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INTERTWINING MAPS FOR THE WEITZENBÖCK AND CHEBYSHEV DERIVATIONS

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The notions of Chebyshev derivations of the first and the second kind are presented. Explicit forms of the corresponding intertwining maps are found.

1. Introduction. Let us consider a family of polynomials $\{P_n(x)\}$, $\deg P_n(x) = n$ of one variable with rational coefficients. We are interested in finding some polynomial identities for these polynomials, i.e., the identities of the form

$$F(P_0(x), P_1(x), \ldots, P_n(x)) = 0,$$

where $F(x_0, x_1, \ldots, x_n)$ is a rational polynomial of $n+1$ variables. It is proved (see, for example, [1]) that the Bernoulli polynomials $B_n(x)$ satisfy the following identities

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i(x) B_{n-i}(x) + (n-1)B_n = 0, \quad n \in \mathbb{N}.$$

Thus, in this case we have

$$F(x_0, x_1, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_i x_{n-i} + (n-1)B_n,$$

where B_n is the Bernoulli number.

In this paper, we propose a general way of finding such polynomial identities for an arbitrary family $\{P_n(x)\}$ of polynomials.

The basic idea of this method is an application of already known polynomial identities which were found by the first author in [1] using methods of the classical invariant theory and the theory of locally nilpotent derivations to the Appel polynomials.

Let us consider a polynomial sequence $\{A_n(x)\}$, where for all $A_n(x)$ the following condition takes place $A_n'(x) = nA_{n-1}(x), n = 0, 1, 2, \ldots$.

The polynomials which possess the property are called the Appel type polynomials, see [2]. In particular, Bernoulli, Euler and Hermite polynomials are the Appel type polynomials.

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The locally nilpotent differentiation of algebra of the polynomials $\mathbb{Q}[x_0, x_1, \ldots, x_n]$ acts under the following rule $\mathcal{D}_A(x_i) = n x_{i-1}$ is closely connected to the Appel polynomials. The kernel of this derivation is called the *subalgebra* $\ker \mathcal{D}_A \subset \mathbb{Q}[x_0, x_1, \ldots, x_n]$ such that $\mathcal{D}_A(f) = 0$ for all polynomials $f \in \ker \mathcal{D}_A$.

The derivation $\mathcal{D}_A$ is so called *basis Weitzenböck derivation.* The kernel of Weitzenböck derivation is well-studied, it is isomorphic to SL_2-invariant algebra, see [3], [4]. As it was shown in [1], for every element of $\mathcal{D}_A$ there exists some polynomial identity for the Appel polynomials. A sequence of polynomials $\{P_n(x)\}$ can be associated with a derivation $\mathcal{D}_P$ of algebra $\mathbb{Q}[x_0, x_1, \ldots, x_n]$ which is defined as following

$$\mathcal{D}_P(x_i) = \alpha_{i,0} x_0 + \alpha_{i,1} x_1 + \cdots + \alpha_{i,i-1} x_{i-1}, i = 1, \ldots, n,$$

where the numbers $\alpha_{i,j}$ are found from the conditions

$$\frac{d}{dx} P_i(x) = \alpha_{i,0} P_0(x) + \alpha_{i,1} P_1(x) + \cdots + \alpha_{i,i-1} P_{i-1}(x), i = 1, \ldots, n.$$

Since the polynomial sequence $\{P_n(x)\}$, $\deg P_n(x) = n$ forms a basis in the vector space of the polynomials of one variable, the numbers $\alpha_{i,j}$ are well-defined.

Let us remark that in the vector space $\langle x_0, x_1, \ldots, x_n \rangle$ derivations $\mathcal{D}_P$ and $\mathcal{D}_A$ act as common linear nilpotent operators. Moreover, Jordan normal form of the matrix of the operator $\mathcal{D}_A$ consists of a single Jordan block with zero main diagonal. In this case, if the Jordan normal form of the matrix of the operator $\mathcal{D}_A$ is formed with a single Jordan block, then there exists a basis in which the action of the matrix of the operator $\mathcal{D}_P$ is the same as the action of operator $\mathcal{D}_A$.

