УДК 519.1

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# METRICALLY RAMSEY ULTRAFILTERS

I. V. Protasov, K. D. Protasova. *Metrically Ramsey ultrafilters*, Mat. Stud. **49** (2018), 115–121.

Given a metric space $(X, d)$, we say that a mapping $\chi\colon [X]^2 \longrightarrow \{0, 1\}$ is an isometric coloring if $d(x, y) = d(z, t)$ implies $\chi(\{x, y\}) = \chi(\{z, t\})$. A free ultrafilter $\mathcal{U}$ on an infinite metric space $(X, d)$ is called metrically Ramsey if, for every isometric coloring $\chi$ of $[X]^2$, there is a member $U \in \mathcal{U}$ such that the set $[U]^2$ is $\chi$-monochrome. We prove that each infinite ultrametric space $(X, d)$ has a countable subset $Y$ such that each free ultrafilter $\mathcal{U}$ on $X$ satisfying $Y \in \mathcal{U}$ is metrically Ramsey. On the other hand, it is an open question whether every metrically Ramsey ultrafilter on the natural numbers $\mathbb{N}$ with the metric $|x - y|$ is a Ramsey ultrafilter. We prove that every metrically Ramsey ultrafilter $\mathcal{U}$ on $\mathbb{N}$ has a member with no arithmetic progression of length 2, and if $\mathcal{U}$ has a thin member then there is a mapping $f\colon \mathbb{N} \longrightarrow \omega$ such that $f(\mathcal{U})$ is a Ramsey ultrafilter.

*For any finite coloring of the set* $[\mathbb{N}]^2$ *of edges of the complete graph on the set* $\mathbb{N}$ *of natural numbers, there exists an infinite subset* $A \subseteq \mathbb{N}$ *such that the set* $[A]^2$ *is monochrome.*

This elegant statement is a graph version of Ramsey theorem, one of the milestone of *Ramsey Theory*. For history (with exposition of the original paper of Frank Ramsey) and foundations of this branch of *Combinatorics*, see [2]. Some isometric versions of Ramsey theorem can be found in [7], [8].

We recall that a family $\mathfrak{F}$ of subsets of a set $X$ is a *filter* if $\varnothing \notin \mathfrak{F}$, $\mathfrak{F}$ is closed under finite intersections and if $A \in \mathfrak{F}$ and $A \subseteq B$ then $B \in \mathfrak{F}$. The family of all filters on $X$ is ordered by inclusion $\subseteq$, and a filter maximal in this ordering is called an *ultrafilter*. A filter $\mathcal{U}$ is an ultrafilter if and only if, for any finite partition $\mathcal{P}$ of $X$, there exists $A \in \mathcal{P}$ such that $A \in \mathcal{U}$. An ultrafilter $\mathcal{U}$ is called *free* if $\bigcap \mathcal{U} = \varnothing$.

Is there a free ultrafilter $\mathcal{U}$ of $\mathbb{N}$ such that, for each coloring $\chi\colon [\mathbb{N}]^2 \longrightarrow \{0, 1\}$, there exists $A \in \mathcal{U}$ such that $[A]^2$ is $\chi$-monochrome? These ultrafilters are called *Ramsey ultrafilters*. The question can not be answered in the system ZFC of axioms of *Set Theory* without additional set-theoretical assumptions. If we accept the Continuum Hypothesis, the answer is positive. On the other hand, there are models of ZFC without Ramsey ultrafilters.

Let $X$ be an infinite set and let $\mathfrak{F}$ be some family of $\{0, 1\}$-colorings of the set $[X]^2$ of all two-element subsets of $X$. We say that a free ultrafilter $\mathcal{U}$ on $X$ is *Ramsey with respect to* $\mathfrak{F}$ if, for any coloring $\chi \in \mathfrak{F}$, there exists $U \in \mathcal{U}$ such that $[U]^2$ is $\chi$-monochrome. In the case in which $\mathfrak{F}$ is the family of all $\{0, 1\}$-colorings of $[X]^2$, we get above definition of Ramsey ultrafilters. It is well-known that $\mathcal{U}$ is a Ramsey ultrafilter if and only if $\mathcal{U}$ is *selective*, i.e. for every partition $\mathcal{P}$ of $X$ either $P \in \mathcal{U}$ for some $P \in \mathcal{P}$ or there exists $U \in \mathcal{U}$ such that $|U \bigcap P| \leq 1$ for each $P \in \mathcal{P}$.

