WIMAN’S TYPE INEQUALITY FOR MULTIPLE POWER SERIES
IN AN UNBOUNDED CYLINDER DOMAIN


In this paper we prove some analogues of Wiman’s inequality for analytic \( f(z) \) and random analytic functions \( f(z, t) \) on \( T = \mathbb{D}^l \times \mathbb{C}^{p-l}, \) \( l \in \mathbb{N}, 1 \leq l < p, I = \{1, \ldots, l\}, J = \{l+1, \ldots, p\} \) of the form \( f(z) = \sum_{|n|=0}^{+\infty} a_n z^n, \) \( f(z, t) = \sum_{|n|=0}^{+\infty} a_n Z_n(t) z^n, \) respectively. Here \( Z = (Z_n) \) is a multiplicative system of random variables on the Steinhaus probability space, uniformly bounded by the number 1. In particular, we prove the following statements: For every \( \varepsilon > 0 \) there exist sets \( E_1 = E_1(\delta, f), E_2 = E_2(\delta, f) \subset [0, 1]^l \times (1, +\infty)^{p-l} \) of asymptotically \( \log \) measure, such that the inequalities

\[
M_f(r) \leq \mu_f(r) \sum_{i \in I} \frac{1}{(1 - r_i)^{1+\varepsilon}} \left( \mu_f(r) \prod_{j \in J} \frac{1}{1 - r_j} \right)^{p+\varepsilon},
\]

\[
M_f(r, t) \leq \mu_f(r) \sum_{i \in I} \frac{1}{(1 - r_i)^{1/2+\varepsilon}} \left( \mu_f(r) \prod_{j \in J} \frac{1}{1 - r_j} \right)^{p/2+\varepsilon}.
\]

hold for all \( r \in T \setminus E_1 \) and for all \( r \in T \setminus E_2 \) a.s. in \( t, \) respectively. Also sharpness of the obtained inequalities is proved.

1. Introduction and the main result. Let \( f \) be an analytic function in the disc \( \mathbb{D}_R = \{ z : |z| < R \}, \) \( 0 < R \leq +\infty, \) represented by the power series

\[
f(z) = \sum_{n=0}^{+\infty} a_n z^n.
\]

Put \( \mathbb{D} = \mathbb{D}_1, \) \( \mathbb{C} = \mathbb{D}_+ \). For \( r \in (0, R) \) we denote

\[
M_f(r) = \max\{|f(z)| : |z| = r\}, \quad \mu_f(r) = \max\{|a_n|r^n : n \geq 0\}.
\]

It is well known ([1], [2, p. 9], [3, 4], [5, p. 28], [6, 7, 8]) that for each nonconstant entire function \( f(z) \) and for every \( \varepsilon > 0 \) there exists a set \( E(\varepsilon, f) \subset [1, +\infty) \) such that Wiman’s inequality

\[
M_f(r) \leq \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon}
\]

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holds for all \( r \in [1, +\infty) \setminus E(\varepsilon, f) \), where the set \( E(\varepsilon, f) \) has \textit{finite logarithmic measure} on \( \mathbb{R}_+ \), i.e.
\[
\int_{E(\varepsilon, f)} \frac{dr}{r} < +\infty.
\]

Also, new descriptions of the magnitude of the exceptional set \( E(\varepsilon, f) \) in inequality (2) were received in papers [6, 7].

Let \( f(z) \) be an analytic function in the unit disc \( \mathbb{D} \) of form (1). For a such function \( f(z) \) and for every \( \delta > 0 \) there exists a set \( E_f(\delta) \subset (0, 1) \) of finite logarithmic measure on \( (0, 1) \), i.e.
\[
\int_{E_f(\delta)} \frac{dr}{1-r} < +\infty,
\]
such that for all \( r \in (0, 1) \setminus E_f(\delta) \) the inequality
\[
M_f(r) \leq \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}
\]
holds ([12]). One can find similar inequality for analytic functions in the unit disc in [11].

Also in [12] was noted that for the function \( g(z) = \sum_{n=1}^{+\infty} \exp\{\varepsilon n\} z^n, \varepsilon \in (0, 1) \) one has
\[
\lim_{r \to 1-0} \frac{M_g(r)}{\mu_g(r)} = C > 0.
\]

In [13] it is proved that by some conditions inequality (2) holds for every analytic function of form (1) in the unit disc outside an exceptional set.

Some analogues of Wiman’s inequality for entire functions of several complex variables can be found in [14]–[23], for analytic functions in the polydisc \( \mathbb{D}^p, p \geq 2 \), in [24, 25, 26], for analytic functions in the unbounded cylinder \( \mathbb{D} \times \mathbb{C} \) in [27, 29], respectively.

In [14] there is considered entire functions of \( p \) complex variables
\[
 f(z) = f(z_1, \ldots, z_p) = \sum_{\|n\|=0}^{+\infty} a_n z^n,
 \tag{3}
\]
where \( z^n = z_1^{n_1} \ldots z_p^{n_p}, p \in \mathbb{N}, n = (n_1, \ldots, n_p) \in \mathbb{Z}_+^p, \|n\| = \sum_{j=1}^p n_j. \) For \( r = (r_1, \ldots, r_p) \in \mathbb{R}_+^p \) denote
\[
\Delta_{r_0} = \{ t \in \mathbb{R}_+^p : t_j \geq r_0^j, j \in \{1, \ldots, p\}, \ln_2 x = \ln \ln x, r^\wedge = \min_{1 \leq i \leq p} r_i, \}
\]
\[
M_f(r) = \max\{|f(z)| : |z_1| = r_1, \ldots, |z_p| = r_p\},
\]
\[
\mu_f(r) = \max\{|a_n r_1^{n_1} \ldots r_p^{n_p} : n \in \mathbb{Z}_+^p\}, \mathcal{M}_f(r) = \sum_{\|n\|=0}^{+\infty} |a_n|r^n.
\]

By \( \Lambda^p \) we denote the class of entire functions of form (3) such that \( \frac{\partial}{\partial z_j} f(z) \neq 0 \) in \( \mathbb{C}^p \) for any \( j \in \{1, \ldots, p\}. \) A subset \( E \) of \( \mathbb{R}_+^p \) is a \textit{set of asymptotically finite logarithmic measure} on \( \mathbb{R}_+^p \) if \( E \) is Lebesgue measurable in \( \mathbb{R}_+^p \) and there exists an \( R \in \mathbb{R}_+^p \) such that \( E \cap \Delta_{r_0} \) is a set of finite logarithmic measure, i.e.
\[
\int_{E \cap \Delta_{r_0}} \cdots \int_{\mathbb{R}_+^p} \prod_{j=1}^p \frac{dr_j}{r_j} < +\infty.
\]
Theorem A ([14]). Let \( f \in \mathcal{A}^p \) and \( \delta > 0 \).

a) Then there exist \( r_0 \in \mathbb{R}^p \) and a subset \( E \) of \( \triangle_{r_0} \) of finite logarithmic measure such that for \( r \in \triangle_{r_0} \setminus E \) we have

\[
\mathcal{M}_f(r) \leq \mu_f(r) \left( \prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r) \right)^{1/2+\delta}.
\]

b) If for some \( \alpha \in \mathbb{R}^p_+ : \mathcal{M}(r) \geq \exp(r^\alpha) = \exp(r_1^{\alpha_1} \cdots r_p^{\alpha_p}) \) as \( r^\wedge \to +\infty \) or, more generally, for each \( \beta > 0 \)

\[
\int \cdots \int_{B(S)} \frac{\prod_{i=1}^{p} dr_i}{r_1 r_2 \cdots r_p \ln^\beta \mathcal{M}_f(r)} < +\infty, \quad \text{as } S^\wedge \to +\infty,
\]

then there exist \( r_0 \in \mathbb{R}^p \) and a subset \( E \) of \( \triangle_{r_0} \) of finite logarithmic measure such that for \( r \in \triangle_{r_0} \setminus E \) we have

\[
\mathcal{M}_f(r) \leq \mu_f(r) \ln^{p/2+\delta} \mu_f(r).
\]

In [25] there were proved some analogues of Wiman’s inequality for analytic functions \( f \) represented by the series of form (3) with the domain of convergence

\[
\mathbb{D}^p = \{ z \in \mathbb{C}^p : |z_j| < 1, j \in \{1, \ldots, p\} \}.
\]

By \( \mathcal{A}_0^p \) we denote the class of such analytic functions that for any \( i \in \{1, \ldots, p\} \) we have

\[
r_i \frac{\partial}{\partial r_i} \ln \mathcal{M}_f(r) + \ln r_i > 1 \quad \text{for } r \in (t_0, 1)^p.
\]

Theorem B ([25]). Let \( f \in \mathcal{A}_0^p \). For every \( \delta > 0 \) there exists a set \( E = E(f, \delta) \subset [0, 1]^p \) of asymptotically finite logarithmic measure such that for all \( r \in [0, 1]^p \setminus E \) we have

\[
M_f(r) \leq \mu_f(r) \prod_{j=1}^{p} \frac{1}{(1 - r_j)^{1+\delta}} \cdot \ln^{p/2+\delta} \left\{ \mu_f(r) \prod_{j=1}^{p} \frac{1}{1 - r_j} \right\}.
\] (4)

In [27] one can find some analogues of Wiman’s inequality for analytic functions represented by the series

\[
f(z) = f(z_1, z_2) = \sum_{n+m=0}^{+\infty} a_{nm} z_1^n z_2^m
\] (5)

with the domain of convergence \( \mathbb{D} \times \mathbb{C} = \{ z \in \mathbb{C}^2 : |z_1| < 1, z_2 \in \mathbb{C} \} \).