Thus, there exists an isomorphism φ_{AP} from the vector space $\langle x_0, x_1, \ldots, x_n \rangle$ to the vector space $\langle x_0, x_1, \ldots, x_n \rangle$, and it could be extended to the corresponding endomorphism $\varphi_{AP} \colon \mathbb{Q}[x_0, x_1, \ldots, x_n] \to \mathbb{Q}[x_0, x_1, \ldots, x_n]$. The map φ_{AP} is called $(\mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{P}})$-*intertwining map.* Any such map induces an isomorphism from $\ker \mathcal{D}_{\mathcal{A}}$ to $\ker \mathcal{D}_{\mathcal{P}}$. Therefore, the main idea of obtaining polynomials identities for a polynomial family $\{P_n(x)\}$ is to find the explicit form of $(\mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{P}})$-intertwining map. Applying the action of the φ_{AP} map to the elements of the kernel $\ker \mathcal{D}_{\mathcal{A}}$, we always obtain the elements of the kernel $\ker \mathcal{D}_{\mathcal{P}}$. As a consequence, if the polynomial $F(x_0, x_1, \ldots, x_n)$ belongs to the kernel $\ker \mathcal{D}_{\mathcal{A}}$, then the element

$$\varphi_{AP}(F(x_0, x_1, \ldots, x_n)) = F(\varphi_{AP}(x_0), \varphi_{AP}(x_1), \ldots, \varphi_{AP}(x_n))$$

always belongs to the kernel of the derivation $\mathcal{D}_{\mathcal{P}}$ and this polynomial of one variable

$$F(\varphi_{AP}(P_0(x)), \varphi_{AP}(P_1(x)), \ldots, \varphi_{AP}(P_n(x))),$$

defines a polynomial identity for the polynomial family $P_n(x)$. The same approach was used by the first author in [5], [6] to find new polynomial identities in the case of the Fibonacci, Lucas and Kravchuk polynomials.

The subject of this study is finding out an explicit form of $(\mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{P}})$-intertwining map for Chebyshev polynomials of the first and the second kind.

2. Intertwining map for the Chebyshev derivation of the first kind. The Chebyshev polynomials of the first kind $T_n(x)$ and the Chebyshev polynomials of the second kind $U_n(x)$ are defined by the following ordinary generating functions

$$\mathcal{G}(T_n(x), t) = \frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x) t^n, \quad \mathcal{G}(U_n(x), t) = \frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} U_n(x) t^n.$$

The derivatives of the polynomials can be expressed in the terms of the polynomials as follows:

$$\frac{d}{dx}T_n(x) = n\Big(\sum_{k=1}^{n-1}\left(1-(-1)^k\right)T_{n-k}(x) + \frac{1-(-1)^n}{2}T_0(x)\Big),$$

$$\frac{d}{dx}U_n(x) = \sum_{k=1}^{n}\left(1-(-1)^{n-k}\right)(n-k+1)U_{n-k}(x) = \sum_{k=0}^{[n/2]}(n-2k)U_{n-2k-1}(x),$$

see [7], [8].

This motivates us to give the following

Definition 1. Derivations of $\mathbb{Q}[x_0, x_1, x_2, \ldots, x_n]$ defined by

$$D_{\mathcal{T}}(x_0) = 0, D_{\mathcal{T}}(x_n) = n\left(\sum_{k=1}^{n-1}\left(1-(-1)^k\right)x_{n-k} + \frac{1-(-1)^n}{2}x_0\right),$$

$$D_{\mathcal{U}}(x_0) = 0, D_{\mathcal{U}}(x_n) = \sum_{k=1}^{n-1}\left(1+(-1)^{n-k+1}\right)(k+1)x_k,$$

are called the *Chebyshev derivation of the first kind* and the *Chebyshev derivation of the second kind*, respectively.