2010 *Mathematics Subject Classification:* 05D10.
*Keywords:* selective ultrafilter; metrically Ramsey ultrafilter; ultrametric space.
doi:10.15330/ms.49.2.115-121

Given a metric space $(X, d)$, we say that a mapping $\chi\colon [X]^2 \longrightarrow \{0,1\}$ is an *isometric coloring* if $d(x,y) = d(z,t)$ implies $\chi(\{x,y\}) = \chi(\{z,t\})$. We note that every isometric coloring $\chi$ is uniquely defined by some mapping $f\colon d(X,X) \setminus \{0\} \longrightarrow \{0,1\}$. Indeed, we take an arbitrary $r \in d(X,X) \setminus \{0\}$, choose $\{x,y\} \in [X]^2$ such that $d(x,y) = r$ and put $f(r) = \chi(\{x,y\})$. On the other hand, for $f\colon d(X,X) \setminus \{0\} \longrightarrow \{0,1\}$, we define $\chi$ by $\chi(\{x,y\}) = f(d(x,y))$.

We say that a free ultrafilter on an infinite metric space $(X,d)$ is *metrically Ramsey* if $\mathcal{U}$ is Ramsey with respect to all isometric colorings of $[X]^2$.

Let $G$ be a group and let $X$ be a $G$-space with the action $(G,X) \longrightarrow X$, $(g,x) \longmapsto gx$. A coloring $\chi\colon [X]^2 \longrightarrow \{0,1\}$ is called *$G$-invariant* if $\chi(\{x,y\}) = \chi(\{gx,gy\})$ for all $\{x,y\} \in [X]^2$ and $g \in G$. A free ultrafilter $\mathcal{U}$ of $X$ is called *$G$-Ramsey* if $\mathcal{U}$ is Ramsey with respect to the family of all $G$-invariant colorings of $[X]^2$.

We consider the special case: $X$ is a metric space and $G$ is a group of isometries of $X$. Clearly, every isometric coloring of $[X]^2$ is $G$-invariant. If $G$ is metrically 2-transitive (if $d(x,y) = d(z,t)$ then there is $g \in G$ such that $g\{x,y\} = \{z,t\}$) then every $G$-invariant coloring of $[X]^2$ is an isometric coloring.

We take the group $\mathbb{Z}$ of integers, put $X = \mathbb{Z}$ and consider the action $\mathbb{Z}$ on $X$ by $(g,x) = g + x$. *Is every $\mathbb{Z}$-Ramsey ultrafilter selective?* This question appeared in [5] and, to our knowledge, remains open. We endow $\mathbb{Z}$ with the metric $d(x,y) = |x-y|$. By above paragraph an ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ is $\mathbb{Z}$-Ramsey if and only if $\mathcal{U}$ is metrically Ramsey. *Is every metrically Ramsey ultrafilter on $\mathbb{Z}$ selective?* This is an equivalent form of the above question. The case of $\mathbb{Z}$ evidently equivalent to the case of $\mathbb{N}$.

Surprisingly or not, the case of ultrametric spaces is cardinally different and much more easy to explore. We recall that a metric $d$ is an *ultrametric* if $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ for all $x,y,z \in X$. We prove that every infinite ultrametric space $X$ has a countable subset $Y$ such that any ultrafilter $\mathcal{U}$ on $X$ satisfying $Y \in \mathcal{U}$ is metrically Ramsey.

**1. Equidistance subsets.** We say that a subset $Y$ of a metric space $(X,d)$ is an *equidistance subset* if there is $r \in \mathbb{R}^+$ such that $d(x,y) = r$ for all distinct $x,y \in Y$. If $Y$ is an equidistance subset of $(X,d)$ then every free ultrafilter $\mathcal{U}$ on $X$ such that $Y \in \mathcal{U}$ is metrically Ramsey.