By \( \mathcal{A}_1^2 \) we denote the class of analytic functions of form (5) with the domain of convergence \( \mathbb{D} \times \mathbb{C} \) and

\[
\frac{\partial}{\partial z_2} f(z_1, z_2) \neq 0 \quad \text{in } \mathbb{D} \times \mathbb{C}, \quad r_1 \frac{\partial}{\partial r_1} \ln \mathcal{M}_f(r) + \ln r_1 > 1 \quad \forall (r_1, r_2) \in (r_0^1, 1) \times (r_0^2, +\infty).
\]

We say that \( E \subset T_1 = (0, 1) \times \mathbb{R}_+ \) is set of asymptotically finite logarithmic measure on \( T_1 \) if there exists \( r_0 \in T_1 \) such that

\[
\nu_{\ln}(E \cap \triangle_{r_0}) = \iint_{E \cap \triangle_{r_0}} \frac{dr_1 dr_2}{(1 - r_1) r_2} < +\infty,
\]
i.e. the set $E \cap \Delta_{r_0}$ is a set of finite logarithmic measure on $T_1$.

**Theorem C** ([27]). Let $f \in A^2_T$. For every $\delta > 0$ there exists a set $E = E(\delta, f) \subset T_1$ of asymptotically finite logarithmic measure such that for all $r \in T_1 \setminus E$ we obtain

$$M_f(r) \leq \frac{\mu_f(r)}{(1 - r_1)^{1+\delta}} \ln^{1+\delta} \frac{\mu_f(r)}{1 - r_1} \cdot \ln^{1/2+\delta} r_2.$$

By $A^p_0(T_i)$, $p \geq 1$, $1 \leq l \leq p$, denote the class of analytic functions of form (3) with the domain of convergence

$$\mathbb{T}_l = \{z = (z_1, \ldots, z_p) \in \mathbb{C}^p: |z_k| < 1, z_j \in \mathbb{C}, k \in \{1, \ldots, l\}, j \in \{l + 1, \ldots, p\}\} = \mathbb{D}^l \times \mathbb{C}^{p-l},$$

and by $A^p(\mathbb{T})$ denote a subclass of the functions $f \in A^p_0(\mathbb{T}_i)$ such that

$$\frac{\partial}{\partial z_j} f(z_1, \ldots, z_p) \neq 0 \quad (\forall z \in \mathbb{T}_l \text{ and } \forall j \in \{l + 1, \ldots, p\})$$

and there exists $r_0 \in \mathbb{R}^p_+$ such that for any $k \in \{1, \ldots, l\}$

$$r_k \frac{\partial}{\partial r_k} \ln \mathfrak{M}_f(r) + \ln r_k > 1 \quad (\forall r \in (r_0^1, 1)^l \times (r_0^2, +\infty)^{p-l}).$$

The aim of this paper is to prove some analogues of Wiman’s inequality for analytic functions $f$ represented by the series of the form (3) with the domain of convergence $\mathbb{T} := \mathbb{T}_l = \mathbb{D}^l \times \mathbb{C}^{p-l}$, $l \in \mathbb{N}$, $1 \leq l < p$. A natural **problem** arises: to prove sharp analogues of the Wiman type inequalities for analytic functions in $\mathbb{T}$.

2. **Wiman’s type inequality for analytic functions in $\mathbb{T}_l$**. For $r = (r_1, \ldots, r_p) \in T := [0, 1)^l \times [0, +\infty)^{p-l}$ and a function $f \in A^p(\mathbb{T}_l)$ we denote

$$\Delta_r = \{(t_1, \ldots, t_p) \in T: t_j > r_j, j \in \{1, \ldots, p\}, I = \{1, \ldots, l\}, J = \{l + 1, \ldots, p\}\},

M_f(r) = \max\{|f(z)|: |z_j| \leq r_j, j \in \{1, \ldots, p\}\},

\mu_f(r) = \max\{|a_n|r^n: (n_1, \ldots, n_p) \in \mathbb{Z}_+^p\}, \quad \mathfrak{M}_f(r) = \sum_{\|n\|=0}^{+\infty} |a_n|r^n.$$

We say that $E \subset T$ is set of **asymptotically finite logarithmic measure on $T$** if there exists $r_0 \in T$ such that

$$\nu_\infty(E \cap \Delta_{r_0}) := \int \cdots \int_{E \cap \Delta_{r_0}} \prod_{i \in I} \frac{dr_i}{1 - r_i} \prod_{j \in J} \frac{dr_j}{r_j} < +\infty.$$

The set of all such sets is denoted by $\Upsilon$.

**Theorem 1.** Let $f \in A^p(\mathbb{T})$. For every $\delta > 0$ there exists a set $E = E(\delta, f) \subset T$, $E \in \Upsilon$ such that for all $r \in T \setminus E$ we obtain

$$M_f(r) \leq \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)^{1+\delta}} \ln^{p/2+\delta} \left( \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \right) \left( \prod_{j \in J} \ln r_j \right)^{p+\delta}. \quad (6)$$

In order to prove Theorem 1 we need the following auxiliary results.
Lemma 1. Let $F$ be a function of the form

$$F(\sigma) = \int_{\mathbb{R}^p_+} a(x)e^{(\sigma,x)}\nu(dx),$$

(7)

where $a(x): \mathbb{R}_+^p \to \mathbb{R}_+$ is a $\nu$-measurable function, and $\nu$ is a countable-additive measure on $\mathbb{R}_+^p$ with unbounded support. We assume that for some fixed $\sigma \in \mathbb{R}_+^p$ there exist

$$\frac{\partial}{\partial \sigma_j} F(\sigma) = \int_{\mathbb{R}^p_+} x_j a(x)e^{(\sigma,x)}\nu(dx), \quad \frac{\partial^2}{\partial \sigma_j^2} F(\sigma) = \int_{\mathbb{R}^p_+} x_j^2 a(x)e^{(\sigma,x)}\nu(dx).$$

Then

$$F(\sigma) \leq \frac{c}{c-1} \int_{X_0(\sigma)} a(x)e^{(\sigma,x)}\nu(dx).$$

(8)

where $c = c(\sigma) > 1$ is arbitrary,

$$a_j = \frac{\partial}{\partial \sigma_j} \ln F(\sigma), \quad b_j = \frac{\partial^2}{\partial \sigma_j^2} \ln F(\sigma),$$

$$X_0(\sigma) = \{x = (x_1, \ldots, x_p) \in \mathbb{R}_+^p : |x_j - a_j| \leq \sqrt{cpb_j}, 1 \leq j \leq p\}.$$

Proof of Lemma 1. We repeat the arguments from the proof of Lemma 1 from [20]. Consider random variables $\xi_j = x_j$, $1 \leq j \leq p$ on the probability space $\mathbb{R}_+^p$ with the following measure

$$P_{\sigma}(dx) = P(dx) = \frac{a(x)}{F(\sigma)} e^{(\sigma,x)}\nu(dx).$$

So, the expectation of the variable $\xi_j$

$$E\xi_j = \int_{\mathbb{R}_+^p} x_j a(x)e^{(\sigma,x)}P(dx) = \frac{1}{F(\sigma)} \frac{\partial F(\sigma)}{\partial \sigma_j} = a_j,$$

and the variance of $\xi_j$

$$D\xi_j = M\xi_j^2 - (M\xi_j)^2 = \frac{1}{F(\sigma)} \frac{\partial^2 F(\sigma)}{\partial \sigma_j^2} - \left( \frac{1}{F(\sigma)} \frac{\partial F(\sigma)}{\partial \sigma_j} \right)^2 = \frac{\partial}{\partial \sigma_j} \left( \frac{1}{F(\sigma)} \frac{\partial F(\sigma)}{\partial \sigma_j} \right) = b_j.$$

We put $d_j = \sqrt{pcb_j}$, $A_j = \{x \in \mathbb{R}_+^p : |\xi_j - E\xi_j| \geq d_j\}$, $A = \bigcup_{j=1}^p A_j$. We have $X_0(\sigma) = A$. Using $p$ times the Bienayme-Chebyshev inequality (see [35, 36]) $P(A_j) = P\{x : |\xi_j(x) - E\xi_j| \geq d_j\} \leq D\xi_j/d_j^2$ we obtain

$$P(A) \leq \sum_{j=1}^p P(A_j) \leq \sum_{j=1}^p \frac{1}{d_j^2} D\xi_j = \frac{1}{c},$$

Therefore,

$$F(\sigma) = \int_{X_0(\sigma)} f(x)e^{(\sigma,x)}\nu(dx) + F(\sigma) \int_A f(x) \frac{e^{(\sigma,x)}}{F(\sigma)}\nu(dx) =$$
Lemma 3. The proof of Lemma 2 is complete.