We have

$$\begin{array}{ll}
D_{\mathcal{T}}(x_0) = 0, & D_{\mathcal{U}}(x_0) = 0,\\
D_{\mathcal{T}}(x_1) = x_0, & D_{\mathcal{U}}(x_1) = 2x_0,\\
D_{\mathcal{T}}(x_2) = 4x_1, & D_{\mathcal{U}}(x_2) = 4x_1,\\
D_{\mathcal{T}}(x_3) = 6\,x_2 + 3\,x_0, & D_{\mathcal{U}}(x_3) = 2(3\,x_2 + x_0),\\
D_{\mathcal{T}}(x_4) = 8\,x_3 + 8\,x_1, & D_{\mathcal{U}}(x_4) = 2(4\,x_3 + 2\,x_1),\\
D_{\mathcal{T}}(x_5) = 10\,x_4 + 10\,x_2 + 5\,x_0, & D_{\mathcal{U}}(x_5) = 2(5\,x_4 + 3\,x_2 + x_0),\\
D_{\mathcal{T}}(x_6) = 12\,x_5 + 12\,x_3 + 12\,x_1, & D_{\mathcal{U}}(x_6) = 2(6\,x_5 + 4\,x_3 + 2x_1),\\
D_{\mathcal{T}}(x_7) = 14\,x_6 + 14\,x_4 + 14\,x_2 + 7\,x_0, & D_{\mathcal{U}}(x_7) = 2(7\,x_6 + 5\,x_4 + 3\,x_2 + x_0),\\
D_{\mathcal{T}}(x_8) = 16\,x_7 + 16\,x_5 + 16\,x_3 + 16x_1, & D_{\mathcal{U}}(x_8) = 2(8\,x_7 + 6\,x_5 + 4x_3 + 2\,x_1).
\end{array}$$

It is obvious that these derivations are triangular and, consequently, locally nilpotent. Also, since $D_{\mathcal{T}}^n(x_n) = 0, D_{\mathcal{T}}^{n-1}(x_n) \neq 0$, and $D_{\mathcal{U}}^n(x_n) = 0, D_{\mathcal{U}}^{n-1}(x_n) \neq 0$, the Jordan normal form of the matrices of the operators $D_{\mathcal{T}}$ and $D_{\mathcal{U}}$ consists of a single Jordan block.

Let us find a $(\mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{T}})$-intertwining map. Define the map ψ_{AT} by

$$\psi_{AT}(x_n) = \alpha_n^{(0)}x_n + \alpha_n^{(1)}x_{n-2} + \alpha_n^{(2)}x_{n-4} + \ldots + \alpha_n^{(i)}x_{n-2i} + \ldots + \alpha_n^{([\frac{n-1}{2}])}x_{n-2[\frac{n-1}{2}]}.$$

We have

$$\begin{gathered}
D_{\mathcal{T}}(\psi_{AT}(x_n)) = 2n\alpha_n^{(0)}(x_{n-1} + x_{n-3} + x_{n-5} + x_{n-7} + \ldots) + \alpha_n^{(1)}2(n-2)(x_{n-3} + x_{n-5} + \\
+ \ldots) + \alpha_n^{(2)}2(n-4)(x_{n-5} + x_{n-7} + \ldots) + \ldots = x_{n-1}(2n\alpha_n^{(0)}) + \\
+x_{n-3}(\alpha_n^{(0)}2n + \alpha_n^{(1)}2(n-2)) + x_{n-5}(\alpha_n^{(0)}2n + \alpha_n^{(1)}2(n-2) + 2(n-4)\alpha_n^{(2)}) + \cdots).
\end{gathered}$$

On the other hand,

$$D_{\mathcal{T}_1}(\psi_{AT}(x_n)) = \psi_{AT}(\mathcal{D}_{\mathcal{A}}(x_n)) = n\psi_{AT}(x_{n-1}) = n(\alpha_{n-1}^{(0)}x_{n-1} + \alpha_{n-1}^{(1)}x_{n-3} + \alpha_{n-1}^{(2)}x_{n-5} + \ldots).$$

