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VECTOR-VALUED BIVARIATE ENTIRE FUNCTIONS OF BOUNDED INDEX SATISFYING A SYSTEM OF DIFFERENTIAL EQUATIONS

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The concept of complex valued bivariate entire functions of bounded index is extended to $\mathbb{C}^n$ valued bivariate entire functions by replacing the absolute value in the definition of an entire function of bounded index by the maximum of the absolute values of the components. If the components of a $\mathbb{C}^n$-valued bivariate entire function are of bounded index, then the function is also of bounded index. We present sufficient conditions providing index boundedness of bivariate vector-valued entire solutions of certain system of partial differential equations.

1. Introduction. If $f(z, w)$ is a bivariate entire function then at a point $(a, b) \in \mathbb{C}^2$ the function $f(z, w)$ has a bivariate Taylor expansion

$$f(z, w) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{kl}(z-a)^k(w-b)^l,$$

where

$$c_{kl} = \frac{1}{k!l!}\left[\frac{\partial^{k+l} f(z, w)}{\partial w^k \partial z^l}\right]_{z=a;w=b} = \frac{1}{k!l!} f^{(k,l)}(a, b).$$

If the bivariate series is absolutely convergent then $\lim_{k,l} |c_{kl}| = 0$.

Similar to Gross ([6]) we presented in [9–11] the following notion of bivariate entire function of bounded index.

Definition 1. A bivariate entire function f is said to be of *bounded index* provided that there exist integers M and N independent of z and w such that

$$\max_{0\le k\le M; 0\le l\le N}\left\{\frac{|f^{(k,l)}(z, w)|}{k!l!}\right\} \ge \frac{|f^{(i,j)}(z, w)|}{i!j!}$$

for all i and j.

We shall say that f is of *index* (M, N) if N ad M are the smallest integers for which above inequality holds. But it leads to the following question: What is the index of the function of f if the corresponding inequality holds for (N, M) and (M, N)? Thus, the index of the function is not uniquely defined. It is better to define an index of a bivariate function as

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height of the vector (N, M) i.e. $N+M$. The least such integer $N+M$ is called the index of the function f and is denoted by $N(f)$. A bivariate entire function which is not of bounded index is said to be of *unbounded index*. The primary definition of entire function of several variables of bounded index was introduced in [1]. Next, for simplicity the case of bivariate entire functions is considered in [14, 15].

In this paper, we are concerned with one possible extension of this concept to vector-valued bivariate functions. Let A^T be a transpose of a $n\times n$ matrix A. If $f_i(z,w)$ $i=1,2,\ldots,n$, are complex-valued bivariate entire functions, then

$$F(z,w) = (f_1(z,w), f_2(z,w), \ldots, f_n(z,w))^T,$$

is a $\mathbb{C}^n$-valued bivariate entire vector function and we write

$$\|F(z,w)\| = \max_{1\le i\le n}\{|f_i(z,w)|\}$$

and for $i,\ j\in\mathbb{Z}_+$

$$F^{(i,j)}(z,w) = \frac{\partial^{i+j}F(z,w)}{\partial w^i \partial z^j} = \left(\frac{\partial^{i+j}f_1(z,w)}{\partial w^i\partial z^j}, \frac{\partial^{i+j}f_2(z,w)}{\partial w^i\partial z^j}, \ldots, \frac{\partial^{i+j}f_n(z,w)}{\partial w^i\partial z^j}\right)^T.$$

2. Main Result.

Definition 2. A vector valued bivariate entire function $F(z,w)$ is said to be of *bounded index* provided that there exist integers M and N independent of z and w such that

$$\max_{0\le k\le M; 0\le l\le N}\left\{\frac{\|F^{(k,l)}(z,w)\|}{k!l!}\right\} \ge \frac{\|F^{(i,j)}(z,w)\|}{i!j!} \tag{1}$$

for all i and j.

The least such integer $N+M$ is called the *index of the vector function* F and is denoted by $N(F)$.