**Proposition 1.** *Every infinite metric space with finite scale $d(X,X)$, $d(X,X) = \{d(x,y)\colon x,y \in X\}$ has a countable equidistance subset.*

*Proof.* We define a coloring $\chi\colon [X]^2 \longrightarrow d(X,X)$ by $\chi(\{x,y\}) = d(x,y)$ and apply the classical Ramsey theorem [2, p.16]. □

For an ultrametric space $(X,d)$ and $r \in d(X,X)$, we use the equivalence $\sim_r$ defined by $x \sim_r y$ if and only if $d(x,y) \leq r$. Then $X$ is partitioned into classes of $r$-equivalence $X = \bigcup_{\alpha<\lambda} X_\alpha$. If $x,y \in X_\alpha$ then $d(x,y) \leq r$. If $x \in X_\alpha$, $y \in X_\beta$ and $\alpha \neq \beta$ then $d(x,y) > r$.

**Proposition 2.** *Let $(X,\alpha)$ be an infinite ultrametric space with finite scale $d(X,X)$. If $|X|$ is regular then $X$ has an equidistance subset $Y$ of cardinality $|Y| = |X|$. If $|X|$ is singular then, for every cardinal $\kappa < |X|$, there is an equidistance subset of cardinality $\kappa$.*

*Proof.* Let $d(X,X) = \{0, r_1, \dots, r_n\}$, $0 < r_1 < \ldots < r_n$. We proceed by induction on $n$. For $n = 1$, the statement is evident: $Y = X$.

To make the inductive step from $n$ to $n+1$, we partition $X$ into classes of $r_n$-equivalence $X = \bigcup_{\alpha<\lambda} X_\alpha$. If $\lambda = |X|$ then we pick one element $y_\alpha \in X_\alpha$ and put $Y = \{y_\alpha\colon \alpha < |X|\}$, so $d(x,y)=r_{n+1}$ for all distinct $x,y\in Y$. Assume that $\lambda<|X|$. If $|X|$ is regular, we take $\alpha$ so that $|X_\alpha|=X$ and apply the inductive assumption to $X_\alpha$. If $|X|$ is singular then we take $X_\alpha$ such that $|X_\alpha|>\kappa$ and use the inductive assumption. □

**Remark 1.** If in Proposition 1.2 $|X|$ is singular, we cannot state that there is an equidistance subset $Y$ of cardinality $X$. We take an arbitrary singular cardinal $\kappa$, put $X=\kappa$ and partition $X=\bigcup_{\alpha<\lambda}X_\alpha$ so that $\lambda<\kappa$ and $|X_\alpha|<\kappa$ for each $\alpha<\kappa$. We define an ultrametric $d$ on $X$ by $d(x,x)=0$, $d(x,y)=1$ if $x,y\in X_\alpha$, $x\neq y$ and $d(x,y)=2$ if $x\in X_\alpha$, $y\in X_\beta$, $\alpha\neq\beta$. If $Y$ is an equidistance subset of $(X,\alpha)$ then either $Y\subseteq X_\alpha$ for some $\alpha<\lambda$, or $|Y\bigcap X_\alpha|\leq 1$ for each $\alpha<\lambda$. Hence, $|Y|<\kappa$.

**Proposition 3.** *For every infinite cardinal $\kappa$, there exists a metric space $(X,d)$ such that $|X|=2^\kappa$, $d(X,X)=\{0,1,2\}$ and every equidistance subset $Y$ of $(X,d)$ is of cardinality $|Y|\leq\kappa$.*

*Proof.* We put $X=2^\kappa$ and apply [4, Theorem 6.2] to define a coloring $\chi\colon[X]^2\longrightarrow\{1,2\}$ with no monochrome $[Z]^2$ for $|Z|>\kappa$. Then we define a metric $d$ on $X$ by $d(x,x)=0$ and $d(x,y)=\chi(\{x,y\})$ for all distinct $x,y\in X$. □

**Proposition 4.** *Let $(X,d)$ be a metric space with infinite scale $d(X,X)$, $|d(X,X)|=\kappa$. If $|X|\geq(2^\kappa)^+$ then there is an equidistance subset $Y$ of $(X,d)$ such that $|Y|=\kappa^+$.*

*Proof.* We define a coloring $\chi\colon[X]^2\longrightarrow d(X,X)$ by $\chi(\{x,y\})=d(x,y)$ and apply the Erdös-Rado theorem [4, Theorem 6.4]. □

**Proposition 5.** *For every infinite metric space $(X,d)$, there exists an injective sequence $(x_n)_{n\in\omega}$ in $X$ such that one of the following conditions is satisfied:*