Thus

Lemma 1. *The sequences* $\alpha_n^{(0)}, \alpha_n^{(1)}, \ldots, \alpha_n^{([\frac{n-1}{2}])}$ *satisfy the following system of recurrence equations:*

$$\begin{cases} \alpha_n^{(0)} 2n = n\alpha_{n-1}^{(0)} \\ \alpha_n^{(0)} 2n + \alpha_n^{(1)} 2(n-2) = n\alpha_{n-1}^{(1)}, \\ \alpha_n^{(0)} 2n + \alpha_n^{(1)} 2(n-2) + 2(n-4)\alpha_n^{(2)} = n\alpha_{n-1}^{(2)}, \\ \ldots\ldots\ldots \\ \alpha_n^{(0)} 2n + \alpha_n^{(1)} 2(n-2) + 2(n-4)\alpha_n^{(2)} + \cdots + 2(n-2\left[\frac{n-1}{2}\right])\alpha_n^{([\frac{n-1}{2}])} = n\alpha_{n-1}^{([\frac{n-1}{2}])}. \end{cases}$$

After simplifying, we obtain

$$\begin{cases} \alpha_n^{(0)} 2n = n\alpha_{n-1}^{(0)}, \alpha_1^{(0)} = 0, \\ \alpha_n^{(1)} 2(n-2) = n(\alpha_{n-1}^{(1)} - \alpha_{n-1}^{(0)}), \alpha_2^{(1)} = 0, \\ 2(n-4)\alpha_n^{(2)} = n(\alpha_{n-1}^{(2)} - \alpha_{n-1}^{(1)}), \alpha_4^{(2)} = 0, \\ \ldots \\ 2(n-2i)\alpha_n^{(i)} = n(\alpha_{n-1}^{(i)} - \alpha_{n-1}^{(i-1)}), \alpha_{2i}^{(i)} = 0 \\ \ldots \\ 2(n-2\left[\frac{n-1}{2}\right])\alpha_n^{([\frac{n-1}{2}])} = n\left(\alpha_{n-1}^{([\frac{n-1}{2}])} - \alpha_{n-1}^{([\frac{n-1}{2}]-1)}\right) \end{cases}$$

Let g_n be a fixed sequence of polynomials and consider the auxiliary recurrence equation

$$2(n-a)x_n = n(x_{n-1} - g_{n-1}),\ x_a = 0,\ n \geq a.$$

Lemma 2. *Let*

$$x_n = -\frac{n^{\underline{a}}}{2^n}\sum_{i=a}^{n-1}\frac{2^i g_i}{i^{\underline{a}}},$$

where $n^{\underline{a}} := n(n-1)(n-2)\cdots(n-(a-1))$*. Then the sequence* x_n *is a solution of the auxiliary recurrence equation.*

Proof. In fact, suppose that

$$x_n = -\frac{n^{\underline{a}}}{2^n}\sum_{i=a}^{n-1}\frac{2^i g_i}{i^{\underline{a}}}.$$

Then we have

$$n(x_{n-1} - g_{n-1}) = n\left(-\frac{(n-1)^{\underline{a}}}{2^{n-1}}\sum_{i=a}^{n-2}\frac{2^i g_i}{i^{\underline{a}}} - g_{n-1}\right) =$$

$$= n\left(-\frac{(n-1)^{\underline{a}}}{2^{n-1}}\sum_{i=a}^{n-2}\frac{2^i g_i}{i^{\underline{a}}} - \frac{2^{n-1}(n-1)^{\underline{a}} g_{n-1}}{2^{n-1}(n-1)^{\underline{a}}}\right) = \frac{n(n-1)^{\underline{a}}}{2^{n-1}}\left(-\sum_{i=a}^{n-2}\frac{2^i g_i}{i^{\underline{a}}} - \frac{2^{n-1} g_{n-1}}{(n-1)^{\underline{a}}}\right) =$$

$$= -\frac{n(n-1)^{\underline{a}}}{2^{n-1}}\sum_{i=a}^{n-1}\frac{2^i g_i}{i^{\underline{a}}} = -2(n-a)\frac{n^{\underline{a}}}{2^n}\sum_{i=a}^{n-1}\frac{2^i g_i}{i^{\underline{a}}} = 2(n-a)x_n.$$

□

In particular,

$$\alpha_n^{(0)} = \frac{1}{2^{n-1}}, \quad \alpha_n^{(1)} = -\frac{n(n-2)}{2^{n-1}}, \quad \alpha_n^{(2)} = \frac{(3n^2-19n+28)n(n-1)}{2^{n+1}}.$$