A concept of bounded index for vector-valued entire function of one variable (i.e. an entire curve) is considered in [7, 12, 13]. Particularly, they presented examples of entire curves of bounded index without all of their components being of bounded index. Also there are two papers about analytic curves of bounded l-index ([5, 16]). The mentioned papers are devoted to analytic theory of systems of ordinary differential equations. But we will study entire functions $F\colon \mathbb{C}^2\to\mathbb{C}^n$ and our definition of bounded index uses partial derivatives as in recent papers about **L**-index in joint variables ([2, 3]). Note that there is another multidimensional approach using directional derivatives in the definition so called functions of bounded **L**-index in direction ([4]). It is possible to use various norms in inequality (1). Particularly, instead the sup-norm we can consider the Euclidean norm. But M. Bordulyak and M. Sheremeta ([5]) proved that an analytic curve $F\colon\mathbb{D}_R\to C^n$ is of bounded l-index by the sup-norm if and only if F is of bounded l-index by the Euclidean norm, where $\mathbb{D}_R=\{z\in C\colon |z|<R\},\ R\in(0,\infty]$. Therefore, we will consider only the sup-norm.

Lemma 1. *If $f_1(z,w)$, $f_2(z,w)$, …, $f_n(z,w)$ are bivariate entire functions of bounded index with indexes $N(f_1)=N_1$, $N(f_2)=N_2$, …, $N(f_n)=N_n$ respectively, then*

$$F(z,w) = (f_1(z,w), f_2(z,w), \ldots, f_n(z,w))$$

is of bounded index M such that $M\le N=\max\{N_1,N_2,\ldots,N_n\}$.

Proof. For $p \in \{1, 2, \ldots, n\}$, we write

$$\frac{|f_p^{(i,j)}(z,w)|}{i!j!} \le \max_{0\le k+l\le N_p}\left\{\frac{|f_p^{(k,l)}(z,w)|}{k!l!}\right\} =$$
$$\max_{0\le k+l\le N}\left\{\frac{|f_p^{(k,l)}(z,w)|}{k!l!}\right\} \le \max_{0\le k+l\le N}\left\{\frac{\|F^{(k,l)}(z,w)\|}{k!l!}\right\}.$$

Therefore, (1) is satisfied. □

If $A = [a_{ij}]$ is an $n \times n$ matrix, we use the norm

$$\|A\| = \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|.$$

Theorem 1. *If a vector valued bivariate entire function $F\colon \mathbb{C}^2 \to \mathbb{C}^n$ is a solution of the following system of partial differential equations*

$$F^{(1,0)}(z,w) = A_{10}F(z,w), \quad F^{(0,1)}(z,w) = A_{01}F(z,w),$$

where A_{10}, A_{01} are constant $n \times n$ matrices, then $F(z,w)$ is an entire function of bounded index with index $N(F) \le P$ where

$$P = p_1 + q_1, \quad p_1 = \min\{p \in \mathbb{Z}_+\colon \|A_{10}\| \le (p+1)\}, \quad q_1 = \min\{p \in \mathbb{Z}_+\colon \|A_{01}\| \le (q+1)\}.$$

Proof. For all integers $p \ge 1$ and $q \ge 1$ and all $(z,w) \in \mathbb{C}^2$ we have

$$F^{(p_1+p,q)}(z,w) = A_{10}F^{(p_1+p-1,q)}(z,w), \quad F^{(p,q_1+q)}(z,w) = A_{01}F^{(p,q_1+q-1)}(z,w)$$

Therefore

$$\frac{\|F^{(p_1+p,q)}(z,w)\|}{(p_1+p)!q!} \le \frac{\|A_{10}\|}{(p_1+p)}\frac{\|F^{(p_1+p-1,q)}(z,w)\|}{(p_1+p-1)!q!},$$
$$\frac{\|F^{(p,q_1+q)}(z,w)\|}{p!(q_1+q)!} \le \frac{\|A_{01}\|}{(q_1+q)}\frac{\|F^{(p,q_1+q-1)}(z,w)\|}{p!(q+1+q-1)!}$$

and so

$$\frac{\left\|F^{(i,j)}(z,w)\right\|}{i!j!} \le \max_{0\le k+l\le p_1+q_1}\left\{\frac{\left\|F^{(k,l)}(z,w)\right\|}{k!l!}\right\}$$

for all $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$ and all $(z,w) \in \mathbb{C}^2$. □

In the proof of Theorem 2 we need the following lemma.

