УДК 517.5

S. Bardyla

ON LOCALLY COMPACT SEMITOPOLOGICAL GRAPH INVERSE SEMIGROUPS

S. Bardyla. *On locally compact semitopological graph inverse semigroups*, Mat. Stud. **49** (2018), 19–28.

In this paper we investigate locally compact semitopological graph inverse semigroups. Our main result is the following: if a directed graph E is strongly connected and has finitely many vertices, then any Hausdorff shift-continuous locally compact topology on the graph inverse semigroup $G(E)$ is either compact or discrete. This result generalizes results of Gutik and Bardyla who proved the above dichotomy for Hausdorff locally compact shift-continuous topologies on polycyclic monoids $\mathcal{P}_1$ and $\mathcal{P}_\lambda$, respectively.

1. Introduction and background. In this paper all topological spaces are assumed to be Hausdorff. We shall follow the terminology of [11, 14, 19, 26]. A semigroup S is called an *inverse semigroup* if for each element $a \in S$ there exists a unique element $a^{-1} \in S$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. The element a^{-1} is called the *inverse* of a. The map $S \to S$, $x \mapsto x^{-1}$ assigning to each element of an inverse semigroup its inverse is called the *inversion*.

A *directed graph* $E = (E^0, E^1, r, s)$ consists of sets E^0, E^1 of *vertices* and *edges*, respectively, together with functions $s, r\colon E^1 \to E^0$, called the *source* and the *range* functions, respectively. In this paper we refer to directed graphs simply as “graphs". A *path* $x = e_1 \dots e_n$ in a graph E is a finite sequence of edges $e_1, \dots, e_n$ such that $r(e_i) = s(e_{i+1})$ for each positive integer $i < n$. We extend the source and range functions s and r on the set $\mathrm{Path}(E)$ of all pathes in graph E as follows: for each $x = e_1 \dots e_n \in \mathrm{Path}(E)$ put $s(x) = s(e_1)$ and $r(x) = r(e_n)$. By $|x|$ we denote the length of the path x. We consider each vertex being a path of length zero. An edge e is called a *loop* if $s(e) = r(e)$. A path x is called a *cycle* if $s(x) = r(x)$ and $|x| > 0$. Let $a = e_1 \dots e_n$ and $b = f_1 \dots f_m$ be two paths such that $r(a) = s(b)$. Then by ab we denote the path $e_1 \dots e_n f_1 \dots f_m$. A path x is called a *prefix* (resp. *suffix*) of a path y if there exists path z such that $y = xz$ (resp. $y = zx$). A graph E is called *finite* if the sets E^0 and E^1 are finite and *infinite* in the other case. A graph E is called *strongly connected* if for each pair of vertices $e, f \in E^0$ there exist paths $u, v \in \mathrm{Path}(E)$ such that $s(u) = r(v) = e$ and $s(v) = r(u) = f$.

A topological (inverse) semigroup is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively). If S is a semigroup (an inverse semigroup) and τ is a topology on S such that (S, τ) is a topological (inverse) semigroup, then we shall call τ a (*inverse*) *semigroup topology* on S. A semitopological semigroup is a Hausdorff topological space together with a separately continuous semigroup operation. For

2010 *Mathematics Subject Classification:* 20M18, 22A15, 54D45.
Keywords: locally compact space; semitopological semigroup; polycyclic monoid; graph inverse semigroup.
doi:10.15330/ms.49.1.19-28

each element x of a semigroup S the map $l_x(s)\colon s \to xs$ ($r_x(s)\colon s \to sx$, resp.) is called a left (right, resp.) shift on the element x. Observe that semigroup S endowed with a topology is semitopological iff for each element $x \in S$ left and right shifts are continuous. A topology τ on a semigroup S is called shift-continuous if (S, τ) is a semitopological semigroup. A semitopological inverse semigroup S is called quasi-topological if the inversion map $S \to S$, $x \mapsto x^{-1}$, is continuous.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subject to the condition $pq = 1$. The bicyclic semigroup admits only the discrete semigroup topology [13]. In [10] this result was extended over the case of semitopological semigroups. The closure of a bicyclic semigroup in a locally compact topological inverse semigroup was described in [13]. In [15] Gutik proved the following theorem.

Theorem A ([15, Theorem 1]). *Any locally compact shift-continuous topology on the bicyclic monoid with adjoined zero is either compact or discrete.*

In [6] Gutik's theorem was generalized over the α-bicyclic monoid.

One of generalizations of the bicyclic semigroup is a λ-polycyclic monoid. For a non-zero cardinal λ, the λ-polycyclic monoid $\mathcal{P}_\lambda$ is the semigroup with identity and zero given by the presentation:

$$\mathcal{P}_\lambda = \left\langle \{p_i\}_{i\in\lambda}, \{p_i^{-1}\}_{i\in\lambda} \mid p_i^{-1}p_i = 1, p_j^{-1}p_i = 0 \text{ for } i \neq j \right\rangle.$$

Polycyclic monoid $\mathcal{P}_k$ over a finite cardinal k was introduced in [24]. Algebraic properties of a semigroup $\mathcal{P}_k$ were investigated in [20] and [21]. Algebraic and topological properties of the λ-polycyclic monoid were investigated in [8] and [9]. In particular, it was proved that for every non-zero cardinal λ the only locally compact semigroup topology on the λ-polycyclic monoid is the discrete topology. Observe that the bicyclic semigroup with an adjoined zero is isomorphic to the polycyclic monoid $\mathcal{P}_1$. Hence Gutik's Theorem A can be reformulated in the following way: any locally compact shift-continuous topology on the polycyclic monoid $\mathcal{P}_1$ is either compact or discrete. In [7] Theorem A was generalized as follows.