To find the general solution of the system let us consider the sequence $\alpha_n^{(s)}$ in the basis of the falling powers. Put $\alpha_n^{(s)} = \frac{1}{2^n}(\beta_0^{(s)} n^{\underline{s}} + \beta_1^{(s)} n^{\underline{s+1}} + \cdots + \beta_s^{(s)} n^{\underline{2s}})$,

It is easy to see that $(n-s)n^{\underline{s}} = n^{\underline{s+1}}, n(n-1)^{\underline{s}} = n^{\underline{s+1}}$. Then

$$2(n-2s)\alpha_n^{(s)} = (n-2s)\frac{1}{2^{n-1}}(\beta_0^{(s)} n^{\underline{s}} + \beta_1^{(s)} n^{\underline{s+1}} + \cdots + \beta_s^{(s)} n^{\underline{2s}}) =$$

$$= \frac{1}{2^{n-1}}\left(((n-s)-s)\beta_0^{(s)} n^{\underline{s}} + ((n-(s+1))-(s-1)))\beta_1^{(s)} n^{\underline{s+1}} + \cdots + (n-2s)\beta_s^{(s)} n^{\underline{2s}}\right) =$$

$$= \frac{1}{2^{n-1}}\left(\sum_{i=0}^{s} \beta_i^{(s)} n^{\underline{s+i+1}} - \sum_{i=0}^{s-1}(s-i)\beta_i^{(s)} n^{\underline{s+i}}\right).$$

On the other hand,

$$n(\alpha_{n-1}^{(s)} - \alpha_{n-1}^{(s-1)}) = \frac{1}{2^{n-1}}\left(\sum_{i=0}^{s} n\beta_i^{(s)}(n-1)^{\underline{s+i}} - \sum_{i=0}^{s-1} n\beta_i^{(s-1)}(n-1)^{\underline{s-1+i}}\right) =$$

$$= \frac{1}{2^{n-1}}\left(\sum_{i=0}^{s} \beta_i^{(s)} n^{\underline{s+i+1}} - \sum_{i=0}^{s-1} \beta_i^{(s-1)} n^{\underline{s+i}}\right).$$

After equating the corresponding coefficients, we obtain that $\beta_i^{(s)} = \frac{\beta_i^{(s-i)}}{s-i}, i = 0\ldots s-1$. The coefficient $\beta_s^{(s)}$ is found from the initial condition $a_{2s}^{(s)} = 0$. We have

$$a_{2s}^{(s)} = \frac{1}{2^{2s}}\left(\sum_{i=0}^{s-1} \frac{\beta_i^{(s-1)}}{s-i}(2s)^{\underline{s+i}} + \beta_s^{(s)}(2s)!\right) = 0.$$

It follows that

$$\beta_s^{(s)} = -\frac{1}{(2s)!}\sum_{i=0}^{s-1}\frac{\beta_i^{(s-1)}}{s-i}(2s)^{\underline{s+i}} = \sum_{i=0}^{s-1}\frac{\beta_i^{(s-1)}}{(s-i)(s-i)!} = \sum_{i=0}^{s-1}\frac{\beta_i^{(s)}}{(s-i)!}.$$

Therefore, we get the following recurrence relations for the sequences $\beta_n^{(s)}$:

$$\beta_0^{(s)} = \frac{\beta_0^{(s-1)}}{s}, \quad \beta_1^{(s)} = \frac{\beta_1^{(s-1)}}{s-1}, \quad \beta_2^{(s)} = \frac{\beta_1^{(s-1)}}{s-2}, \quad \ldots \quad \beta_{s-1}^{(s)} = \beta_1^{(s-1)}, \quad \beta_s^{(s)} := b_s = \sum_{i=0}^{s-1}\frac{\beta_i^{(s)}}{(s-i)!}.$$

Thus, $\beta_i^{(s)} = \frac{1}{(s-i)!}b_i$, for $i = 0, \ldots, s-1$. Therefore, we get the following recurrence relation for the sequence b_s :

$$\sum_{i=0}^{n}\frac{1}{(n-i)!^2}b_n = 0, n > 0.$$

Recall the definition of the Bessel function $J_\alpha(x)$:

$$J_\alpha(z) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i\Gamma(i+\alpha+1)} \left(\frac{z}{2}\right)^{2i+\alpha}.$$

Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{n!^2} z^n = J_0(\sqrt{-4z}\,),$$

and the ordinary generating function $G(b_n, z) = \sum_{n=0}^{\infty} b_n x^n$. The last relation implies the following one $J_0(\sqrt{-4z}\,)G(b_n, z) = 1$.