Lemma 2 ([14, 15])**.** *An entire bivariate function f is of bounded index if and only if there exist integers $M > 0, N > 0$ and a constant $C > 0$ such that*

$$\max_{0\le i\le M, 0\le j\le N}\{|f^{(M+1,i)}(z,w)|, |f^{(j,N+1)}(z,w)|\} \le C\max_{0\le i\le M; 0\le j\le N}\left\{\left|f^{(i,j)}(z,w)\right|\right\}$$

for all $(z,w) \in \mathbb{C}^2$.

Theorem 2. *Let $f(z,w)$ be a transcendental entire bivariate function and satisfying the system of partial differential equations*

$$P_{m0}(z,w)f^{(m,0)}(z,w)+\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}P_{ij}(z,w)f^{(i,j)}(z,w)=g(z,w),$$
$$Q_{0n}(z,w)f^{(0,n)}(z,w)+\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}Q_{ij}(z,w)f^{(i,j)}(z,w)=h(z,w),$$

where $P_{ij}(z,w), Q_{ij}(z,w), i=0,1,2,\dots,m; j=0,1,...,n$ are polynomials, $P_{m0}(z,w)$ and $Q_{0n}(z,w)$ are polynomials with separable variables of degree in each variable not less than degree of any $P_{ij}(z,w)$ and $Q_{ij}(z,w)$ in the same variable accordingly and $g(z,w)$, $h(z,w)$ are entire functions of bounded index. Then $f(z,w)$ is of bounded index.

Proof. If $g(z,w)$ is of bounded index then there exist integers $M\geq 0$ and $N\geq 0$ and constant $C>0$ such that

$$\max_{0\leq i\leq M, 0\leq j\leq N}\{|g^{(M+1,i)}(z,w)|,|g^{(j,N+1)}(z,w)|\}\leq C\max_{0\leq i\leq M;0\leq j\leq N}\left\{\left|g^{(i,j)}(z,w)\right|\right\}$$

for all $(z,w)\in\mathbb{C}^2$. For convenience we assume that $P_{i,n}(z,w)\equiv 0$ and $P_{m,j}(z,w)\equiv 0$ for i, $j\in\mathbb{N}$. Thus

$$\left|g^{(M+1,i)}(z,w)\right|=\left|\sum_{k=0}^{m}\sum_{j=0}^{n}\frac{\partial^{M+i+1}}{\partial z^{M+1}\partial w^{i}}P_{kj}(z,w)f^{(k,j)}(z,w)\right|=$$
$$=\left|\sum_{k=0}^{m}\sum_{j=0}^{n}\sum_{t=0}^{M+1}\sum_{l=0}^{i}\binom{M+1}{t}\binom{i}{l}P_{kj}^{(t,l)}(z,w)f^{(k+M+1-t,j+i-l)}(z,w)\right|\leq$$
$$\leq C\max_{0\leq k\leq M;0\leq j\leq n}\left\{\left|g^{(k,j)}(z,w)\right|\right\}.$$

Hence

$$\left|P_{00}(z,w)f^{(m+M+1,i)}(z,w)\right|\leq$$
$$\leq\left|\sum_{k=0}^{m}\sum_{j=0}^{n}\sum_{t=0}^{M+1}\sum_{l=0}^{i}\binom{M+1}{t}\binom{i}{l}P_{kj}^{(t,l)}(z,w)f^{(k+M+1-t,j+i-l)}(z,w)\right.$$
$$\left.+\sum_{t=1}^{M+1}\sum_{l=1}^{i}\binom{M+1}{t}\binom{i}{l}P_{m0}^{(t,l)}(z,w)f^{(m+M+1-t,n+i-l)}(z,w)\right|+$$
$$+C\max_{0\leq k\leq M;0\leq j\leq N}\left\{\left|\sum_{t=0}^{m}\sum_{l=0}^{n}\sum_{r=0}^{k}\sum_{s=0}^{j}\binom{k}{r}\binom{j}{s}P_{tl}^{(r,s)}(z,w)f^{(t+k-r,l+j-s)}(z,w)\right|\right\}\leq$$
$$\leq(C+1)(m+1)(n+1)(M+2)(M+1)!(N+2)(N+1)!\times$$
$$\times\max_{0\leq k\leq m;0\leq j\leq n;0\leq t\leq M+1;0\leq l\leq i}\{|P_{kj}^{(t,l)}(z,w)|\}\max_{0\leq i\leq M+m;0\leq j\leq i+n+1}\left\{\left|f^{(k,j)}(z,w)\right|\right\}.$$