Theorem B ([7, Main Theorem]). *Any locally compact shift-continuous topology on the λ-polycyclic monoid $\mathcal{P}_\lambda$ is either compact or discrete.*

For a directed graph $E = (E^0, E^1, r, s)$ the graph inverse semigroup (or simply GIS) $G(E)$ over E is a semigroup with zero generated by the sets E^0, E^1 together with the set $E^{-1} = \{e^{-1} \mid e \in E^1\}$ satisfying the following relations for all $a, b \in E^0$ and $e, f \in E^1$:

(i) $a \cdot b = a$ if $a = b$ and $a \cdot b = 0$ if $a \neq b$;

(ii) $s(e) \cdot e = e \cdot r(e) = e$;

(iii) $e^{-1} \cdot s(e) = r(e) \cdot e^{-1} = e^{-1}$;

(iv) $e^{-1} \cdot f = r(e)$ if $e = f$ and $e^{-1} \cdot f = 0$ if $e \neq f$.

Graph inverse semigroups are generalizations of the polycyclic monoids. In particular, for every non-zero cardinal λ, the λ-polycyclic monoid is isomorphic to the graph inverse semigroup over the graph E which consists of one vertex and λ distinct loops. However, in [4] it was proved that the λ-polycyclic monoid is a universal object in the class of graph inverse semigroups. More precisely, each GIS $G(E)$ embeds as an inverse subsemigroup into the λ-polycyclic monoid $\mathcal{P}_\lambda$ with $\lambda \geq |G(E)|$.

According to [16, Chapter 3.1], each non-zero element of the graph inverse semigroup $G(E)$ is of the form uv^{-1} where $u, v \in \operatorname{Path}(E)$ and $r(u) = r(v)$. A semigroup operation in $G(E)$ is defined by the formulas:

$$u_1v_1^{-1} \cdot u_2v_2^{-1} = \begin{cases} u_1wv_2^{-1}, & \text{if } u_2 = v_1w \text{ for some } w \in \operatorname{Path}(E); \\ u_1(v_2w)^{-1}, & \text{if } v_1 = u_2w \text{ for some } w \in \operatorname{Path}(E); \\ 0, & \text{otherwise,} \end{cases}$$

$$uv^{-1} \cdot 0 = 0 \cdot uv^{-1} = 0 \cdot 0 = 0.$$

Simple verifications show that $G(E)$ is an inverse semigroup and $(uv^{-1})^{-1} = vu^{-1}$.

We shall say that GIS $G(E)$ satisfies condition $(\star)$ if for each infinite subset $A \subset \operatorname{Path}(E)$ there exists an infinite subset $B \subset A$ and an element $\mu \in G(E)$ such that for each $x \in B$, $\mu \cdot x \in \operatorname{Path}(E)$ and $|\mu \cdot x| > |x|$.

Graph inverse semigroups play an important role in the study of rings and C^*-algebras (see [1, 3, 12, 18, 25]). Algebraic properties of graph inverse semigroups were studied in [2, 4, 16, 17, 20, 22]. In [23] it was showed that a locally compact topological GIS $G(E)$ over a finite graph E is discrete. In [5, Theorem 1] the author characterized graph inverse semigroups admitting only discrete locally compact semigroup topology:

Theorem C. *The discrete topology is the only locally compact semigroup topology on a graph inverse semigroup $G(E)$ if and only if $G(E)$ satisfies the condition $(\star)$.*

Further we shall often use the following fact proved in [23, Lemma 1]:

Lemma 1. *For any $a, b \in G(E) \setminus \{0\}$, the sets $\{x \in G(E) \mid x \cdot a = b\}$ and $\{x \in G(E) \mid a \cdot x = b\}$ are finite.*

2. Main results. Let $G(E)$ be the graph inverse semigroup over a graph E. Fix an arbitrary vertex $e \in E^0$ and let $C^e := \{u \in \operatorname{Path}(E) \mid s(u) = r(u) = e\}$. Put

$$C_1^e := \{u \in C^e \mid r(v) \neq e \text{ for each non-trivial prefix } v \text{ of } u\}.$$

By $\langle C^e \rangle$ (resp. $\langle C_1^e \rangle$) we denote the inverse subsemigroup of $G(E)$ which is generated by the set C^e (resp. C_1^e). Observe that $e \in C_1^e$ and e is the identity in $\langle C^e \rangle$.