Thus, we prove the following statement.

Theorem 1. *A Weitzenböck-Chebyshev intertwining map ψ_{AT} of the first kind has the following form*

$$\psi_{AT}(x_n) = x_n + \alpha_n^{(1)} x_{n-2} + \alpha_n^{(2)} x_{n-4} + \ldots + \alpha_n^{(i)} x_{n-2i} + \ldots + \alpha_n^{([\frac{n-1}{2}])} x_{n-2[\frac{n-1}{2}]},$$

where

$$\alpha_n^{(s)} = \frac{1}{2^n}\left(\frac{1}{s!} b_0 n^{\underline{s}} + \cdots + \frac{1}{(s-i)!} b_i n^{\underline{s+i}} + \cdots + b_s n^{\underline{2s}}\right),$$

and the generating function for $b_0, b_1, \ldots, b_n, \ldots$ is

$$\sum_{i=0}^{\infty} b_i z^i = J_0^{-1}(\sqrt{-4z}\,).$$

3. Intertwining map for the Chebyshev derivation of the second kind. Let us define the linear map ψ_{AU} by

$$\psi_{AU}(x_n) = \alpha_n^{(0)} x_n + \alpha_n^{(1)} x_{n-2} + \alpha_n^{(2)} x_{n-4} + \ldots + \alpha_n^{(i)} x_{n-2i} + \ldots + \alpha_n^{([\frac{n-1}{2}])} x_{n-2[\frac{n-1}{2}]}.$$

Let us consider the following statement.

Lemma 3. *The sequences $\alpha_n^{(0)}, \alpha_n^{(1)}, \ldots, \alpha_n^{([\frac{n-1}{2}])}$ satisfy the following system of recurrence equations:*

$$\begin{cases} \alpha_n^{(0)} = \alpha_{n-1}^{(0)} \\ (n-2)(\alpha_n^{(0)} + \alpha_n^{(1)}) = n\alpha_{n-1}^{(1)}, \\ (n-4)(\alpha_n^{(0)} + \alpha_n^{(1)} + \alpha_n^{(2)}) = n\alpha_{n-1}^{(2)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (n-2\left[\frac{n-1}{2}\right]+1)(\alpha_n^{(0)} + \alpha_n^{(1)}2 + \alpha_n^{(2)} + \cdots + \alpha_n^{([\frac{n-1}{2}])} = n\alpha_{n-1}^{([\frac{n-1}{2}])}, \end{cases}$$

or, equivalently,

$$\begin{cases} \alpha_n^{(0)} = \alpha_{n-1}^{(0)}, \alpha_1^{(0)} = 1, \\ \alpha_n^{(1)} = \dfrac{n}{n-2}\alpha_{n-1}^{(1)} - \alpha_{n-1}^{(0)}, \alpha_2^{(1)} = 0, \\ \alpha_n^{(2)} = \dfrac{n}{n-4}\alpha_{n-1}^{(2)} - \dfrac{n}{n-2}\alpha_{n-1}^{(1)}, \alpha_4^{(2)} = 0, \\ \dots, \\ \alpha_n^{(i)} = \dfrac{n}{n-2i}\alpha_{n-1}^{(i)} - \dfrac{n}{n-2(i-1)}\alpha_{n-1}^{(i-1)}, \alpha_{2i}^{(i)} = 0 \\ \dots, \\ \alpha_n^{([\frac{n-1}{2}])} = \dfrac{n}{n-2\left[\frac{n-1}{2}\right])}\alpha_{n-1}^{([\frac{n-1}{2}])} - \dfrac{n}{n-2(\left[\frac{n-1}{2}\right]-1))}\alpha_{n-1}^{([\frac{n-1}{2}]-1)}. \end{cases}$$

The proof is similar to that of Lemma 1.