Hence, we have inequality (8). □

Lemma 2. Let \( f \in \mathcal{A}^p(\mathbb{T}) \) be a function of form (3),

\[
a_j = r_j \frac{\partial}{\partial r_j} \ln \mathcal{M}_f(r), \quad b_j = r_j \frac{\partial}{\partial r_j} \left( r_j \frac{\partial}{\partial r_j} \ln \mathcal{M}_f(r) \right), \quad C = C(r) > 1,
\]

\[
X_0(r) = \{ x \in \mathbb{R}^p_+ : |x_j - a_j| \leq \sqrt{cpb_j}, \ 1 \leq j \leq p \}.
\]

Then for all \( r \in T \) we have

\[
\mathcal{M}_f(r) \leq \frac{C}{C - 1} \sum_{n \in X_0(r)} |a_n| r^n.
\]

Proof. For \( \sigma = (\sigma_1, \ldots, \sigma_p) \), \( r = (r_1, \ldots, r_p) \), \( r_j = e^{\sigma_j} \) we have

\[
\mathcal{M}_f(r) = \sum_{|n| = 0}^{+\infty} |a_n| r^n = \int_{\mathbb{R}^p_+} a(x) e^{(\sigma,x)} \nu(dx) = F(\sigma),
\]

where \( a(x) = |a_n| \) for \( x = n \in \mathbb{Z}^p_+ \) and \( \nu \) is the measure such that

\[
\nu(E) = \sum_{n \in E} \delta_n(E), \quad \delta_n(E) = \begin{cases} 1, & n \in E, \\ 0, & n \notin E, \end{cases}
\]

for any bounded set \( E \subset \mathbb{R}^p_+ \).

Remark that

\[
\frac{\partial \ln F(\sigma)}{\partial \sigma_j} = r_j \frac{\partial \ln \mathcal{M}_f(r)}{\partial r_j}.
\]

Thus, using Lemma 1 with \( c = C \), \( r = e^\sigma \) and \( X_0^*(r) = X_0(\ln \sigma) \) we get

\[
\mathcal{M}_f(r) = F(\sigma) \leq \frac{c}{c - 1} \int_{X_0(\sigma)} a(x) e^{(\sigma,x)} \nu(dx) = \frac{C}{C - 1} \sum_{n \in X_0^*(r)} |a_n| r^n.
\]

The proof of Lemma 2 is complete. □

Lemma 3. Let \( \delta > 0 \). If \( f \in \mathcal{A}^p(\mathbb{T}) \) then there exists a set \( E \subset T \) \( (E \in \mathcal{T}) \), such that for all \( r \in T \setminus E \) the inequalities

\[
\frac{\partial}{\partial r_m} \ln \mathcal{M}_f(r) \leq \frac{1}{1 - r_m} \prod_{i \in I, i \neq m} \frac{1}{(1 - r_i)^\delta} \left( \ln \mathcal{M}_f(r) \prod_{j \in J} \ln r_j \right)^{1+\delta}, \ m \in I; \tag{9}
\]

\[
\frac{\partial}{\partial r_k} \ln \mathcal{M}_f(r) \leq \prod_{i \in I} \frac{1}{(1 - r_i)^\delta} \left( \ln \mathcal{M}_f(r) \prod_{j \in J, j \neq k} \ln r_j \right)^{1+\delta}, \ k \in J \tag{10}
\]

hold.
Proof of Lemma 3. Without loss generality we prove inequality (9) for \( i = 1 \). Suppose that 
\( E_1^* \subset T \) is the set for which inequality (9) with \( i = 1 \) does not hold. Then we choose \( r^0 \in T \) such that \( \ln \mathcal{M}_f(r^0) > 1 \) and \( r_j > e, j \in J, r_i > 1/2, i \in I \)

\[
\nu_n(E_1^* \cap \triangle_r, 0) \leq \int \ldots \int_{E_1^* \cap \triangle_r} \prod_{l \notin I} 1 - r_i \prod_{j \in J} \frac{1}{r_j} \, dr_1 \ldots dr_p \leq
\]

\[
\int \ldots \int_{E_1^* \cap \triangle_r} \frac{(1 - r_i) \partial}{\partial r_i} \ln \mathcal{M}_f(r) \prod_{l \in I} \ln(1 - r_i) \prod_{j \in J} \frac{1}{r_j} (\ln(1 - r_i))^{1+\delta} \prod_{j \in J} \frac{1}{r_j} \, dr_1 \ldots dr_p \leq
\]

Consider the mapping \( V: T \to \mathbb{R}^p_+ \), where \( V = (v_1(r), \ldots, v_p(r)), v_1(r) = \ln \mathcal{M}_f(r) \), \( v_2(r) = r_2, \ldots, v_{l-1}(r) = r_{l-1}, v_l(r) = \ln r_l, v_p(r) = \ln r_p \).
So

\[
J_2 := \frac{D(v_1(r), \ldots, v_p(r))}{D(r_1, \ldots, r_p)} = \begin{vmatrix} \frac{\partial v_1}{\partial r_1} & \cdots & \frac{\partial v_1}{\partial r_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_p}{\partial r_1} & \cdots & \frac{\partial v_p}{\partial r_p} \end{vmatrix} = \prod_{j \in J} \frac{1}{r_j} \frac{\partial}{\partial r_l} \ln \mathcal{M}_f(r).
\]

Therefore

\[
\nu_n(E_1^* \cap \triangle_r, 0) \leq \int \ldots \int_{U(E_2^* \cap \triangle_r)} v_1^{1+\delta}(1 - v_2)^{1-\delta} \ldots (1 - v_{l-1})^{1-\delta} \cdot v_l^{1+\delta} \ldots v_p^{1+\delta} < +\infty.
\]

Without loss generality we prove inequality (9) for \( j = p \). Let \( E_p \subset T \) be a set for which inequality (9) does not hold with \( j = p \)

\[
\nu_n(E_1^* \cap \triangle_r, 0) \leq \int \ldots \int_{E_p \cap \triangle_r} \frac{r_p \frac{\partial}{\partial r_p} \ln \mathcal{M}_f(r) \prod_{l \in I} \ln(1 - r_i) \prod_{j \in J} \frac{1}{r_j} \prod_{j \in J} \frac{1}{r_j} \, dr_1 \ldots dr_p}{(\ln(1 - r_i) \prod_{j \in J} \frac{1}{r_j} \prod_{j \in J} \frac{1}{r_j})^{1+\delta}} \leq
\]

Let \( r^0 \) be such that \( \ln \mathcal{M}_f(r^0) > 1 \). Define the mapping \( W: T \to T \), where \( W = (w_1(r), \ldots, w_p(r)) \), and
\( w_1(r) = r_1, \ldots, w_l(r) = r_l, \ldots, w_{l+1}(r) = r_{l+1}, \ldots, w_{p-1}(r) = \ln r_{p-1}, \ldots, w_p(r) = \ln \mathcal{M}_f(r) \).
So,

\[
J_3 := \begin{vmatrix} \frac{\partial w_1}{\partial r_1} & \cdots & \frac{\partial w_1}{\partial r_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_p}{\partial r_1} & \cdots & \frac{\partial w_p}{\partial r_p} \end{vmatrix} = \prod_{j \in J} \frac{1}{r_j} \frac{\partial}{\partial r_p} \ln \mathcal{M}_f(r).
\]

Therefore

\[
\nu_n(E_1^* \cap \triangle_r, 0) \leq \int \ldots \int_{E_p \cap \triangle_r} \frac{\frac{\partial}{\partial r_p} \ln \mathcal{M}_f(r) \prod_{l \in I} \ln(1 - r_i) \prod_{j \in J} \frac{1}{r_j} \prod_{j \in J} \frac{1}{r_j} \, dr_1 \ldots dr_p}{(\ln(1 - r_i) \prod_{j \in J} \frac{1}{r_j} \prod_{j \in J} \frac{1}{r_j})^{1+\delta}} \leq
\]

\[
\int \ldots \int_{E_p \cap \triangle_r} \frac{dw_1 \ldots dw_p}{\prod_{i \in I} (1 - w_i)^{1-\delta} \prod_{j \in J} w_j^{1+\delta}} < +\infty.
\]

It remains to remark, that if \( E_j \in \Upsilon, j \in \{1, \ldots, p\} \) then \( \bigcup_{j=0}^p E_j \in \Upsilon \). □
Remark that the exponent \(1 + \delta\) cannot be replaced a number smaller than 1. Suppose that one can replace this exponent \(\mu\) with \(\mu = \mu \times M\), where \(M = \prod_{i \in I} (1 - r_i)^\delta\). Then

\[
\mathcal{M}_f(r) \leq \frac{C}{C + 1} \mu_f(r) \prod_{j=1}^p (2\sqrt{cp}b_j + 2) \leq C^* \mu_f(r) \prod_{j=1}^p \sqrt{\frac{\partial^2}{\partial^2 r_j}} \ln \mathcal{M}_f(r) \leq \]

\[
\leq C^* \mu_f(r) \prod_{m \in B} \left[ \frac{1}{1 - r_m} \prod_{i \in I, i \neq m} (1 - r_i)^\delta \left( \frac{\partial}{\partial r_m} \ln \mathcal{M}_f(r) \prod_{j \in J} \ln r_j \right)^{1+\delta} \right]^{1/2} \times
\]

\[
\times \prod_{k \in D} \left[ \prod_{i \in I} \frac{1}{1 - r_i} \left( \frac{\partial}{\partial r_k} \ln \mathcal{M}_f(r) \prod_{j \in J \setminus k} \ln r_j \right)^{1+\delta} \right]^{1/2} \leq
\]

\[
\leq \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)^{1+p2\delta}} \ln p/2p2\delta \mathcal{M}_f(r) \prod_{j \in J} \left( \ln r_j \right)^{p+2p2\delta}.
\]

