, \lambda_{(n)}) \geq 0$ for $\|n\|$ large enough, we have

$$(\forall \omega \in B)(\forall \varepsilon > 0)(\exists k^*(\omega) \in \mathbb{N})(\forall \|n\| \geq k^*(\omega))\colon \ |f_{(n)}(\omega)| < e^{(-1+\varepsilon)(\sigma,\lambda_{(n)})},$$

Denote

$$A_{(n)} := \big\{\omega\colon |f_{(n)}(\omega)| \geq e^{(-1+\varepsilon)(\sigma,\lambda_{(n)})}\big\}.$$

Then, $B \subset C := \bigcup_{N=0}^{\infty} \bigcap_{\|n\|=N}^{\infty} \overline{A}_{(n)}$ and $P(C) = 1$; C is the event "$(A_{(n)})$ finitely often". The pairwise independence of the events $(A_{(n)})$ follows from the pairwise independence of the random variables $(|f_{(n)}(\omega)|)$. In this case, from the Refined Second Lemma Borel-Cantelli we obtain (see [21, 22], [23, p. 84])

$$\sum_{k=0}^{+\infty} P(A_{(n)}) < +\infty. \tag{20}$$

But,

$$P(A_{(n)}) = 1 - P(\overline{A}_{(n)}) = 1 - F_{(n)}\big(e^{(-1+\varepsilon)(\sigma,\lambda_{(n)})}\big).$$

Next, since for each $\omega \in B$ there exist a sequence $m_k \to +\infty$ and n such that $\|n\| = m_k$ and

$$\ln|f_{(n)}(\omega)| > (-1 - \varepsilon)(\sigma, \lambda_{(n)}),$$

therefore, $\omega \in \bigcap_{N=0}^{\infty} \bigcup_{\|n\|=N} A^1_{(n)} := C^1$, where $A^1_{(n)} = \{\omega\colon |f_{(n)}(\omega)| > e^{(-1-\varepsilon)(\sigma,\lambda_{(n)})}\}$. Then, $B \subset C^1$ and, thus, $P(C^1) = 1$; C^1 is the event "$(A^1_{(n)})$ infinitely often". Let us now prove that the following condition is fulfilled

$$\sum_{\|n\|=0}^{+\infty} \Big(1 - F_{(n)}\big(e^{(-1-\varepsilon)(\sigma,\lambda_{(n)})} + 0\big)\Big) = \sum_{\|n\|=0}^{+\infty} P(A^1_{(n)}) = +\infty. \tag{21}$$

Assume the contrary

$$\sum_{\|n\|=0}^{+\infty} P(A^1_{(n)}) < +\infty.$$

Then by the first part of the Borel-Cantelli Lemma $P(\overline{C}^1) = 1$. Thus, $P(C^1) = 0$ and we obtain the contradiction. Therefore, condition (21) holds.

Sufficiency. We keep the notation from the proof of the necessity. From the convergence of series (20) by the first part of the Borel-Cantelli Lemma we obtain that the probability of the event "$(A_{(n)})$ finitely often" is equal to 1, i.e. $P(C) = 1$. Therefore, for each $\omega \in C$ there exists $k_0(\omega)$ such that $\omega \in \overline{A}_{(n)}$ for all (n) satisfying $\|n\| \geq k_0(\omega)$. This implies that for all $\omega \in C$ and (n) such that $\|n\| \geq k_0(\omega)$ the inequality $|f_{(n)}(\omega)| < e^{(-1+\varepsilon)(\sigma,\lambda_{(n)})}$ holds. But, $(\sigma, \lambda_{(n)}) \geq 0$ for every $\|n\|$ large enough, therefore, for all $\omega \in C$ and (n) such that $\|n\| \geq k_1(\omega)$

$$\varliminf_{\|n\|\to+\infty} \frac{-\ln|f_{(n)}(\omega)|}{(\sigma, \lambda_{(n)})} \geq 1 - \varepsilon.$$

Similarly, from the assumption we have (21). Thus, by the Second Borel-Cantelli Lemma we obtain that the probability the event "$(A^1_{(n)})$ infinitely often" is equal 1. Therefore, for every $\omega \in C^1$, $P(C^1) = 1$, there exists a sequence $m_k \to +\infty$ such that $\omega \in A^1_{(n)}$ as $\|n\| = m_k$, i.e. $\ln|f_{(n)}(\omega)| > (-1-\varepsilon)(\sigma, \lambda_{(n)})$. Hence one has

$$\varliminf_{\|n\|\to+\infty} \frac{-\ln|f_{(n)}(\omega)|}{(\sigma, \lambda_{(n)})} \leq 1 + \varepsilon.$$