Let g_n be some fixed sequence and consider the auxiliary recurrence equation

$$x_n = n\left(\frac{x_{n-1}}{n-s} - \frac{g_{n-1}}{n-(s-2)}\right), x_s = 0.$$

Lemma 4. *The solution of this auxiliary recurrence equation is*

$$x_n = -n^{\underline{s}}\sum_{i=s}^{n-1}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))}.$$

Proof. Using the relations

$$\frac{n(n-1)^{\underline{s}}}{n-1} = n^{\underline{s}}, n(n-1)^{\underline{s-1}} = n^{\underline{s}}.$$

we have

$$n\left(\frac{x_{n-1}}{n-s} - \frac{g_{n-1}}{n-(s-2)}\right) = n\left(-\frac{(n-1)^{\underline{s}}}{n-s}\sum_{i=s}^{n-2}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))} - \frac{g_{n-1}}{n-(s-2)}\right) =$$

$$= n\left(-\frac{(n-1)^{\underline{s}}}{n-s}\sum_{i=s}^{n-2}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))} - \frac{(n-1)^{\underline{s-1}}\,g_{n-1}}{((n-1)^{\underline{s-1}})((n-1)-(s-3))}\right) =$$

$$= -\frac{n(n-1)^{\underline{s}}}{n-s}\sum_{i=s}^{n-2}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))} - \frac{n(n-1)^{\underline{s-1}}\,g_{n-1}}{((n-1)^{\underline{s-1}})((n-1)-(s-3))} =$$

$$= n^{\underline{s}}\left(-\sum_{i=s}^{n-2}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))} - \frac{g_{n-1}}{((n-1)^{\underline{s-1}})((n-1)-(s-3))}\right) =$$

$$= -n^{\underline{s}}\sum_{i=s}^{n-1}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))}.$$

□

By using the result of Lemma 4, we obtain

$$\alpha_n^{(0)} = 1,\ \alpha_n^{(1)} = -\frac{1}{2}(n-1)(n-2),\ \alpha_n^{(2)} = \frac{1}{6}\ (n-4)\,(n-3)\,(n-2)\,n.$$

To find a solution for the system consider an unknown sequence $\{\alpha_n^{(s)}\}$ in the basis of the falling powers. Put

$$\alpha_n^{(s)} = (n-(2s-1))\left(\beta_0^{(s)} n^{\underline{s-1}} + \beta_1^{(s)} n^{\underline{s}} + \cdots + \beta_s^{(s)} n^{\underline{2s-1}}\right) = (n-(2s-1))\sum_{i=0}^{s} \beta_i^{(s)} n^{\underline{s-1+i}}.$$

Then we have

$$\alpha_n^{(s)} = (n-(2s-1))\sum_{i=0}^{s} \beta_i^{(s)} n^{\underline{s-1+i}} = \sum_{i=0}^{s-1} \beta_i^{(s)} (n-(s+i-1)-(s-i)) n^{\underline{s+i}} =$$

$$= \sum_{i=0}^{s} \beta_i^{(s)} (n-(s+i-1)) n^{\underline{s+i-1}} - \sum_{i=0}^{s} \beta_i^{(s)} (s-i) n^{\underline{s-1+i}} = \sum_{i=0}^{s} \beta_i^{(s)} n^{\underline{s+i}} - \sum_{i=0}^{s-1} (s-i)\beta_i^{(s)} n^{\underline{s-1+i}}.$$

On the other hand,

$$\alpha_{n-1}^{(s)} = (n-2s)\sum_{i=0}^{s} \beta_i^{(s)} (n-1)^{\underline{s-1+i}}, \quad \alpha_{n-1}^{(s-1)} = (n-(2s-2))\sum_{i=0}^{s-1} \beta_i^{(s-1)} (n-1)^{\underline{s+i-2}}.$$

Consequently,

$$\alpha_n^{(s)} = n\left(\frac{\alpha_{n-1}^{(s)}}{n-2s} - \frac{\alpha_{n-1}^{(s-1)}}{n-(2s-2)}\right) = \sum_{i=0}^{s} n\beta_i^{(s)} (n-1)^{\underline{s-1+i}} - \sum_{i=0}^{s-1} n\beta_i^{(s-1)} (n-1)^{\underline{s-2+i}} =$$

$$= \sum_{i=0}^{s} \beta_i^{(s)} n^{\underline{s+i}} - \sum_{i=0}^{s-1} \beta_i^{(s-1)} n^{\underline{s-1+i}}.$$