Then

\[
\ln \mathcal{M}_f(r) \leq \ln \mu_f(r) + (1 + p2\delta) \sum_{i \in I} \ln \frac{1}{1 - r_i} + \left( \frac{p}{2} + 2p2\delta \right) \ln \ln \mathcal{M}_f(r) + (p + 2p2\delta) \sum_{j \in J} \ln \ln r_j,
\]

\[
\ln \mathcal{M}_f(r) - (p/2 + 2p2\delta) \ln \ln \mathcal{M}_f(r) \leq (p + 2p2\delta) \ln \left( \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)} \prod_{j \in J} \ln r_j \right),
\]

\[
\ln \mathcal{M}_f(r) \leq (2p + 2p2\delta) \ln \left( \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)} \prod_{j \in J} \ln r_j \right),
\]

\[
\mathcal{M}_f(r) \leq \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)^{1+p2\delta}} \ln p/2p2\delta \left( \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)} \prod_{j \in J} \ln r_j \right)^{p+2p2\delta} \leq
\]

\[
\leq \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)^{1+p2\delta}} \ln p/2p2\delta \left( \sum_{j \in J} \ln \ln r_j \right)^{p+2p2\delta} \prod_{j \in J} \left( \ln r_j \right)^{p+2p2\delta} =
\]

\[
= \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)^{1+p2\delta}} \ln p/2p2\delta \left( \sum_{j \in J} \ln \ln r_j \right)^{p+2p2\delta} \prod_{j \in J} \left( \ln r_j \right)^{p+2p2\delta} \leq
\]

\[
\leq \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)^{1+p2\delta}} \ln p/2p2\delta \left( \sum_{j \in J} \ln \ln r_j \right)^{p+2p2\delta}.
\]

\[\Box\]

3. Sharpness of Theorem 1. Remark that the exponent \(1 + \delta\) at \(\prod_{i \in I} \frac{1}{1 - r_i}\) in inequality (6) cannot be replaced a number smaller than 1. Suppose that one can replace this exponent
by $\sigma \in (0,1)$. Consider the function

$$f(z) = f(z_1, \ldots, z_p) = \prod_{i\in I} \frac{1}{1-r_i} \prod_{j\in J} \gamma_j(z_j),$$

where $\gamma_j(z_j)$ are entire functions such that $\ln \mu_{\gamma_j}(r_j) < r_j$, $r \to +\infty$, $j \in J$. Then $M_f(r) = \prod_{i\in I} \frac{1}{1-r_i} \prod_{j\in J} M_{\gamma_j}(z_j)$ and $\mu_f(r) = \prod_{j\in D} \mu_{\gamma_j}(r_j)$. Denote $r'(\varepsilon,p)$ such that for all $r_j > r'(\varepsilon,p), i \in I$ we have

$$\left(\frac{1}{(1-r_i)^{1-\varepsilon}}\right)^{2/p} - \ln \frac{1}{1-r_i} > \varepsilon, \quad i \in I$$

and there exists $r_0 \in T$ such that

$$\Delta_{r_0} \cap B' = \left\{ r \in [r'(\varepsilon); 1]^l \times [\varepsilon; +\infty)^{p-l}, (\forall i \in I, \forall j \in J) : \left(\frac{1}{(1-r_i)^{1-\varepsilon}}\right)^{2/p} - \ln \frac{1}{1-r_i} > r_j \right\} \cap \Delta_{r_0} \subset \Delta_{r_0} \cap \Delta_{r_0} \subset \left\{ r \in [r'(\varepsilon); 1]^l \times [\varepsilon; +\infty)^{p-l}, (\forall i \in I, \forall j \in J) : \frac{1}{1-r_i} > \left(\frac{1}{(1-r_i)^{1-\varepsilon}}\right)^{p/2} \left(\ln \frac{1}{1-r_i} + r_j\right)^{p/2} \left(\ln r_j\right)^2 \right\} \cap \Delta_{r_0} \subset \left\{ r \in [r'(\varepsilon); 1]^l \times [\varepsilon; +\infty)^{p-l}, (\forall i \in I, \forall j \in J) : \prod_{i\in I} \frac{1}{1-r_i} = \prod_{j\in J} \mu_{\gamma_j}(r_j) > \prod_{i\in I} \frac{1}{1-r_i} \prod_{j\in J} \mu_{\gamma_j}(r_j) \ln^{p/2} \left(\prod_{i\in I} \frac{1}{1-r_i} \prod_{j\in J} \mu_{\gamma_j}(r_j) \right) \left(\ln r_j\right)^2 \right\} \cap \Delta_{r_0} \subset \left\{ r \in [r'(\varepsilon); 1]^l \times [\varepsilon; +\infty)^{p-l}, (\forall i \in I, \forall j \in J) : M_f(r) > \mu_f(r) \prod_{i\in I} \frac{1}{1-r_i} \prod_{j\in J} \left(\ln r_j\right)^2 \right\} \cap \Delta_{r_0} = B \cap \Delta_{r_0}.$$

Then the measure of the set $B$

$$\nu_{\infty}(B) = \int_{B' \cap \Delta_{r_0}} \int_{B' \cap \Delta_{r_0}} \prod_{i\in I} \frac{dr_i}{1-r_i} \prod_{j\in J} \frac{dr_j}{r_j} \geq \int_{B' \cap \Delta_{r_0}} \int_{B' \cap \Delta_{r_0}} \prod_{i\in I} \frac{dr_i}{1-r_i} \prod_{j\in J} \frac{dr_j}{r_j} \geq \int_{r'(\varepsilon,p)} \int_{r'(\varepsilon,p)} \left(\frac{1}{(1-r_i)^{1-\varepsilon}}\right)^{2/p} - \ln \frac{1}{1-r_i} \prod_{j\in J} \frac{dr_j}{r_j} \prod_{i\in I} \frac{dr_i}{1-r_i} = \frac{p-l}{2} \int_{r'(\varepsilon,p)} \int_{r'(\varepsilon,p)} \ln \left(\frac{1}{1-r_i}\right)^{2/p} - \ln \frac{1}{1-r_i} \prod_{i\in I} \frac{dr_i}{1-r_i} \geq \frac{p-l}{2} \int_{r'(\varepsilon,p)} \int_{r'(\varepsilon,p)} \prod_{i\in I} \frac{dr_i}{1-r_i} = +\infty.$$
We remark that any of the exponents 1 and \( p/2 \) in the inequality of Theorem 4 cannot be replaced by a smaller number. It follows from such a statement.

**Theorem 2.** There exist a function \( f \in \mathcal{A}^p(\mathbb{T}_l) \) and a constant \( C > 0 \) such that

\[
E = \left\{ r \in T : M_f(r) > C \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \ln^{p/2} \left( \frac{\mu_f(r)}{1 - r_i} \right) \right\}
\]

has asymptotically infinite logarithmic measure.

**Proof of Theorem 2.** Let us consider the function

\[
f(z) = f(z_1, \ldots, z_p) = \prod_{i \in D} \prod_{n_i=0}^{\infty} z_i^{n_i} \prod_{i \in I} \sum_{n_j=0}^{\infty} e^{n_j} z_j^{n_j} = \prod_{i \in I} \psi(z_i) \prod_{j \in J} \varphi(z_j), \quad \varepsilon \in (0, 1).
\]

Remark, that for this function there exists \( r_0 \) such that

\[
\Delta_{r_0} \cap \left\{ r \in T : M_f(r) > \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)^{1+\delta}} \ln^{p/2+\delta} \left( \frac{\mu_f(r)}{1 - r_i} \right) \right\} \subset
\]

\[
\subset \Delta_{r_0} \cap \left\{ r \in T : M_f(r) > \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)^{1+\delta}} \ln^{p/2+\delta} \left( \frac{\mu_f(r)}{1 - r_i} \right) \right\}.
\]

Then

\[
M_\varphi(r_i) \geq C_1(\varepsilon) \frac{\mu_\varphi(r_i)}{1 - r_i} \ln^{1/2} \frac{\mu_\varphi(r_i)}{1 - r_i}, \quad r_i > r_i^0, \quad i \in I,
\]

\[
M_\psi(r_j) \geq (\sqrt{2\pi} - \delta) \mu_\psi(r_j) \ln^{1/2} \mu_\psi(r_j), \quad \delta > 0, \quad r_j > r_j^0, \quad j \in J.
\]

So, \( M_f(r_1, r_2) = \prod_{i \in I} M_\varphi(r_i) \prod_{j \in J} M_\psi(r_j) \) and for \( r \in (r_i^0, 1)^l \times (r_j^0, +\infty)^{p-l} \) we have

\[
M_f(r) \geq (\sqrt{2\pi} - \delta)^{p-l} C_1^l(\varepsilon) \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \left( \prod_{i \in I} \ln \frac{\mu_\varphi(r_i)}{1 - r_i} \prod_{j \in J} \ln \mu_\psi(r_j) \right)^{1/2}.
\]

Consider positive increasing functions \( g_1(t) = \ln \frac{\mu_\varphi(t)}{1 - t} \) and \( g_2(t) = \ln \mu_\psi(t) \). We define

\[
A = \left\{ r \in T : g_1(r_{i1}) \ldots g_1(r_{i_l}) \cdot g_2(r_{i1+1}) \ldots g_2(r_p) > \frac{1}{2^l(2p-l)^p} (g_1(r_1) + \ldots + g_1(r_l) + g_2(r_{l+1}) + \ldots + g_2(r_p))^p \right\}
\]

\[
\sup \left\{ r \in T : (\forall i \in I)(\forall j \in J) \frac{1}{2} < \frac{g_1(r_i)}{g_2(r_j)} < 2 \right\} = E^*.
\]