We put $C(\varepsilon) = C \cap C^1$, $\varepsilon \in \{\frac{1}{k}\colon k \in \mathbb{N}\}$. Thus, $P(\cap_{k\geq1} C(1/k)) = 1$ and, therefore,

$$\varliminf_{\|n\|\to+\infty} \frac{-\ln|f_{(n)}(\omega)|}{(\sigma, \lambda_{(n)})} = 1 \text{ for all } \omega \in \bigcap_{k\geq1} C(1/k).$$

b) By the assumption $(\exists B \in \mathcal{A}, P(B) = 1)(\forall \omega \in B)$:

$$\varlimsup_{\|n\|\to+\infty} \frac{-\ln|f_{(n)}(\omega)|}{(\sigma, \lambda_{(n)})} = 1.$$

Taking into account that $(\sigma, \lambda_{(n)}) \leq 0$ for all $\|n\|$ large enough, from the definition of $\varlimsup$ we have

$$(\forall \omega \in B)(\forall \varepsilon > 0)(\exists k^*(\omega) \in \mathbb{N})(\forall \|n\| \geq k^*(\omega))\colon\ |f_{(n)}(\omega)| < e^{(-1-\varepsilon)(\sigma,\lambda_{(n)})}.$$

We denote

$$A_{(n)} := \big\{\omega\colon |f_{(n)}(\omega)| \geq e^{(-1-\varepsilon)(\sigma,\lambda_{(n)})}\big\}.$$

Then, $B \subset C := \bigcup_{N=0}^{\infty} \bigcap_{\|n\|=N}^{\infty} \overline{A}_{(n)}$ and thus, $P(C) = 1$.

Similarly as in the proof of **a)**, for each $\omega\in B$ there exists a sequence $m_k\to+\infty$ such that for $\|n\|=m_k$

$$\ln|f_{(n)}(\omega)|>(-1+\varepsilon)(\sigma,\lambda_{(n)}).$$

Hence, $B\subset\bigcap_{N=0}^{\infty}\bigcup_{\|n\|=N}A^1_{(n)}:=C^1$, where $A^1_{(n)}=\{\omega\colon|f_{(n)}(\omega)|>e^{(-1+\varepsilon)(\sigma,\lambda_{(n)})}\}$. Then again $P(C^1)=1$.

Practically a verbatim repetition of the reasoning from the proof of **a)** gives

$$\sum_{\|n\|=0}^{+\infty}P(A_{(n)})<+\infty\ \wedge\ \sum_{\|n\|=0}^{+\infty}P(A^1_{(n)})=+\infty,$$

whence we get the required conclusion. The proof of the sufficiency also is almost a verbatim repetition of the reasoning from the proof of **a)**.
c) By Propositions 3 and 2,

$$G_a=\mathbb{R}^p\text{ a.s.}\iff\alpha_0^*(\omega):=\varliminf_{\|n\|\to+\infty}\frac{-\ln|f_{(n)}(\omega)|}{\|\lambda_{(n)}\|}=+\infty\text{ a.s.}$$

We denote $A_{(n)}=\{\omega:|f_{(n)}(\omega)|\ge\exp(-\Delta\|\lambda_{(n)}\|)\}$. Because,

$$\alpha_0^*(\omega)=+\infty\iff\ (\forall\Delta>0)(\exists k_0(\omega))(\forall n,\|n\|\ge k_0(\omega))\colon\quad-\ln|f_{(n)}(\omega)|>\Delta\|\lambda_{(n)}\|,$$

then as above $P(C)=1$ for $C=\bigcup_{N=0}^{\infty}\bigcap_{\|n\|=N}^{\infty}\overline{A}_{(n)}$. Using of the Second Borel-Cantelli Lemma gives $\sum P(A_{(n)})<+\infty$, thus,

$$\sum_{\|n\|=0}^{+\infty}\big(1-F\big(\exp(-\Delta\|\lambda_{(n)}\|)\big)\big)=\sum_{\|n\|=0}^{+\infty}P(A_{(n)})<+\infty.$$