*(i) the sequence $(d(x_0,x_n))_{n\in\omega}$ is increasing;*

*(ii) the sequence $(d(x_0,x_n))_{n\in\omega}$ is decreasing;*

*(iii) for every $n\in\omega$ and all $i,j$, $i>n$, $j>n$, we have $d(x_n,x_i)=d(x_n,x_j)$.*

*Proof.* We assume that there exists $x_0\in X$ such that the set $d(x_0,X)$ is infinite, $d(x_0,X)=\{d(x_0,x)\colon x\in X\}$. We choose a countable subset $Y$ of $X$ such that $x_0\notin Y$ and $d(x_0,y)\neq d(x_0,z)$ for all distinct $y,z\in Y$. The set $d(x_0,Y)$ contains either increasing or decreasing sequence $(r_{n+1})_{n\in\omega}$. For each $n\in\omega$, we choose $x_{n+1}$ such that $d(x_0,x_{n+1})=r_n$. Then the sequence $(d(x_0,x_n))_{n\in\omega}$ satisfies either $(i)$ or $(ii)$.

In the alternative case, the set $d(x,X)$ is finite for each $x\in X$. We fix $x_0\in X$ and choose a countable subset $X_1$ such that $|d(x_0,X_1)|=1$. We pick $x_1\in X_1$ and choose a countable subset $X_2\subseteq X_1$ such that $|d(x_1,X_2)|=1$ and so on. After $\omega$ steps, we get the sequence $(x_n)_{n\in\omega}$ satisfying $(iii)$. □

**Proposition 6.** *Let $(X,d)$ be an infinite metric space and let $\{X_n\colon n\in\omega\}$ be a family of non-empty pairwise disjoint subsets of $X$ such that $d(X_i,X_j)\bigcap d(X_n,X_n)=\varnothing$ for all $n$ and distinct $i,j$. Let $\mathcal{U}$ be a metrically Ramsey ultrafilter on $X$ such that $\bigcup_{n<\omega}X_n\in\mathcal{U}$ and $X_n\notin\mathcal{U}$ for each $n<\omega$. Then the following statements hold:*

*(i) there exists $U\in\mathcal{U}$ such that $|U\bigcap X_n|\leq 1$ for each $n<\omega$;*

*(ii) if $d(X_i,X_j)\bigcap d(X_k,X_l)=\varnothing$ for all distinct $\{i,j\},\{k,l\}\in[\omega]^2$ then there is a mapping $\varphi\colon X\longrightarrow\omega$ such that the ultrafilter $\varphi(\mathcal{U})$ is selective;*

*(iii) if for each $n < \omega$ there exists $m > n$, $|X_m| > n$, then there exists an ultrafilter $\mathcal{V}$ on $X$ such that $\bigcup_{n<\omega} X_n \in \mathcal{V}$ and $\mathcal{V}$ is not metrically Ramsey.*

*Proof.* $(i)$ By the assumption, the sets $A = \bigcup_{n<\omega} d(X_n, X_n)$ and $B = \bigcup_{i\neq j} d(X_i, X_j)$ are disjoint. We take an arbitrary mapping $f\colon \mathbb{R}^+ \longrightarrow \{0,1\}$ such that $f|_A \equiv 0$, $f|_B \equiv 1$, and consider the isometric coloring $\chi$ of $[X]^2$ defined by $f$. Since $\mathcal{U}$ is metrically Ramsey, there is $U \in \mathcal{U}$ such that $[U]^2$ is $\chi$-monochrome. Clearly, $|U \bigcap X_n| \leq 1$ for each $n < \omega$.
$(ii)$ We define $\varphi$ by the rule: if $x \in X_i$ then $\varphi(x) = i$, if $x \in X \setminus \bigcup_{n<\omega} X_n$, then $\varphi(x) = 0$. We take an arbitrary coloring $\chi'\colon [\omega]^2 \longrightarrow \{0,1\}$ and define a coloring $\chi\colon [\bigcup_{n<\omega} X_n]^2 \longrightarrow \{0,1\}$ as follows. If $x \in X_i$, $y \in X_j$, $i \neq j$ then $\chi(\{x,y\}) = \chi'(\{x,y\})$. If $x, y \in X_n$ then $\chi(\{x,y\}) = 0$. By the assumption, the coloring $\chi$ is isometric. We choose $U \in \mathcal{U}$ such that $U \subseteq \bigcup_{n<\omega} X_n$ and $[U]^2$ is $\chi$-monochrome and $|U \bigcap X_n| \leq 1$ for each $n < \omega$. Then $\varphi(U) \in \varphi(\mathcal{U})$ and $[\varphi(U)]$ is $\chi'$-monochrome, so $\varphi(\mathcal{U})$ is a Ramsey ultrafilter.
$(iii)$ We consider the family of all filters $\mathfrak{F}$ on $X$ such that $\bigcup_{n<\omega} X_n \in \mathfrak{F}$ and, for every $n \in \omega$ and $F \in \mathfrak{F}$, there exists $m \in \omega$ such that $|F \bigcap X_m| > n$. By the Zorn Lemma, this family has maximal by inclusion element $\mathcal{V}$. It is easy to verify that $\mathcal{V}$ is ultrafilter. By $(i)$, $\mathcal{V}$ is not metrically Ramsey. □