Indeed, if \( r \in A \) then

\[
g_1(r_{i1}) \ldots g_1(r_{i_l}) g_2(r_{i1+1}) \ldots g_2(r_p) \geq \frac{g_2^2(r_p)}{2^l} \times \frac{1}{2^l(2p-l)^p} (g_1(r_1) + \ldots + g_1(r_l) + g_2(r_{l+1}) + \ldots + g_2(r_p))^p \geq
\]

\[
\geq \frac{1}{2^l(2p-l)^2} (g_1(r_1) + \ldots + g_1(r_l) + g_2(r_{l+1}) + \ldots + g_2(r_p))^p.
\]
There exists the inverse function \( g_2^{-1}: \mathbb{R}_+ \to (r_0, 1) \), which is also increasing. For \( r_1 \in (r_1^0, 1) \) we define \( r_1^* \) and \( r_2^* \) such that

\[
r_1^* = g_2^{-1}\left(\frac{g_1(r_1)}{2}\right), \quad r_2^* = g_2^{-1}(2g_1(r_1)).
\]

In [27] it was proved that

\[
r < g_2^{-1}(r) < \frac{3}{2} r, \quad r \to +\infty. \tag{13}
\]

Then from inequality (12) for all \( r \in E^* \) we deduce

\[
M_f(r) \geq \frac{(\sqrt{2\pi - \delta})^{p-1} C_1^0(\varepsilon)}{2^{l(2p - l)}^2} \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \left( \sum_{i \in I} \ln \mu_\varphi(r_i) + \sum_{j \in J} \ln \mu_\psi(r_j) \right)^{p/2} =
\]

\[
= \frac{(\sqrt{2\pi - \delta})^{p-1} C_1^0(\varepsilon)}{2^{l(2p - l)}^2} \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \ln^{p/2} \left\{ \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \right\}.
\]

It remains to prove that the set \( E^* \) is a set of infinite asymptotically logarithmic measure. Using (13) we get

\[
\nu_\ln(E^*) = \int_{E^* \cap \Delta r_1^0} \prod_{i \in I} \frac{dr_i}{1 - r_i} \prod_{j \in J} \frac{dr_j}{r_j} = \int_{r_0^1}^{1} \int_{r_0^1}^{1} \prod_{i \in I} \frac{dr_i}{1 - r_i} \prod_{j \in J} \frac{dr_j}{r_j} =
\]

\[
= \int_{r_0^1}^{1} \int_{r_0^1}^{1} \left( \ln r_2^* - \ln r_1^* \right)^{p-1} \prod_{i \in I} \frac{dr_i}{1 - r_i} =
\]

\[
= \int_{r_0^1}^{1} \int_{r_0^1}^{1} \left( \ln g_2^{-1}(2g_1(r_1)) - \ln g_2^{-1}(\frac{g_1(r_1)}{2}) \right) \prod_{i \in I} \frac{dr_i}{1 - r_i} \geq
\]

\[
= \int_{r_0^1}^{1} \int_{r_0^1}^{1} \left( \ln(2g_1(r_1)) - \ln(\frac{3g_1(r_1)}{2}) \right) \prod_{i \in I} \frac{dr_i}{1 - r_i} =
\]

\[
= \int_{r_0^1}^{1} \int_{r_0^1}^{1} \ln^{p-1} \frac{8}{3} \prod_{i \in I} \frac{dr_i}{1 - r_i} = +\infty.
\]

\[
\square
\]

**Proposition 1.** For any \( \varepsilon > 0 \) and for a function \( f \in \mathcal{A}^p(T_1) \) from the proof of Theorem 2 there exists \( r_0 \in T \) such that for all \( r \in \Delta r_0 \) we have

\[
\mu_f(r) 2^{-l} \geq 1, \quad \left( \ln \left( \frac{2}{3} \sum_{j \in J} r_j \right) \right)^{2-l-\varepsilon} \geq 1, \quad \left( \ln \ln \mu_f(r) \right)^a \geq \left( \prod_{j \in J} \ln r_j \right)^p,
\]

and the set
\[ E_1 = \left\{ r \in T: M_f(r) > 0 \right\} \]
\[ \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \ln^{n/2} \left( \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \right) \left( \prod_{j \in J} \ln r_j \right)^{p/2} (\ln \mu_f(r))^{-a} ; \quad (14) \]

has asymptotically infinite logarithmic measure, where \( a = p(p - l) + \varepsilon. \)

**Proof.** Consider the function
\[ f(z) = f(z_1, \ldots, z_p) = \prod_{i \in I} \left( \sum_{n_i=0}^{+\infty} e^{\frac{zn_i}{n_i+\varepsilon}} \right) \prod_{j \in J} \left( \sum_{n_j=0}^{+\infty} z_j^n \right) = \prod_{i \in I} \psi(z_i) \prod_{j \in J} \phi(z_j). \]

Remark that \( \ln \mu_\varphi(u) \geq \frac{2}{3} u, \ u \to +\infty \) (see [27]). Then there exists \( r_0 \in \mathbb{R}_+^p \) such that for all \( r \in \Delta_{r_0} \) we have
\[ \left( \frac{1}{2} \prod_{j \in J} \ln r_j \right)^{a - \varepsilon} \geq \left( \prod_{j \in J} \ln r_j \right)^{a - \frac{a}{2}}, \quad \left( \ln \left( \frac{2}{3} \sum_{j \in J} r_j \right) \right)^{\varepsilon} 2^{-l - \varepsilon} \geq 1, \quad \sum_{j \in J} \frac{2}{3} r_j > l \ln 2. \]

Thus, for all \( r \in \Delta_{r_0} \) we get
\[ \ln \mu_f(r) = \sum_{i \in I} \ln \mu_\psi(r_i) + \sum_{j \in J} \ln \mu_\phi(r_j) \geq \sum_{j \in J} \ln \mu_\phi(r_j) \geq \sum_{j \in J} \frac{2}{3} r_j > l \ln 2. \]

i.e. \( \mu_f(r) 2^{-l} \geq 1, \)

\[ (\ln \ln \mu_f(r))^a \geq \ln^a \left( \sum_{j \in J} \frac{2}{3} r_j \right) \geq \prod_{j \in J} \ln r_j + \ln \frac{2}{3} \geq \left( \frac{1}{2} \prod_{j \in J} \ln r_j \right)^a \geq \left( \prod_{j \in J} \ln r_j \right)^{\frac{a}{2}} = \left( \prod_{j \in J} \ln r_j \right)^{p}. \]

Therefore, \( E \cap \Delta_{r_0} \subset E_1 \cap \Delta_{r_0}. \) It implies that the set \( E_1 \) has asymptotically infinite logarithmic measure. \( \square \)

**4. Wiman’s inequality for random analytic functions in \( \mathbb{T} = \mathbb{T}_I \) and analytic functions with rapidly oscillating coefficients.** Let \( \Omega = [0, 1] \) and \( P \) be the Lebesgue measure on \( \mathbb{R}. \) We consider the Steinhaus probability space \( (\Omega, \mathcal{A}, P), \) where \( \mathcal{A} \) is the \( \sigma \)-algebra of Lebesgue measurable subsets of \( \Omega. \)

Let \( X = (X_n(t))_{\mathbb{Z}_+^p} \) be a real multiplicative system (MS) of random variables uniformly bounded by the number 1. That is for all \( n \in \mathbb{Z}_+^p \) and \( t \in [0, 1] \) we have \( |X_n(t)| \leq 1 \) for almost all \( t \in [0; 1] \) and
\[ (\forall k \in \mathbb{N}) (\forall \{i_1, i_2, \ldots, i_k \}: i_s \neq i_t \ (s \neq t) \in \mathbb{Z}_+^p): E(X_{i_1}X_{i_2}\cdots X_{i_k}) = 0, \]

where \( E \xi \) is the expectation of a random variable \( \xi \).

Let \( Z = (Z_n(t)) \) be a sequence of random complex variables \( Z_n(t) = X_n(t) + iY_n(t) \) such that both \( X = X_n(t) \) and \( Y = Y_n(t) \) are real MS.
Let $Z = (Z_n(t))$ be some sequence of complex-valued random variables defined in the space $(\Omega, \mathcal{A}, P)$. For an analytic function $f$ of the form (3) by $\mathcal{K}(f, Z)$ we denote the class of random analytic functions of the form

$$f(z, t) = \sum_{\|n\|=0}^{+\infty} a_n Z_n(t) z^n. \quad (15)$$

In the sequel, the notion “almost surely” (a.s.) will be used in the sense that the corresponding property holds almost everywhere with respect to the Lebesgue measure $P$ on $\Omega$. We say that some relation holds almost surely in the class $\mathcal{K}(f, Z)$ if it holds for each analytic function $f(z, t)$ of the form (15) almost surely in $t$.

In the case when $Z = (X_n(t))$ is the Rademacher sequence, under additional assumptions about the entire function $f$ P. Levy ([30]) proved that in the classical Wiman’s inequality a.s. in the class $K(f, Z)$ we can replace the constant $1/2$ by $1/4$. In the case when $Z = (X_n(t))$ is the Rademacher sequence P. Levy ([30]) proved that for any entire function we can replace the constant $1/2$ by $1/4$ in the Wiman’s inequality a.s. in the class $K(f, Z)$ (Levy’s phenomenon). P. Erdős and A. Rényi ([9]) proved the same result for the class $K(f, H)$, where $H = (e^{2\pi i \omega_n(t)}, (\omega_n(t)))$ is a sequence of independent uniformly distributed random variables on $[0, 1]$. This statement is true also for any class $K(f, X)$, where $X = (X_n(t))$ is a MS uniformly bounded by the number 1 ([31, 33, 34]).