Now we prove the converse implication. From the condition $\sum P(A_{(n)})<+\infty$ by the first part of the Borel-Cantelli Lemma, the probability of the event "$(A_{(n)})$ finitely often" is equal to 1. Hence, we have that for any $\Delta>0$ there exists $B=B(\Delta),\ P(B)=1$ such that for all $\omega\in B$ there exists $k_0(\omega)$ such that $n,\|n\|\ge k_0(\omega)$ the following inequality $-\ln|f_{(n)}(\omega)|>\Delta\|\lambda_{(n)}\|$ holds, thus

$$(\forall\omega\in B(\Delta))\colon\quad\varliminf_{\|n\|\to+\infty}\frac{-\ln|f_{(n)}|}{\|\lambda_{(n)}\|}\ge\Delta.\tag{22}$$

We consider now the sequence $\Delta_k=k$, $k\ge1$. By the proved, for every $k\ge1$ there exist $B(\Delta_k),\ P(B(\Delta_k))=1$, such that inequality (22) is fulfilled with $\Delta=k$. Therefore, for every $\omega\in B_0=\bigcap_{k=1}^{+\infty}B(\Delta_k)$ we have

$$(\forall k\ge1)\colon\quad\varliminf_{\|n\|\to+\infty}\frac{-\ln|f_{(n)}|}{\|\lambda_{(n)}\|}\ge k.$$

It follows that $\alpha_0^*(\omega)=+\infty$ for all $\omega\in B_0$. Is remains to note that the condition $(\forall k\ge1)\colon\ P(B(\Delta_k))=1$ implies that $P(B_0)=1$. ☐

3. Dirichlet series with random exponents. In this section we consider the Dirichlet series of the form

$$F(s) = F_\omega(s) = F(s,\omega) = \sum_{\|n\|=0}^{+\infty} a_{(n)} \exp\{(\lambda_{(n)}(\omega), s)\}, \tag{23}$$

where $(a_{(n)})$ is a sequence of complex numbers, and $\Lambda^p = (\lambda_{(n)}(\omega))_{n\in\mathbb{Z}_+^p}$, $\lambda_{(n)}(\omega)\colon \Omega \to \mathbb{R}_+^p$ is the random vector-sequence with non-negative random components (random variables) on the probability space $\big(\Omega, \mathcal{A}, P\big)$. By $\mathcal{D}_{ex}^p$ denote the class of Dirichlet series of the form (23) such that $F_\omega \in D^p(\Lambda^p)$ a.s.

Everywhere in this section, we assume that the following condition is fulfilled:

$$\ln\|n\| = o\big(\ln|a_{(n)}|\big) \quad (\|n\| \to +\infty).$$

In this section we prove similar theorem to assertion **c)** from Theorem 1 for Dirichlet series with random exponents

$$\Lambda^p = (\lambda_{(n)}(\omega))_{n\in\mathbb{Z}_+^p}$$

and deterministic coefficients $\mathbf{f} = (f_k)$, $f_k \in \mathbb{C}$, $k \geq 0$.

From Propositions 3, 7 and 10 we obtain following assertions.

Proposition 13. *(Prop. 3) Let $F \in \mathcal{D}_{ex}^p$ of the form* (23) *and* $\lambda(\omega) := \varliminf\limits_{\|n\|\to+\infty} \|\lambda_{(n)}(\omega)\| > 0$ *a.s. In order that $G_\mu = \mathbb{R}^p$ a.s., it is necessary and sufficient that*

$$\alpha_0(\omega) := \lim_{\|n\|\to+\infty} \frac{-\ln|a_{(n)}|}{\|\lambda_{(n)}(\omega)\|} = +\infty \quad \textit{a.s.}$$

(Prop. 7) If $F \in \mathcal{D}_{ex}^p$ and $\lambda(\omega) > 0$ a.s., then $G_\mu \cap \mathbb{R}^p = G_1 \cap \mathbb{R}^p$, $\mathbb{R}^p \cap \partial G_\mu = \mathbb{R}^p \cap G_1^*$ a.s.
(Prop. 10) Let $F \in \mathcal{D}_{ex}^p$. If $\ln\|n\| = o\big(\ln|a_{(n)}|\big)$ $(\|n\| \to +\infty)$, then $\Pi_{\alpha_0(\omega)e_1} \subset G_a$ a.s. and $G_c = G_1 = G_\mu = G_a$ a.s.