Now since degree of P_{00} is greater than degree of P_{kj} in each variable, $k=0.,1,2,\dots,m$; $j=0,1,2,\dots,n$, there exist $K>0$, $R>0$ and $Q>0$ such that

$$|P_{m0}(z,w)|\geq K\left|P_{kj}^{(t,l)}(z,w)\right|\ \ k=0,1,2,\dots,M+1\text{ and }l=0,1,2,\dots,i$$

and for all $|z| \geq R$ and $|w| \geq Q$. Thus,

$$\left|f^{(m+M+1,n+i)}(z,w)\right| \leq$$
$$\leq (C+1)(m+1)(n+1)(M+2)(M+1)!(i+1)i! \max_{0\leq k\leq m+M;0\leq j\leq i-1+n} \left\{\left|f^{(k,j)}(z,w)\right|\right\}$$

for $|z| > R$ and $|w| > Q$. Hence, for

$$S_1 = (C+1)(m+1)(n+1)(M+2)(M+1)!(i+1)i!,\ T = m+M,\ U = n+i,$$
$$\left|f^{(T+1,U)}(z,w)\right| \leq S_1 \max_{0\leq k\leq T;0\leq j\leq U} \left\{\left|f^{(k,j)}(z,w)\right|\right\}$$

for $|z| \geq R$ and $|w| \geq Q$. Similarly, it can be proved that

$$\left|f^{(T,U+1)}(z,w)\right| \leq S_2 \max_{0\leq k\leq T;0\leq j\leq U} \left\{\left|f^{(k,j)}(z,w)\right|\right\}.$$

These inequalities hold for all $|z| < R$ and $|w| < Q$. Choosing $S \geq \max\{S_1, S_2\}$ by Lemma 2, we have f is bounded index. $\square$

Theorem 3. *If a vector valued bivariate entire function $F\colon \mathbb{C}^2 \to \mathbb{C}^n$ is a solution of system of partial differential equations*

$$P_{m0}(z,w)F^{(m,0)}(z,w) + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{ij}(z,w)F^{(i,j)}(z,w) = g(z,w),$$
$$Q_{0n}(z,w)F^{(0,n)}(z,w) + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} Q_{ij}(z,w)F^{(i,j)}(z,w) = h(z,w),$$

where $P_{ij}(z,w), Q_{ij}(z,w), i = 0,1,2,\ldots,m; j = 0,1,...,n$ are polynomials and $P_{m0}(z,w)$, $Q_{0n}(z,w)$ are polynomials with separable variables of degree in each variable not less than that of degree of any $P_{ij}(z,w)$ and $Q_{ij}(z,w)$ in each variable respectively and $g(z,w)$, $h(z,w)$ are entire functions of bounded index, then each component of $F(z,w)$ is of bounded index.

Proof. Since each component of $F(z,w)$ satisfies a system of partial differential equations of the form in Theorem 2, each component is of bounded index. $\square$

Theorem 4. *If a vector valued bivariate entire function $F\colon \mathbb{C}^2 \to \mathbb{C}^n$ is a solution of the following system of partial differential equation*

$$F^{(1,0)}(z,w) = A_{10}F(z,w) + R_1(z,w),\quad F^{(0,1)}(z,w) = A_{01}F(z,w) + R_2(z,w),$$

where A_{10}, A_{01} are constant $n \times n$ matrices, then $F(z,w)$ is an entire function of bounded index with index $N(F) \leq P$ where

$$P = p_1 + q_1,\quad p_1 = \min\{p \in \mathbb{Z}_+ \colon \|A_{10}\| \leq (p+1) \text{ and } p-1 \geq \max_{1\leq i\leq n} \deg r_{i1}\},$$
$$q_1 = \min\{q \in \mathbb{Z}_+ \colon \|A_{01}\| \leq (q+1) \text{ and } q-1 \geq \max_{1\leq i\leq n} \deg r_{i2}\},$$
$$R_1(z,w) = (r_{11}(z,w), r_{21}(z,w), \ldots, r_{n1}(z,w)),$$
$$R_2(z,w) = (r_{12}(z,w), r_{22}(z,w), \ldots, r_{n2}(z,w))$$

are vector valued polynomials.