**2. The ultrametric case.**

**Proposition 7.** *For every infinite ultrametric space $(X, d)$, there exists a countable subset $Y$ of $X$ such that every free ultrafilter $\mathcal{U}$ on $X$ satisfying $Y \in \mathcal{U}$ is metrically Ramsey.*

*Proof.* We choose the sequence $(x_n)_{n\in\omega}$ given by Proposition 1.5, put $Y = \{x_n \colon n \in \omega\}$, fix an arbitrary mapping $f\colon \mathbb{R}^+ \longrightarrow \{0,1\}$ and take an arbitrary free ultrafilter $\mathcal{U}$ satisfying $Y \in \mathcal{U}$.

We assume that either $(i)$ or $(ii)$ of Proposition 1.5 holds for $(x_n)_{n\in\omega}$. We define a mapping $h\colon Y \longrightarrow \mathbb{R}^+$ by $h(x_n) = d(x_0, x_n)$ and choose $k \in \{0,1\}$ such that $(fh)^{-1}(k) \in \mathcal{U}$. Since $d$ is an ultrametric, in the case $(i)$ we have $d(x_i, x_n) = d(x_0, x_n)$ for all $i < n$, and in the case $(ii)$ we have $d(x_i, x_n) = d(x_0, x_i)$ for all $i < n$. In both cases, if $\{x_i, x_n\} \in [(fh)^{-1}(k)]^2$ then $f(d(x_i, x_n)) = k$.

If $(x_n)_{n\in\omega}$ satisfies $(iii)$ of Proposition 1.5 then we define a mapping $h\colon Y \longrightarrow \mathbb{R}^+$ by $h(x_n) = d(x_n, x_i)$, $i > n$ and repeat the above arguments. □

**Proposition 8.** *For a free ultrafilter $\mathcal{U}$ on an infinite set $X$, the following statements are equivalent:*

*(i) $\mathcal{U}$ is selective;*

*(ii) $\mathcal{U}$ is metrically Ramsey for each ultrametric $d$ on $X$ such that $d(X,X) = \{0,1,2\}$.*

*Proof.* The implication $(i) \Longrightarrow (ii)$ is evident. To show $(ii) \Longrightarrow (i)$, we assume that $\mathcal{U}$ is not selective and choose a partition $\mathcal{P}$ of $X$ such that $P \notin \mathcal{U}$ for each $P \in \mathcal{P}$, and for every $U \in \mathcal{U}$, there is $P \in \mathcal{P}$ such that $|P \bigcap U| > 1$. We define an ultrametric $d$ on $X$ by $d(x,x) = 0$, $d(x,y) = 1$ if $x \neq y$, $x, y \in P$ for some $P \in \mathcal{P}$, and $d(x,y) = 2$ if $x, y$ belong to different cells of the partition $\mathcal{P}$. We define a coloring $\chi\colon [X]^2 \longrightarrow \{1,2\}$ by $\chi(\{x,y\}) = d(x,y)$. Then the set $[U]^2$ is not $\chi$-monochrome for each $U \in \mathcal{U}$ so $\mathcal{U}$ is not metrically Ramsey and $(ii) \Longrightarrow (i)$. □

**Proposition 9.** *Let $(X, d)$ be an infinite ultrametric space with finite scale $d(X,X) = \{0, r_1, \ldots, r_n\}$, $0 < r_1 < \ldots < r_n$. Then the following statements are equivalent:*