In the spring of 1996 during the report of P. V. Filevych on the Lviv seminar of the theory of analytic functions professors A. A. Goldberg and M. M. Sheremeta posed the following question (see [21]). Does Levy’s phenomenon take place for analogues of Wiman’s inequality for entire functions of several complex variables?

In the papers [18, 19] one can find an affirmative answer to this question on Fenton’s inequality ([15]) for random entire functions of two complex variables, in [23] about an inequality from [14] for random entire functions of several complex variables, in [24] in the case of analytic functions in the polydisc.

### 4.1. Random analytic functions in $\mathcal{A}^p(\mathbb{T}_l)$.

Consider the class of random analytic functions $\mathcal{K}(f, Z)$ of the form (15) for an analytic function $f \in \mathcal{A}^p(\mathbb{T}_l)$ of the form (3) and a MS $Z = (Z_n(t))$. For $r = (r_1, \ldots, r_p)$ and a function $f(z, t)$ we denote $M_f(r, t) := \max\{|f(z, t)| : |z_1| = r_1, \ldots, |z_p| = r_p\}$.

We prove the following statement.

**Theorem 3.** Let $f \in \mathcal{A}^p$, $Z$ be a MS uniformly bounded by the number 1, $\delta > 0$. Then almost surely in $t$ there exists a set $E = E(f, t, \delta), E \subset \mathbb{Y}$ such that for all $r \in T \setminus E$ we have

$$M_f(r, t) \leq \mu_f(r) \sum_{i=1}^{1} \frac{1}{1 - r_i}^{1/2 + \delta} \ln^{p/4 + \delta} \left( \mu_f(r) \prod_{i=1}^{1} \left( 1 - r_i \right) \right) \left( \prod_{j \in J} \ln r_j \right)^{p/2 + \delta}. \quad (16)$$

To prove the Theorem 3 we need the following lemmas.

**Lemma 4** ([22]). Let $X = (X_n(t))$ be a MS uniformly bounded by the number 1. Then for all $\beta > 0$ there exists a constant $A_{3p} > 0$, which depends on $p$ and $\beta$ only such that for all $N \geq N_1(p) = \max\{p, 4\pi\}$ and $\{c_n: \|n\| \leq N\} \subset \mathbb{C}$ we have

$$P\left\{ t: \max \left\{ \sum_{\|n\|=0}^{N} c_n X_n(t) e^{i(n, \psi)} : \psi \in [0, 2\pi] \right\} \geq A_{3p} S_N \ln^{\frac{1}{p}} N \right\} \leq \frac{1}{N^{\beta}} \quad (17)$$
Proof of Lemma 5. We remark

\[ \sum_{||n||=0}^{+\infty} ||n|| |a_n|r^n \leq \mu_f(r) \prod_{i \in I} \frac{1}{(1-r_i)^{2+\delta}} \ln^{p/2+1+\delta} \left( \mu_f(r) \prod_{i \in I} \frac{1}{1-r_i} \right) \prod_{j \in J} (\ln r_j)^{p+1+\delta}. \]

Proof of Theorem 3. Without loss of generality we may suppose that \( Z = X = (X_{nm}(t)) \) is a real MS. For \( k, m \in \mathbb{Z}_+ \) and \( l \in \mathbb{Z} \) such that \( k > -l \) we denote

\[ G_{kl} = \left\{ r = (r_1, r_2) \in T : k \leq \sum_{i \in I} \ln \frac{1}{1-r_i} \leq k+1, \ l \leq \ln \mu_f(r) \leq l+1 \right\}, \]

\[ G_{klm} = \left\{ r = (r_1, r_2) \in G_{kl} : m \leq \sum_{j \in J} \ln r_j \leq m+1 \right\}, \ G_{kl}^+ = \bigcup_{i=k}^{+\infty} \bigcup_{j=l}^{+\infty} G_{ij}. \]
Remark that
\[ E_0 = \left\{ r \in T : \sum_{i \in I} \ln \frac{1}{1 - r_i} + \ln \mu_f(r) < 1 \right\} = \left\{ r \in T : \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} < e \right\} \in \mathcal{T}, \]

because there exists \( r_0 \) such that \( E_0 \cap \Delta_{r_0} = \emptyset \).

By Lemma 5 there exists a set \( E_1 \supseteq E_0, E_1 \in \mathcal{T} \) such that for all \( r \in T \setminus E_1 \) we have
\[
\sum_{\|n\| \geq d} |a_n|r^n \leq \sum_{\|n\| \geq d} \frac{\|n\|}{d} |a_n|r^n \leq \frac{1}{d} \sum_{\|n\| = 0}^{\infty} \|n\| |a_n|r^n \leq \frac{1}{d} m_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \prod_{j \in J} (\ln r_j)^{p+1+\delta} \leq \mu_f(r), \quad (18)
\]
where
\[ d = d(r) = \prod_{i \in I} \frac{1}{1 - r_i^{2+\delta}} \ln^{p/2+1+\delta} \left( \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \prod_{j \in J} (\ln r_j)^{p+1+\delta}. \right) \]

Let \( G_{kl}^* = G_{kl} \setminus E_2, I = \{(i, j) : G_{ij}^* \neq \emptyset\}, \]
\[ E_2 = E_0 \cup E_1 \cup \left( \bigcup_{(i, j) \notin I} G_{ij} \right). \]

Then \( \#I = +\infty \). For \((k, l) \in I\) we choose a sequence \( r^{(k, l)} \in G_{kl}^* \) such that
\[ M_f(r^{(k, l)}) = \min \{ M_f(r) : r \in G_{kl}^* \}. \]

So, for all \( r \in G_{kl}^* \) we get
\[
\frac{1}{e} \mu_f(r^{(k, l)}) \leq \mu_f(r) \leq e \mu_f(r^{(k, l)}), \quad (19)
\]
\[
\frac{1}{e} \prod_{i \in I} \frac{1}{1 - r^{(k, l)}_i} \leq \prod_{i \in I} \frac{1}{1 - r_i} \leq e \prod_{i \in I} \frac{1}{1 - r^{(k, l)}_i}, \quad (20)
\]
\[
\frac{1}{e^2} \mu_f(r^{(k, l)}) \prod_{i \in I} \frac{1}{1 - r^{(k, l)}_i} \leq \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \leq e^2 \mu_f(r^{(k, l)}) \prod_{i \in I} \frac{1}{1 - r^{(k, l)}_i}, \quad (21)
\]
and also
\[
\bigcup_{(k, l) \in I} G_{kl}^* = \bigcup_{(k, l) \in I} G_{kl} \setminus E_2 = \bigcup_{k, l = 1}^{+\infty} G_{kl} \setminus E_2 = T \setminus E_2.
\]

Denote \( N_{kl} = \lfloor 2d_1(r^{(k, l)}) \rfloor \), where
\[
d_1(r) = e^{2+\delta} \prod_{i \in I} \frac{1}{1 - r^{(k, l)}_i} \ln^{p/2+1+\delta} \left( e^2 \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \prod_{j \in J} (\ln e r_j)^{p+1+\delta}. \right)
\]

For \( r \in G_{kl}^* \) we put
\[
W_{N_{kl}}(r, t) = \max \left\{ \left| \sum_{\|n\| \leq N_{kl}} a_n r^n X_n(t) e^{i(n, \psi)} \right| : \psi \in [0, 2\pi]^p \right\}.
\]
For a Lebesgue measurable set $G \subset G^*_{kl}$ and for $(k, l) \in I$ we denote

$$\nu_{kl}(G) = \frac{\text{meas}(G)}{\text{meas}(G^*_{kl})},$$

where $\text{meas}$ denotes the Lebesgue measure on $\mathbb{R}^p$.

Remark that $\nu_{kl}$ is a probability measure defined on the $\sigma$-algebra of Lebesgue measurable subsets of $G^*_{kl}$ ([23]). Let

$$\Omega = \bigcup_{(k,l) \in I} G^*_{kl}$$

and

$k_i, l_{i,j}: (k_i, l_{i,j}) \in I$, $k_i < k_{i+1}$, $l_{i,j} < l_{i,j+1}$, $\forall i, j \in \mathbb{Z}_+$.

For Lebesgue measurable subsets $G$ of $\Omega$ we denote

$$\nu(G) = 2^{k_0} \sum_{i=0}^{+\infty} \left( \frac{1}{2^{k_i}} \left(1 - \left(\frac{1}{2}\right)^{k_{i+1} - k_i}\right) \right) \times \prod_{j=0}^{N_i} \frac{2^{k_{i,j}} - 1 - \left(\frac{1}{l}\right)^{l_{i,j} - 1 - l_{i,j+1}}}{1 - \left(\frac{1}{l}\right)^{l_i, l_{i+1} + l_{i,0}} \nu_{k_{i+1} l_{i+1}, l_{i+1} + l_{i,0}}(G \cap G^*_{kl})},$$

(22)

where $N_i = \max\{j: (k_i, l_{i,j}) \in I\}$. It easy to se that $\nu_{k_{i+1} l_{i+1}}(G^*_{k_{i+1} l_{i+1}}) = \nu(\Omega) = 1$.

Thus $\nu$ is a probability measure, which is defined on measurable subsets of $\Omega$. On $[0, 1] \times \Omega$ we define the probability measure $P_0 = P \otimes \nu$, which is the direct product of the probability measures $P$ and $\nu$. Now for $(k, l) \in I$ we define

$$F_{kl} = \{(t, r) \in [0, 1] \times \Omega: W_{N_{kl}}(r, t) > AS_{N_{kl}}(r) \ln^{1/2} N_{kl}\},$$

$$F_{kl}(r) = \{t \in [0, 1]: W_{N_{kl}}(r, t) > AS_{N_{kl}}(r) \ln^{1/2} N_{kl}\},$$

where

$$S_{N_{kl}}^2(r) = \sum_{\|n\|=0}^{N_{kl}} |a_n|^2 r^{2n},$$

and $A$ is the constant from Lemma 4 with $\beta = 1$.