Proof. Let p and q be two nonnegative integers such that $\|A_{10}\| \leq (p+1)$ and $p-1 \geq \max_{1\leq i\leq n} \deg r_{i1}$, $\|A_{01}\| \leq (q+1)$ and $q-1 \geq \max_{1\leq i\leq n} \deg r_{i2}$. Then

$$F^{(m+p,n)}(z,w) = A_{10}F^{(m+p-1,n)}(z,w) + R^{(m+p-1,n)}(z,w) = A_{10}F^{(m+p-1,n)}(z,w)$$

for all integers $m \geq 1$ and $n \geq 1$ and all $(z,w) \in \mathbb{C}^2$. Therefore,

$$\frac{\|F^{(m+p,n)}(z,w)\|}{(m+p)!n!} \leq \frac{\|A_{10}\|}{(m+p)} \frac{\|F^{(m+p-1,n)}(z,w)\|}{(m+p-1)!n!},$$

similarly

$$\frac{\|F^{(m,n+q)}(z,w)\|}{m!(n+q)!} \leq \frac{\|A_{01}\|}{(n+q)} \frac{\|F^{(m,n+q-1)}(z,w)\|}{m!(n+q-1)!},$$

and so

$$\frac{\|F^{(i,j)}(z,w)\|}{i!j!} \leq \max_{0\leq k+l\leq p+q} \left\{ \frac{\|F^{(k,l)}(z,w)\|}{k!l!} \right\}$$

for all $i \in \mathbb{Z}_+$, $j \in \mathbb{Z}_+$ and all $(z,w) \in \mathbb{C}^2$. □

Theorem 5. *If $F\colon \mathbb{C}^2 \to \mathbb{C}^n$ is an entire solution of the following system of partial differential equations*

$$F^{(1,0)}(z,w) = A_{10}F(z,w) + R_1(z,w), \quad F^{(0,1)}(z,w) = A_{01}F(z,w) + R_2(z.w),$$

where A_{10}, A_{01} are $n \times n$ matrices whose entries are rational bivariate functions which are bounded at infinity and $R_1(z,w)$, $R_2(z,w)$ are vector valued functions whose components are rational functions which are bounded at infinity, then $F(z,w)$ is a function of bounded index.

Proof. Let p_{0i} be the least common denominator of the a_{ij}'s and r_i's where $R_1 = (r_{11}, r_{21}, \ldots, r_{n1})$, a_{ij} are entries of A_{10}. Let $b_{ij} = p_{0i}s_{ij}$ and $p_i = p_{0i}r_i$. Then b_{ij} and p_i are bivariate polynomials of degree $\leq \deg p_{0i}$. Let $h = B_{10} = [b_{ij}]$ and $P = (p_1, p_2, \ldots, p_n)^T$, $P_0 = (p_{01}, \ldots, p_{0n})$ and so F is a solution of $F^{(1,0)}(z,w)P_0 = B_{11}F(z,w) + P$. Differentiating M times with respect to z and N times with respect to w where $M \geq h+1$ and $N \geq h+1$, we obtain

$$\sum_{k=0}^{M}\sum_{l=0}^{N} \binom{M}{k}\binom{N}{l} p_0^{(k,l)} F^{(M+1-k,N-l)}(z,w) =$$
$$= \sum_{k=0}^{M}\sum_{l=0}^{N} \binom{M}{k}\binom{N}{l} B_{11}^{(k,l)}(z,w) F^{(M-k,N-l)}(z,w)$$

which simplifies

$$p_0 F^{(M+1,N)}(z,w) = \sum_{k=0}^{h}\sum_{l=0}^{h} \binom{M}{k}\binom{N}{l} B_{11}^{(k,l)}(z,w) F^{(M-k,N-l)}(z,w) -$$
$$- \sum_{k=1}^{h}\sum_{l=1}^{h} \binom{M}{k}\binom{N}{l} p_0^{(k,l)}(z,w) F^{(M+1-k,N-l)}(z,w)$$