Hence,

$$\sum_{i=0}^{s-1} \beta_i^{(s-1)} n^{\underline{s-1+i}} = \sum_{i=0}^{s-1} (s-i)\beta_i^{(s)} n^{\underline{s-1+i}}.$$

By equating the corresponding coefficients of the $n^{\underline{i}}$, we get

$$\beta_i^{(s)} = \frac{\beta_i^{(s-1)}}{s-i}, i = 0 \ldots s-1.$$

The coefficient $\beta_s^{(s)}$ is determined from the initial condition $\alpha_{2s}^{(s)} = 0$ that is why

$$\alpha_{2s}^{(s)} = \sum_{i=0}^{s} \beta_i^{(s)} (2s)^{\underline{s-1+i}} = \sum_{i=0}^{s-1} \beta_i^{(s)} (2s)^{\underline{s-1+i}} - \beta_s^{(s)} (2s)^{\underline{2s-1}} = 0.$$

Taking into account $(2s)^{\underline{2s-1}} = 2s(2s-1)\ldots 2 = (2s)!$ we have

$$\beta_s^{(s)} = \frac{1}{(2s)!}\sum_{i=0}^{s-1} \beta_i^{(s)} (2s)^{\underline{s-1+i}} = \sum_{i=0}^{s-1} \frac{\beta_i^{(s)}}{(s-i+1)!}.$$

We obtain the following relations for $\beta_n^{(s)}$:

$$\beta_0^{(s)} = \frac{\beta_0^{(s-1)}}{s}, \; \beta_1^{(s)} = \frac{\beta_1^{(s-1)}}{s-1}, \; \beta_2^{(s)} = \frac{\beta_1^{(s-1)}}{s-2}, \; \ldots \beta_{s-1}^{(s)} = \beta_1^{(s-1)}, \; \beta_s^{(s)} := b_s = \sum_{i=0}^{s-1} \frac{\beta_i^{(s)}}{(s-i+1)!}.$$

It yields $\beta_i^{(s)} = \frac{1}{(s-i)!} b_i$, for $i = 0, \dots, s-1$, and

$$b_s = \frac{1}{((s+1)!)^2} + \sum_{i=1}^{s-1} \frac{1}{((s-i)(s-i+1)!)} b_i.$$

Therefore, we get the following recurrence relation for the sequence $\{b_i\}$:

$$\sum_{i=0}^{s} \frac{1}{(s-i)!(s-i+1)!} b_i = 0.$$

Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} z^n = \frac{1}{\sqrt{z}} J_1(\sqrt{-4z}\,),$$

and write

$$G(b_n, z) = \sum_{n=0}^{\infty} b_n z^n.$$

Then, taking into account the recurrence relation we find that

$$G(b_n, z) \frac{J_1(\sqrt{-4z}\,)}{\sqrt{z}} = 1.$$

Thus, we have the following statement.

Theorem 2. *A Weitzenböck-Chebyshev intertwining map ψ_{AU} of the second kind has the following form*

$$\psi_{AU}(x_n) = x_{n+1} + \alpha_n^{(1)} x_{n-1} + \alpha_n^{(2)} x_{n-3} + \ldots + \alpha_n^{(i)} x_{n+1-2i} + \ldots + \alpha_n^{([\frac{n-1}{2}])} x_{n+1-2[\frac{n-1}{2}]},$$

where

$$\alpha_n^{(s)} = (n-2s+1) \left(\frac{(-1)^s}{s!} b_0 n^{\underline{s-1}} + \cdots + \frac{(-1)^{s-i}}{(s-i)!} b_i n^{\underline{s+i}} + \cdots + b_s n^{\underline{2s-1}} \right),$$

and the ordinary generating function for $b_0, b_1, \dots, b_n, \dots$ is the following:

$$\sum_{i=0}^{\infty} b_i z^i = \frac{\sqrt{z}}{J_1(\sqrt{-4z}\,)}.$$

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