*(i)* *every free ultrafilter on $(X, d)$ is metrically Ramsey;*

*(ii)* *for every $i \in \{1, \dots, n\}$, the partition $\mathcal{P}_i$ of $X$ into classes of $r_i$-equivalence has only finite number of infinite classes and there is $m \in \omega$ such that $|C| < m$ for each finite class $C$ from $\mathcal{P}_i$.*

*Proof.* $(i) \Longrightarrow (ii)$. If $\mathcal{P}_i$ has infinitely many infinite classes or the set $\{|C| \colon C$ is a finite class from $\mathcal{P}_i\}$ is infinite we apply Proposition 1.6$(iii)$ to get a free ultrafilter $\mathcal{V}$ on $X$ which is not metrically Ramsey.
$(ii) \Longrightarrow (i)$. We proceed on induction by $n$. For $n = 1$, the statement is evident.

To make the inductive step from $n$ to $n+1$, we take an arbitrary free ultrafilter $\mathcal{U}$ on $X$ and we consider the partition $\mathcal{P}_{n+1}$. Let $X_1, \dots, X_m$ be the set of all infinite classes from $\mathcal{P}_{n+1}$. If $X_1 \bigcup \dots \bigcup X_m \in \mathcal{U}$ then we take $X_i \in \mathcal{U}$ and apply the inductive assumption. If $X \setminus (X_1 \bigcup \dots \bigcup X_m) \in \mathcal{U}$ then we choose $U \in \mathcal{U}$ such that $U \subseteq X \setminus (X_1 \bigcup \dots \bigcup X_m)$ and $|U \bigcap C| \leq 1$ for each finite class $C \in \mathcal{P}_{n+1}$. Then $U$ is an equidistance set so $\mathcal{U}$ is metrically Ramsey. □

**3. The case of $\mathbb{N}$.**

**Proposition 10.** *Let $\mathcal{U}$ be a metrically Ramsey ultrafilter on $\mathbb{N}$ and let $f \colon \mathbb{N} \longrightarrow \mathbb{N}$ be a mapping such that $f(x) > x$ for each $x \in \mathbb{N}$. Then there exists a member $U \in \mathcal{U}$ having no subsets of the form $\{a, a+x, a+f(x)\}$. In particular ( for $f(x) = 2x$), some member of $\mathcal{U}$ has no arithmetic progressions of length 2.*

*Proof.* We consider a directed graph $\Gamma_f$ with the set of vertices $\mathbb{N}$ and the set of edges $\{(x, f(x)) \colon\ x \in \mathbb{N}\}$. Since $f(x) > x$, $\Gamma_f$ is the disjoint union of directed trees $T$ such that each vertex of $T$ has at most one input edge. Using this observation, it is easy to partition $\mathbb{N} = A_1 \bigcup A_2$ so that $f(A_1) \subseteq A_2$, $f(A_2) \subseteq A_1$.

The partition $\mathbb{N} = A_1 \bigcup A_2$ defines an isometric coloring $\chi \colon [\mathbb{N}]^2 \longrightarrow \{1, 2\}$ by $\chi(\{x, y\}) = i$ if and only if $d(x, y) \in A_i$. We take a subset $U \in \mathcal{U}$ such that the set $[U]^2$ is $\chi$-monochrome and assume that $\{a, a+x, a+f(x)\} \subset U$ for some $a, x \in \mathbb{N}$. We note that $d(a, a+x) = x$, $d(a, a+f(x)) = f(x)$, but $x$ and $f(x)$ belong to disjoint subsets $A_1, A_2$, so $\chi(\{a, a+x\}) \neq \chi(\{a, a+f(x)\})$ and we get a contradiction with the choice of $U$. □

Let $\mathcal{U}$ be metrically Ramsey ultrafilter on $\mathbb{N}$. Assume that there is $U \in \mathcal{U}$ such that $d(x, y) \neq d(z, t)$ for all distinct $\{x, y\}, \{z, t\} \in [U]^2$. Then every $\{0, 1\}$-coloring of $[U]^2$ can be extended to some isometric coloring of $[\mathbb{N}]^2$. Hence, $\mathcal{U}$ is a Ramsey ultrafilter.

We say that a subset $T = \{t_n \colon t_n < t_{n+1}, n < \omega\}$ of $\mathbb{N}$ is *thin* if $(t_{n+1} - t_n) \longrightarrow \infty$ as $n \longrightarrow \infty$.