Theorem 2. *Let $f \in \mathcal{D}_{ex}^p$ and $\Lambda = \big(\lambda_{(n)}(\omega)\big)_{(n)\in\mathbb{Z}_+^p}$ be a sequence such that the sequence $\big(\|\lambda_{(n)}(\omega)\|\big)_{(n)\in\mathbb{Z}_+^p}$ is a sequence of pairwise independent random variables with distribution functions $F_{(n)}(x) := P\{\ \omega : \|\lambda_{(n)}(\omega)\| < x\}$, $x \in \mathbb{R}^p$, $(n) \in \mathbb{Z}_+^p$. In order that $G_a = \mathbb{R}^p$ a.s., it is necessary and sufficient that*

$$(\forall\varepsilon > 0)\colon \sum_{\|n\|=0}^{+\infty} \Big(1 - F_{(n)}\big(\exp(-\varepsilon\ln|a_{(n)}|)\big)\Big) < +\infty. \tag{24}$$

Proof of Theorem 2. By Proposition 13, $G_\mu = \mathbb{R}^p$ a.s. if and only if

$$\lim_{\|n\|\to+\infty} \frac{-\ln|a_{(n)}|}{\|\lambda_{(n)}(\omega)\|} = +\infty \quad \text{a.s.}$$

This is equivalent to the following statement: there exists $B \in \mathcal{A}$, $P(B) = 1$, such that for each $\omega \in B$, for any $\Delta > 0$ there exists $k_0(\omega)$ such that for all (n), $\|n\| \geq k_0$ we get $-\ln|a_{(n)}| > \Delta\|\lambda_{(n)}(\omega)\|$. For arbitrary $\varepsilon > 0$ we put $\Delta = 1/\varepsilon$. Denote $A_{(n)} := \Big\{\omega\colon \|\lambda_{(n)}(\omega)\| \geq -\varepsilon\ln|a_{(n)}|\Big\}$. It is clear, that $B \subset \overline{C} := \bigcup_{N=0}^\infty \bigcap_{\|n\|=N}^\infty \overline{A}_{(n)}$, hence $P(\overline{C}) = 1$, and $C=\bigcap_{N=0}^{\infty}\bigcup_{\|n\|=N}^{\infty}A_{(n)}$ is the event "$(A_{(n)})$ infinitely often", i.e. $\overline{C}$ is the event "$(A_{(n)})$ finitely often". From pairwise independence of random variables $(\|\lambda_{(n)}(\omega)\|$ follows pairwise independence of events $(A_{(n)})$. Therefore, by refined Second Borel-Cantelli Lemma ([23, p. 84])

$$\sum_{\|n\|=0}^{+\infty}\big(1-F_{(n)}\big(|a_{(n)}|^{-\varepsilon}\big)\big)=\sum_{\|n\|=0}^{+\infty}P(A_{(n)})<+\infty.$$

If condition (24) is fullfilled then

$$\sum_{\|n\|=0}^{+\infty}P(A_{(n)})=\sum_{\|n\|=0}^{+\infty}\big(1-F_{(n)}\big(|a_{(n)}|^{-\varepsilon}\big)\big)<+\infty,$$

and by of the first part of Borel-Cantelli Lemma, $P(\overline{C})=1$ for each $\varepsilon>0$. We put $\varepsilon=\varepsilon_k=1/k$. Then $P(\overline{C}_k)=1$ for each $k\geq 1$, where

$$\overline{C}_k:=\bigcup_{N=0}^{\infty}\bigcap_{\|n\|=N}^{\infty}\overline{A}_{(n)}^k,\quad A_{(n)}^k=\Big\{\omega\colon \|\lambda_{(n)}(\omega)\|\geq -\varepsilon_k\ln|a_{(n)}|\Big\}.$$

Denote $B=\bigcap_{k=1}^{+\infty}\overline{C}_k$. Then $P(B)=1$ and for every $\omega\in B$ there exists $k_0(\omega)$ such that for all (n), $\|n\|\geq k_0(\omega)$ and for all $k\geq 1$ we have

$$\|\lambda_{(n)}(\omega)\|<-\varepsilon_k\ln|a_{(n)}|.$$

hence, $\varliminf\limits_{\|n\|\to+\infty}\frac{-\ln|a_{(n)}|}{\|\lambda_{(n)}(\omega)\|}\geq k$. Therefore, $\lim\limits_{\|n\|\to+\infty}\frac{-\ln|a_{(n)}|}{\|\lambda_{(n)}(\omega)\|}=+\infty$ and $G_\mu=\mathbb{R}^p$ a.s. □