because $\deg b_{ij} \le h$. Therefore

$$\frac{F^{(M+1,N)}(z,w)}{(M+1)!N!} = \frac{1}{(M+1)} \sum_{k=0}^{h} \sum_{l=0}^{h} \frac{1}{k!l!} \frac{B_{11}^{(k,l)}(z,w)}{p_0} \frac{F^{(M-k,N-l)}(z,w)}{(M-k)!(N-l)!} - \tag{2}$$

$$-\sum_{k=1}^{h} \sum_{l=1}^{h} \frac{1}{k!l!} \left(1 - \frac{k}{M+1}\right) \left(1 - \frac{l}{N}\right) \frac{p_0^{(k,l)}(z,w)}{p_0(z,w)} \frac{F^{(M+1-k,N+1-l)}(z,w)}{(M+1-k)!(N-l)!}.$$

Since

$$\frac{p_0^{(k,l)}(z,w)}{p_0} \to 0 \quad |z| \to \infty, \quad |w| \to \infty \quad \text{for} \quad k = 1,2,\dots,h; l = 1,2,\dots,h$$

there exists $T > 0$ such that

$$\sum_{k=1}^{h} \sum_{l=1}^{h} \left| \frac{p_0^{(k,l)}}{p_0} \right| < \frac{1}{4} \quad \text{if} \quad |z| \ge T, \quad |w| \ge T. \tag{3}$$

The inequality $\deg b_{ij} \le h$ yields

$$\sum_{k=0}^{h} \sum_{l=0}^{h} \frac{1}{k!l!} \frac{\|B_{11}^{(k,l)}(z,w)\|}{|p_0(z,w)|} \le H$$

if $|z| \ge T,\ |w| \ge T$, where H is a constant. Choose $M_1 \ge h+1$ and $N_1 \ge h+1$ such that $\frac{1}{(M_1+1)} H < \frac{1}{4}$. Then

$$\frac{1}{(M+1)} \sum_{k=0}^{h} \sum_{l=0}^{h} \frac{1}{k!l!} \frac{\|B_{11}^{(k,l)}(z,w)\|}{|p_0(z,w)|} < \frac{1}{4} \tag{4}$$

if $M \ge M_1$, $N \ge N_1$, and $|z| \ge T,\ |w| \ge T$. If $M \ge M_1$, $N \ge N_1$, and $|z| \ge T,\ |w| \ge T$, we have

$$\frac{\|F^{(M+1,N)}(z,w)\|}{(M+1)!N!} \le \left[\frac{1}{(M+1)} \sum_{k=0}^{h} \sum_{l=0}^{h} \frac{1}{k!l!} \frac{\|B_{11}^{(k,l)}(z,w)\|}{p_0(z,w)} + \sum_{k=1}^{h} \sum_{l=1}^{h} \frac{1}{k!l!} \left| \frac{p_0^{(k,l)}(z,w)}{p_0} \right| \right] \times$$

$$\times \max_{0 \le i \le M_1; 0 \le j \le N_1} \frac{\|F^{(i,j)}\|}{i!j!}.$$

Therefore

$$\frac{\|F^{(M+1,N)}(z,w)\|}{(M+1)!N!} \le \max_{0 \le i \le M; 0 \le j \le N} \frac{\|F^{(i,j)}(z,w)\|}{i!j!}$$

for all $M \ge M_1$, $N \ge N_1$ and $|z| \ge T,\ |w| \ge T$. But every entire function is of bounded index on a compact set. Thus, there are integers M_2 and N_2 such that

$$\frac{|f^{(i,j)}(z,w)|}{i!j!} \le \max_{0 \le i \le M_2; 0 \le j \le N_2} \frac{|f^{(i,j)}(z,w)|}{i!j!}$$

for all $i = 0,1,2,\dots; j = 0,1,2,\dots; k = 0,1,2,\dots$ and all (z,w) such that $|z| \ge T,\ |w| \ge T$. Choosing $M \ge \max\{M_1, M_2\}$ and $N \ge \max\{N_1, N_2\}$, we have (1) for all $i = 0,1,2,\dots; j = 0,1,2,\dots$ and all $(z,w) \in \mathbb{C}^2$. □

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