**Proposition 11.** *If a metrically Ramsey ultrafilter $\mathcal{U}$ on $\mathbb{N}$ has a thin subset $T \in \mathcal{U}$ then there exists a mapping $\varphi \colon \mathbb{N} \longrightarrow \omega$ such that the ultrafilter $\varphi(\mathcal{U})$ is selective and $\varphi$ is finite-to-one on some member $U \in \mathcal{U}$.*

*Proof.* Let $T = \{t_n \colon t_n < t_{n+1}, n \in \omega\}$. Assume that we have chosen two sequences $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ in $T$ such that
(1) $a_n < b_n < a_{n+1} < b_{n+1}$ for each $n \in \omega$;
(2) $d([a_n, b_n) \cap T, [a_n, b_n) \cap T) \bigcap d([a_i, b_i) \cap T, [a_j, b_j) \cap T) = \varnothing$ for all $n$ and distinct $i, j$ ;
(3) $d([a_i, b_i) \cap T, [a_j, b_j) \cap T) \bigcap d([a_k, b_k) \cap T, [a_l, b_l) \cap T) = \varnothing$ for all distinct $\{i, j\}$, $\{k, l\} \in$

$[\omega]^2$;
(4) $d([b_n, a_{n+1}) \cap T, [b_n, a_{n+1}) \cap T) \bigcap d([b_i, a_{i+1}) \cap T, [b_j, a_{j+l}) \cap T) = \varnothing$ for all $n$ and distinct $i, j$ ;
(5) $d([b_i, a_{i+1}) \cap T, [b_j, a_{j+1}) \cap T) \bigcap d([b_k, a_{k+1}) \cap T, [b_l, a_{l+l}) \cap T) = \varnothing$ for all distinct $\{i, j\}$, $\{k, l\} \in [\omega]^2$.

We put $A = \bigcup_{n\in\omega}([a_n, b_n) \cap T)$, $B = \bigcup_{n\in\omega}([b_n, a_{n+1}) \cap T)$ and note that $A$ and $B$ with the corresponding partitions satisfy Proposition 1.6 (*ii*). Since either $A \in \mathcal{U}$ or $B \in \mathcal{U}$, Proposition 1.6 (*ii*) gives the mapping $\varphi\colon \mathbb{N} \longrightarrow \omega$ such that $\varphi(\mathcal{U})$ is selective. By the construction of $\varphi$, $\varphi$ is finite-to-one on $A$ or $B$ respectively.

It remains to construct $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$. We put $a_0 = t_0$, $b_0 = t_1$ and assume that we have chosen $a_0, b_0, \dots, a_n, b_n$. Since $T$ is thin, we can choose $a_{n+1} \in T$ so that $a_{n+1} > 2b_n$ and $|t - t'| > 2a_n$ for all distinct $t, t' \in T \setminus [1, a_{n+1})$. Then we choose $b_{n+1} \in T$ so that $b_{n+1} > 2a_{n+1}$ and $|t - t'| > 2b_n$ for all distinct $t, t' \in T \setminus [1, b_{n+1})$. After $\omega$ steps, we get the desired $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$. □

**4. Comments and open questions.** 1. In connection with Proposition 3.1, we mention [5, Corollary 2]: every metrically Ramsey ultrafilter on $\mathbb{N}$ has a member $U$ with no subsets of the form $\{x, y, x + y\}$, $x \neq y$.

In connection with Proposition 3.2, we ask

**Question 1.** *Let $\mathcal{U}$ be a metrically Ramsey ultrafilter on $\mathbb{N}$. Does there exist a thin subset $U \in \mathcal{U}$?*

**Question 2.** *Assume that a metrically Ramsey ultrafilter $\mathcal{U}$ on $\mathbb{N}$ has a thin member. Is $\mathcal{U}$ selective?*

2. Let $G$ be an Abelian group. A coloring $\chi\colon [G]^2 \longrightarrow \{0, 1\}$ is called a PS-*coloring* if, for $\{x, y\}, \{z, t\} \in [G]^2$, $x + y = z + t$ implies $\chi(\{x, y\}) = \chi(\{z, t\})$. A free ultrafilter $\mathcal{U}$ on $G$ is called a PS-*ultrafilter* if $\mathcal{U}$ is Ramsey with respect to all PS-*colorings* of $[G]^2$. The PS-ultrafilters were introduced and studied in [6], for an exposition of [6] see [1, Chapter 10].