REFERENCES

1. T. Kojima, *On the double Dirichlet's series*, Sc. Reports Tôhoku Imperial Univ. (1), Math., Phys., Chem., **9** (1920), 351–340.
2. V.P. Gromov, *Multiple series of Dirichlet polynomials*, Sib. Math. J., **10** (1969), №3, 374–386, English transl. from Sibirski Matematicheski Zhurnal, **10** (1969), №3, 522–536.
3. V.P. Gromov, *To the theory of multiple Dirichlet series*, Izv. AN ArmSSR. Mat., **5** (1970), №5, 449–457. (in Russian)
4. V.P. Gromov, *To the theory of multiple Dirichlet series*, Izv. AN ArmSSR. Mat., **7** (1970), №2, 90–103. (in Russian)
5. A.I. Janušauskas, *Double Dirichlet series*, Lit. Mat. Sb., **18** (1978), №3, 201–211, English transl. in Lit. Math. J., **18** (1978), №3, 445-–452.
6. A.I. Janušauskas, *Properties of conjugate abscissas of convergence of double Dirichleet series*, Lit. Mat. Sb., **19** (1979), №1, 213–228, English transl. in Lit. Math. J. **19** (1979), №1, 152–164.
7. O.Yu. Zadorozhna, O.M. Mulyava, *On the conjugate abscissas convergence of the double Dirichlet series*, Mat. Stud., **28** (2007), №1, 29–35. (in Ukrainian)
8. O.Yu. Zadorozhna, O.B. Skaskiv, *On the conjugate abscissas convergence of multiple Dirichlet series*, Carpathian Mathematical Publications, **1** (2009), №2, 152–160. (in Ukrainian)
9. O.Yu. Zadorozhna, O.B. Skaskiv, *On the domains of convergence of the double Dirichlet series*, Mat. Stud., **32** (2009), №1, 81–86. (in Ukrainian)
10. O.B. Skaskiv, O.Yu. Zadorozhna, *On domains of convergence of multiple random Dirichlet series*, Mat. Stud., **36** (2011), №1, 51–57.

11. A.O. Kuryliak, O.B. Skaskiv, N.Yu. Stasiv, *Abscissas convergence of the random multiple Dirichlet series*, Precarpatian visn. NTSh. Chyslo, (2018), №1, (42), 1–11. (in Ukrainian)
12. O.M. Mulyava, *On the convergence abscissa of a Dirichlet series*, Mat. Stud., **9** (1998), №2, 171–176. (in Ukrainian)
13. O.Yu. Zadorozhna, O.B. Skaskiv, *Elementary remarks about abscissas of convergence of the Laplace-Stieltjes integrals*, Bukovyn. Mat. Zhurn., **1** (2013), №3–4, 45–50. (in Ukrainian)
14. O.B. Skaskiv, A.I. Bandura, Asymptotic estimates of positive integrals and entire functions, Lviv–Ivano-Frankivsk: Goliney, 2015, 108 p.
15. Skaskiv O.B., Stasiv N.Yu. *Abscissas of convergence of Dirichlet series with random exponents*, Visn. Lviv. Univ. Ser. mech.-mat., **84** (2017), 76–91. (in Ukrainian)
16. Shapovalovska L.O., Skaskiv O.B. *On the radius of convergence of random gap power series*, Int. Journal of Math. Analysis, **9** (2015), №38, 1889–1893.
17. Skaskiv O.B., Shapovalovska L.O. *On the abscissas of convergence of random Dirichlet series*, Bukovyn. Mat. Zhurn., **3** (2015), №1, 110–114. (in Ukrainian)
18. Kuryliak A.O., Skaskiv O.B., Stasiv O.Yu. *On the abscissas of convergence of Dirichlet series with random pairwise independent exponents*, ArXiv:1703.03280v1[math.CV] 12 Mar 2017, 12 p.
19. Kuryliak A.O., Skaskiv O.B., Stasiv O.Yu. *On the convergence of Dirichlet series with random exponents*, Int. Journal of Appl. Math., **30** (2017), №3, 229–238.
20. Kuryliak A.O., Skaskiv O.B., Stasiv N.Yu. *On the abscissas of convergence of Dirichlet series with the random exponents and coefficients*, Bukovyn. Mat. Zhurn., **5** (2017), №3–4, 90—-97.
21. Erdős P., Rényi A. *On Cantor's series with convergent* $\sum 1/q_n$, Ann. Univ. Sci. Budapest Eőtvős. Sect. Math., **2** (1959), 93–109.
22. Petrov V.V., Sums of idependent random variables, New York: Springer, 1975.
23. Billingsley P., Probability and measure, New York: Wiley, 1986.

Ivan Franko National University of Lviv
Lviv, Ukraine
andriykuryliak@gmail.com
olskask@gmail.com
n-stas@ukr.net

Received 5.12.2017
Revised 26.04.2018