Using Fubini’s theorem and Lemma 4 with $c_n = a_n r^n$ and $\beta = 1$, we get for $(k, l) \in I$

$$P_0(F_{kl}) = \int_{\Omega} \left( \int_{F_{kl}(r)} dP \right) d\nu = \int_{\Omega} P(F_{kl}(r)) d\nu \leq \frac{1}{N_{kl}} \nu(\Omega) = \frac{1}{N_{kl}}.$$
By Borel-Cantelli’s lemma the infinite quantity of the events \( \{ F_{kl} : (k, l) \in I \} \) may occur with probability zero. So,

\[
P_0(F) = 1, \quad F = \bigcup_{s=1}^{+\infty} \bigcup_{m=1}^{+\infty} \bigcap_{k \geq s, \ l \geq m} F_{kl} \subset [0, 1] \times \Omega.
\]

Then for any point \((t, r) \in F\) there exist \(k_0 = k_0(t, r)\) and \(l_0 = l_0(t, r)\) such that for all \(k \geq k_0, \ l \geq l_0, \ (k, l) \in I\) we have

\[
W_{N_{kl}}(r, t) \leq A S_{N_{kl}}(r) \ln^{1/2} N_{kl}.
\]

So, \(\nu(F^\wedge(t)) = 1\) (see [23]).

For any \(t \in F_1\) ([23]) and \((k, l) \in I\) we choose a point \(r_0^{(k,l)}(t) \in \Gamma^*_{kl}\) such that

\[
W_{N_{kl}}(r_0^{(k,l)}(t), t) \geq \frac{3}{4} M_{kl}(t), \quad M_{kl}(t) \stackrel{\text{def}}{=} \sup \{ W_{N_{kl}}(r, t) : r \in \Gamma^*_{kl} \}.
\]

Thus, \(\nu_{kl}(F^\wedge(t) \cap \Gamma^*_{kl}) = 1\) for all \((k, l) \in I\) it follows that there exists a point \(r^{(k,l)}(t) \in \Gamma^*_{kl} \cap F^\wedge(t)\) such that

\[
|W_{N_{kl}}(r_0^{(k,l)}(t), t) - W_{N_{kl}}(r^{(k,l)}(t), t)| < \frac{1}{4} M_{kl}(t),
\]

hence

\[
W_{N_{kl}}(r^{(k,l)}(t), t) - W_{N_{kl}}(r_0^{(k,l)}(t), t) < \frac{1}{4} M_{kl}(t).
\]

Thus,

\[
\frac{3}{4} M_{kl}(t) \leq W_{N_{kl}}(r_0^{(k,l)}(t), t) \leq W_{N_{kl}}(r^{(k,l)}(t), t) + \frac{1}{4} M_{kl}(t).
\]

Since \((t, r^{(k,l)}(t)) \in F\), we obtain

\[
\frac{1}{2} M_{kl}(t) \leq W_{N_{kl}}(r^{(k,l)}(t), t) \leq A S_{N_{kl}}(r^{(k,l)}(t)) \ln^{1/2} N_{kl}.
\]

Now for \(r^{(k,l)} = r^{(k,l)}(t)\) we get

\[
S_{N_{kl}}^2(r^{(k,l)}) \leq \mu_f(r^{(k,l)}) \mathcal{N}_f(r^{(k,l)}) \leq \mu_f^2(r^{(k,l)}) \prod_{i \in I} \frac{1}{1 - r_i^{(k,l)}} \ln^{p/2 + \delta} \left( \mu_f(r^{(k,l)}) \prod_{i \in I} \frac{1}{1 - r_i^{(k,l)}} \left( \prod_{j \in J} \ln r_j^{(k,l)} \right)^{p + \delta} \right).
\]

So, for \(t \in F_1\) and all \(k \geq k_0(t), \ l \geq l_0(t)\), we obtain

\[
S_{N_{kl}}(r^{(k,l)}) \leq \mu_f(r^{(k,l)}) \prod_{i \in I} \frac{1}{1 - r_i^{(k,l)}} \ln^{p/4 + \delta} \left( \mu_f(r^{(k,l)}) \prod_{i \in I} \frac{1}{1 - r_i^{(k,l)}} \left( \prod_{j \in J} \ln r_j^{(k,l)} \right)^{p/2 + \delta} \right). \tag{23}
\]

It follows from (19)–(21) that \(d_1(r^{(k,l)}) \geq d(r)\) for \(r \in \Gamma^*_{kl}\). Then for \(t \in F_1, r \in F^\wedge(t) \cap \Gamma^*_{kl}, \ (k, l) \in I, \ k \geq k_0(t), \ l \geq l_0(t)\) we get

\[
M_f(r, t) \leq \sum_{\|n\| \geq 2d_1(r^{(k,l)})} |a_n| r^n + W_{N_{kl}}(r, t) \leq \sum_{\|n\| \geq 2d(r)} |a_n| r^n + M_{kl}(t).
\]
Finally, for \( t \in F_1, r \in F^\wedge(t) \cap G^*_{kl}, l \geq l_0(t) \) and \( k \geq k_0(t) \) we deduce

\[
M_f(r^{(k,l)}, t) \leq \mu_f(r^{(k,l)}) + 2AS_{N_k}(r^{(k,l)}) \ln^{1/2} N_{kl} \leq \mu_f(r^{(k,l)}) + 2A \mu_f(r^{(k,l)}) \prod_{i \in I} \frac{1}{(1 - r_i^{(k,l)})^{1/2 + \delta}} \ln^{p/4 + \delta} \left( \mu_f(r^{(k,l)}) \prod_{i \in I} \frac{1}{1 - r_i^{(k,l)}} \right) \left( \prod_{j \in J} \ln r_j^{(k,l)} \right)^{p/2 + \delta} \times \]

\[
\times \ln \left( e^{2 + \delta} \prod_{i \in I} \frac{1}{(1 - r_i^{(k,l)})^{2 + \delta}} \ln^{p/2 + 1 + \delta} \left( e^2 \mu_f(r^{(k,l)}) \prod_{i \in I} \frac{1}{1 - r_i^{(k,l)}} \prod_{j \in J} \ln(r_j^{(k,l)}) \right)^{p + 1 + \delta} \right).
\]

Therefore the following inequality

\[
M_f(r, t) \leq \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)^{1/2 + \delta}} \ln^{p/4 + 2\delta} \left( \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \right) \left( \prod_{j \in J} \ln r_j \right)^{p/2 + 2\delta}.
\]

holds a.s. \( (t \in F_1, P(F_1) = 1) \) for all

\[
r \in \left( \bigcup_{(k,l) \in I} (G^*_{kl} \cap F^\wedge(t)) \cap G^*_{kl} \right) \backslash E^* = (T \cap G^+_{kl}) \backslash (E^* \cup G^* \cup E_1) = T \backslash E_2,
\]

where

\[
G^*_{kl} = \bigcup_{i = k}^{+\infty} G_{kli}, \quad E_2 = E_1 \cup G^* \cup E^*, \quad G^* = \bigcup_{(k,l) \in I} (G^*_{kl} \backslash F^\wedge(t)).
\]

It remains to remark that \( \nu(G^*) \) satisfies

\[
\nu(G^*) = \sum_{(k,l) \in I} (\nu_{kl}(G^*_{kl}) - \nu_{kl}(F^\wedge(t))) = 0.
\]

Then for all \( (k,l) \in I \) we obtain

\[
\nu_{kl}(G^*_{kl} \backslash F^\wedge(t)) = \frac{\text{meas}(G^*_{kl} \backslash F^\wedge(t))}{\text{meas}(G^*_{kl})} = 0,
\]

\[
\text{meas}(G^*_{kl} \backslash F^\wedge(t)) = \int_{G^*_{kl} \backslash F^\wedge(t)} \prod_{i \in I} \frac{dr_i}{1 - r_i} \prod_{j \in J} \frac{dr_j}{r_j} = 0.
\]

\[
\square
\]

### 4.2. Analytic functions with rapidly oscillating coefficients.

We consider the class \( \mathcal{K}(f, \theta) \) of analytic functions

\[
f(z, t) = f(z_1, \ldots, z_p, t) = \sum_{|n| = 0}^{+\infty} a_n e^{2\pi i \theta_n t} z^n.
\]

(24)

Here \( \theta = (\theta_n) \) is a sequence of positive integer such that its increasing arrangement \( (\theta^*_k) \) \( \{\theta_n: n \in \mathbb{Z}_+^p\} = \{\theta^*_k: k \in \mathbb{Z}_+\} \), \( \theta^*_k > \theta^*_{k+1} \), satisfies the condition \( \theta \) is Hadamard sequence

\[
\frac{\theta^*_k}{\theta^*_k} > q > 1, k > 0,
\]

(25)
where \( f \in A^p(T_l) \). Remark, that in the case \( q \geq 2 \) analytic functions of the form (24) satisfy the assumptions of Theorem 3, because \((\cos \theta_t), (\sin \theta_t)\) are MS. But in the case \( q > 1 \) the sequence of random variables \((\cos \theta_t, t \in Z^+)\) need not be a MS (see [21]). So the following question arrives naturally: does Levy’s phenomenon hold for the class \( K(f, \theta) \) with \( f \in A^p \) and an Hadamard sequence \( \theta^* \)?