If $G$ has a finite set $B(G) = \{g \in G\colon 2g = 0\}$ of elements of order 2 then every PS-ultrafilter on $G$ is selective. If $B(G)$ is infinite then, under Martin's Axiom, there is a non-selective PS-ultrafilter on $G$. If there exists PS-ultrafilter on some countable group $G$ then there is a $P$-point in $\omega^*$.

Now we consider the countable Boolean group $\mathbb{B}$, $B(\mathbb{B}) = \mathbb{B}$. We note that a coloring $\chi\colon [\mathbb{B}]^2 \longrightarrow \{0, 1\}$ is a PS-coloring if and only if $\chi$ is $\mathbb{B}$-invariant. Thus, a free ultrafilter $\mathcal{U}$ on $\mathbb{B}$ is a PS-ultrafilter if and only if $\mathcal{U}$ is $\mathbb{B}$-Ramsey. By the above paragraph, in the models of ZFC with no P-points in $\omega^*$, there are no $\mathbb{B}$-Ramsey ultrafilters. However, every strongly summable ultrafilter on $\mathbb{B}$ is a $PS$-ultrafilter. For strongly summable ultrafilters on Abelian groups, see [3].

On the other hand, $\mathbb{B}$ is the direct sum $\oplus_{n<\omega}\{0, 1\}_n$ of $\omega$ copies of $\mathbb{Z}_2 = \{0, 1\}$, and has the natural structure of the ultrametric space $(\mathbb{B}, d)$, where $d((x_n)_{n\in\omega}, (y_n)_{n\in\omega}) = \min\{m\colon x_n = y_n,\ n \geq m\}$. By Proposition 2.1, there are plenty metrically Ramsey ultrafilters on $\mathbb{B}$ in ZFC. Applying Proposition 1.6 (iii), we can find ultrafilters on $\mathbb{B}$ which are not metrically Ramsey.

3. By [4, Theorem 6.2], there is a coloring $\chi\colon [\mathbb{R}]^2 \longrightarrow \{0, 1\}$ such that if $X \subset \mathbb{R}$ and $[X]^2$ is $\chi$-monochrome then $|X| \leq \omega$.

We endow $\mathbb{R}$ with the natural metric $d(x, y) = |x - y|$ and we ask

**Question 4.3.** *Does there exist an isometric coloring* $\chi\colon [\mathbb{R}]^2 \longrightarrow \{0,1\}$ *such that if* $[X]^2$ *is monochrome then* $|X| \le \omega$*?*

We endow the Cantor cube $\{0,1\}^\omega$ with the standard metric and we ask

**Question 4.4.** *Does there exist an isometric coloring* $\chi\colon\ [\{0,1\}^\omega]^2 \to \{0,1\}$ *such that if* $[X]^2$ *is monochrome then* $|X| \le \omega$*?*

## REFERENCES


1. M. Filali, I. Protasov, *Ultrafilters and topologies on groups*, Math. Stud. Monogr. Ser., V.13, VNTL, Lviv, 2010.
2. R. Graham, B. Rotschild, J. Spencer, Ramsey Theory, Willey, New York, 1980.
3. N. Hindman, I. Protasov, D. Strauss, *Strongly summable ultrafilters on Abelian groups*, Mat. Stud., **10** (1998), №2, 121–132.
4. K. Kunen, Combinatories, in: Handbook in Mathematical Logic, V.90 (Studies in Logic on Foundations of Mathematics, J. Barwise (editor)), Elsevier, 1982.
5. O. Petrenko, I. Protasov, *Selective and Ramsey ultrafilters on G-spaces*, Notre Dame J. Formal Logic, **58** (2017), 453–459.
6. I. Protasov, *Ultrafilters and partitions of Abelian groups*, Ucr. Mat. Zh., **53** (2001), 85–93; translation in Ukr. Math. J. **53** (2001), 99–107.
7. I. Protasov, *Isometric versions of Ramsey theorem*, EUREKA, **65** (2017), 25–27.
8. I. Protasov, *On colorings and isometries*, Proc. Intern. Geometry Center, **10**, (2017), №2, 1–7; translation in Ukr. Math. J. **53** (2001), 99–107.



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*Received 27.05.2018*