A theorem analogous to Theorem 3 is also valid for analytic functions with rapidly oscillating coefficients.

**Theorem 4.** Let \( \delta > 0, f \in K(f, \theta) \) be an analytic function of the from (24) and a sequence of positive integers \( (\theta_n)_{n \in Z^+} \) satisfy condition (25). Then almost surely for \( t \in R \) there exists \( E(\delta, t) \in Y \) such that for all \( r \in T \setminus E \) we have

\[
M_f(r, t) \leq \mu_f(r) \prod_{i \in I} \frac{1}{(1 - r_i)\sqrt{1 + \delta}} \ln^{p/4 + \delta} \left( \frac{\mu_f(r)}{\prod_{j \in J} \ln r_j} \right)^{(p-1)/4 + \delta}.
\]

(26)

One can find similar inequalities for entire functions of one complex variable in [9, 10], for analytic functions in the unit disc in [13], for entire functions of several variables in [18], [19], [23], [26], [29], and for analytic functions without exceptional sets in [16, 32].

To prove Theorem 4 we need an auxiliary assertions.

**Lemma 6 ([10]).** Let \( (\theta_k^* N)_{k=1} \) be a sequence of integers such that condition (25) holds. Then there exist constants \( A_q, B_q \) (depending only on \( q \)) such that for any \( \{b_k: 1 \leq k \leq N\} \subset C \) and \( \lambda > 0 \) we have

\[
P \left\{ t: \left| \sum_{k=1}^{N} b_k e^{2 \pi i \theta_k^* t} \right| \geq A_q \lambda S_N \right\} \leq B_q e^{-\lambda^2},
\]

where \( S_N^2 = \sum_{k=1}^{n} |b_k|^2, P \) is the Lebesque measure on \([0; 1]\).

**Lemma 7 ([28]).** Let \( \theta = (\theta_n)_{n \in Z^+} \) be a sequence of integers satisfying (25). Then for any \( \beta > 0, p \geq 1, l \in N, l \geq p \) and \( \{c_n: n \in Z^+\} \subset C \) we get

\[
P \left\{ t: \max \left\{ \left| \sum_{|n|=0}^{l} c_n \exp \left( i(n, \psi) + 2 \pi i \theta_n t \right) \right| : \psi \in [0; 2\pi]^p \right\} \geq A_{\beta p} l^{1/2} \right\} \leq \frac{(5\pi + 1)^p B}{l^{\beta}},
\]

where

\[
(n, \psi) = \sum_{s=1}^{p} n_s \psi_s, \quad S_l^2 = \sum_{|n|=0}^{l} |c_n|^2, \quad A = \sqrt{\frac{\beta + p}{2}(3 + p)}A_q + 1
\]

and \( B = B_q (A_q, B_q) \) are constants from Lemma 6).

**Proof of Theorem 4.** To prove Theorem 4, we need to rewrite formally the proof of Theorem 3, using Lemma 7 instead of Lemma 5. We need to use Lemma 5 and Theorem 1 again. The only formal difference is that in the proof of Theorem 3 we need to change the definition \( W_{N,t}(r, t) \) onto

\[
W_{N,t}(r, t) = \max \left\{ \left| \sum_{|n|=0}^{l} a_n r^n e^{i(n, \psi) + 2 \pi i \theta_n t} \right| : \psi \in [0; 2\pi]^p \right\}.
\]

Retention all the other notations from the proof of Theorem 3 and literally rewriting this proof, we obtain the assertion of Theorem 4.
5. Sharpness of Theorems 3 and 4. We will prove that any of exponents \(p/4+\delta\) and \(1/2+\delta\) in inequality (5) cannot be replaced by a number smaller than \(p/4\) and \(1/2\), respectively.

**Theorem 5.** Let \(1 \leq l < p\) and \(Z = (Z_n(t))\) be a sequence of random variables such that \((\forall n): |Z_n(t)| \geq 1\) a.s. in \([0, 1]\). Then for any \(\varepsilon > 0\) there exist an analytic function \(f \in A^p(T_l)\), a constant \(C > 0\) and \(r_0 \in T\) such that \(a.s.\ in\ t\ for all \(r \in \Delta_{r_0}\) we get

\[
M_f(r, t) \geq \mu_f(r) \prod_{i \leq l} \frac{1}{\sqrt{1 - r_i}} \ln^{p/4} \left( \mu_f(r) \prod_{i \leq l} \frac{1}{1 - r_i} \right) \left( \prod_{j \in J} \ln r_j \right)^{p/2} (\ln \ln \mu_f(r))^{-p(p-l)/2 - \varepsilon}.
\]

**Proof.** Consider the functions

\[
g(z) = \prod_{i \leq l} \left( \sum_{n_i = 0}^{+\infty} e^{\sqrt{n_i} z_i} n_i \right) \prod_{j \in J} \left( \sum_{n_j = 0}^{+\infty} \frac{z_j}{n_j!} \right) = \sum_{\|n\| = 0} g_n z^n,
\]

\[
f(z) = \prod_{i \leq l} \left( \sum_{n_i = 0}^{+\infty} e^{\sqrt{n_i} z_i} n_i \right) \prod_{j \in J} \left( \sum_{n_j = 0}^{+\infty} \frac{z_j}{\sqrt{n_j}} \right) = \sum_{\|n\| = 0} f_n z^n,
\]

\[f(z, t) = \sum_{\|n\| = 0} Z_n(t) f_n z^n.
\]

Remark, that for all \(r \in T\) we have

\[
\mu_g(r^2) = \max \left\{ \prod_{i \leq l} \frac{r_i^{2n_i}}{n_i!} \prod_{j \in J} e^{\sqrt{n_j} r_j^{2n_j}} : n \in \mathbb{Z}_+^p \right\} = \max \left\{ \left( \prod_{i \leq l} \frac{r_i^{n_i}}{\sqrt{n_i}} \prod_{j \in J} e^{\sqrt{n_j} r_j^{n_j}} \right)^2 : n \in \mathbb{Z}_+^p \right\} = (\mu_f(r))^2.
\]

Using Parseval’s equality, we get for almost all \(t\)

\[
M_g(r^2) \leq \sum_{\|n\| = 0}^{+\infty} |Z_n(t)|^2 |g_n| r^{2n} = \frac{1}{(2\pi)^p} \int_{|0, 2\pi|^p} \int |f(re^{i\theta}, t)|^2 d\theta \leq (M_f(r, t))^2.
\]

Then by Proposition 1 there exists \(r_0 \in T\) such that for \(r \in \Delta_{r_0} \cup E\)

\[
M_g(r) > \mu_g(r) \prod_{i \leq l} \frac{1}{1 - r_i} \ln^{p/2} \left( \mu_g(r) \prod_{i \leq l} \frac{1}{1 - r_i} \right) \left( \prod_{j \in J} \ln r_j \right)^{p/2} (\ln \mu_g(r))^{-a},
\]

where \(a = p(p-l) + \varepsilon\).

Hence, for all \(r\) such that \(r^2 = (r_1^2, \ldots, r_p^2) \in \Delta_{r_0} \cap E\) one has

\[
(M_f(r, t))^2 \geq M_g(r^2) \geq \mu_g(r^2) \prod_{i \leq l} \frac{1}{1 - r_i^2} \ln^{p/2} \left( \mu_g(r^2) \prod_{i \leq l} \frac{1}{1 - r_i^2} \right) \left( \prod_{j \in J} \ln r_j^2 \right)^{p/2} (\ln \mu_g(r^2))^{-a} \geq \mu_f^2(r) \prod_{i \leq l} \frac{1}{1 - r_i^2} \ln^{p/2} \left( \mu_f^2(r) \prod_{i \leq l} \frac{1}{1 - r_i^2} \right) \left( \prod_{j \in J} \ln r_j^2 \right)^{p/2} (\ln \mu_f^2(r))^{-a}.
\]
But $1 - r_i^2 \leq 2(1 - r_i)$ ($i \in I$), and by Proposition 1

$$\left( \ln \ln \mu_f(r) \right)^\epsilon \geq \left( \ln \left( \frac{2}{3} \sum_{j \in J} r_j \right) \right)^\epsilon \geq 2^{l + \epsilon}.$$  

Thus

$$\left( M_f(r, t) \right)^2 \geq \mu_f^2(r) 2^{-l} \prod_{i \in I} \frac{1}{1 - r_i} \ln^{p/2} \left( \mu_f^2(r) \prod_{i \in I} \frac{1}{1 - r_i} \right) \left( \mu_f(r) \prod_{j \in J} \ln r_j \right)^p \left( \ln \ln \mu_f(r) \right)^{-a} \geq \mu_f^2(r) \prod_{i \in I} \frac{1}{1 - r_i} \ln^{p/2} \left( \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \right) \left( \prod_{j \in J} \ln r_j \right)^p \left( \ln \ln \mu_f(r) \right)^{-a - \epsilon}.$$  

Therefore, for all $r$ such that $r^2 = (r_1^2, \ldots, r_p^2) \in \Delta_{r_0} \cap E$ we obtain

$$M_f(r, t) \geq \mu_f(r) \prod_{i \in I} \frac{1}{\sqrt{1 - r_i}} \ln^{p/4} \left( \mu_f(r) \prod_{i \in I} \frac{1}{1 - r_i} \right) \left( \prod_{j \in J} \ln r_j \right)^{p/2} \left( \ln \ln \mu_f(r) \right)^{-a/2 - \epsilon_1},$$

where $\epsilon_1 = \epsilon/2$.  

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