Lemma 2. *For each vertex $e \in E^0$ of an arbitrary graph E the following statements hold:*

1) *if $C_1^e = \{e\}$ then $\langle C^e \rangle = \{e\}$;*
2) *if $|C_1^e \setminus \{e\}| = 1$ then $\langle C^e \rangle$ is isomorphic to the bicyclic monoid;*
3) *if $|C_1^e \setminus \{e\}| = \lambda > 1$ then $\langle C^e \rangle$ is isomorphic to the λ-polycyclic monoid $\mathcal{P}_\lambda$.*

Proof. Fix an arbitrary vertex $e \in E^0$. The statement 1 is obvious.

Now we prove the statement 3. Suppose that $|C_1^e \setminus \{e\}| = \lambda > 1$. Let $C_1^e \setminus \{e\} = \{u_\alpha\}_{\alpha \in \lambda}$ be an enumeration of $C_1^e \setminus \{e\}$. For convenience we put $e = u_{-1}$. Observe that for each element $v \in C^e$ there exist elements $u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_n} \in C_1^e$ such that $v = u_{\alpha_1}u_{\alpha_2}\ldots u_{\alpha_n}$. Simple verifications show that

$$\langle C_1^e \rangle = \{uv^{-1} \mid u, v \in C^e\} \cup \{0\} = \langle C^e \rangle.$$

Let $G = \{p_\alpha\}_{\alpha\in\lambda} \cup \{p_\alpha^{-1}\}_{\alpha\in\lambda}$ be the set of generators of $\mathcal{P}_\lambda$. We define a map $f\colon C_1^e \to \mathcal{P}_\lambda$ in the following way: $f(u_{-1}) = 1$ and $f(u_\alpha) = p_\alpha$ for each $\alpha \in \lambda$. Extend the map f on the set $\langle C^e\rangle$ in the following way: for each element $u = u_{\alpha_1}u_{\alpha_2}\ldots u_{\alpha_n} \in C^e$ put $f(u) = p_{\alpha_1}p_{\alpha_2}\ldots p_{\alpha_n}$. For each non-zero element $uv^{-1} \in \langle C^e\rangle$ put $f(uv^{-1}) = f(u)f(v)^{-1}$ and $f(0) = 0$. Obviously, f is a bijection. Let us show that f is a homomorphism. Fix arbitrary elements $ab^{-1}, cd^{-1} \in \langle C^e\rangle$, where

$$a = u_{\alpha_1}\ldots u_{\alpha_n}, b = u_{\beta_1}\ldots u_{\beta_m}, c = u_{\gamma_1},\ldots u_{\gamma_k}, d = u_{\delta_1}\ldots u_{\delta_t}.$$

There are three cases to consider:

(1) $ab^{-1}\cdot cd^{-1} = ac_1d^{-1}$, i.e., $c = bc_1$;

(2) $ab^{-1}\cdot cd^{-1} = a(db_1)^{-1}$, i.e., $b = cb_1$;

(3) $ab^{-1}\cdot cd^{-1} = 0$.

Suppose that case (1) holds, i.e., $u_{\gamma_1},\ldots u_{\gamma_k} = u_{\beta_1}\ldots u_{\beta_m}u_{\gamma_{m+1}}\ldots u_{\gamma_k}$. Observe that

$$f(ac_1d^{-1}) = f(u_{\alpha_1}\ldots u_{\alpha_n}u_{\gamma_{m+1}}\ldots u_{\gamma_k})f(u_{\delta_1}\ldots u_{\delta_t})^{-1} = p_{\alpha_1}\ldots p_{\alpha_n}p_{\gamma_{m+1}}\ldots p_{\gamma_k}(p_{\delta_1}\ldots p_{\delta_t})^{-1}.$$

On the other hand

$$f(ab^{-1})\cdot f(cd^{-1}) = p_{\alpha_1}\ldots p_{\alpha_n}(p_{\beta_1}\ldots p_{\beta_m})^{-1}\cdot p_{\beta_1}\ldots p_{\beta_m}p_{\gamma_{m+1}}\ldots p_{\gamma_k}\cdot(p_{\delta_1}\ldots p_{\delta_t})^{-1} =$$
$$= p_{\alpha_1}\ldots p_{\alpha_n}p_{\gamma_{m+1}}\ldots p_{\gamma_k}(p_{\delta_1}\ldots p_{\delta_t})^{-1} = f(ac_1d^{-1}).$$

Case (2) is similar to case (1). Consider case (3). In this case there exists a positive integer i such that $u_{\beta_j} = u_{\gamma_j}$ for every $j < i$ and $u_{\beta_i} \neq u_{\gamma_i}$. Observe that $f(ab^{-1}\cdot cd^{-1}) = f(0) = 0$.

$$f(ab^{-1})\cdot f(cd^{-1}) =$$
$$= p_{\alpha_1}\ldots p_{\alpha_n}p_{\beta_m}^{-1}\ldots p_{\beta_i}^{-1}(p_{\beta_{i-1}}^{-1}\ldots p_{\beta_1}^{-1}\cdot p_{\beta_1}\ldots p_{\beta_{i-1}})p_{\gamma_i}\ldots p_{\gamma_k}\cdot(p_{\delta_1}\ldots p_{\delta_t})^{-1} =$$
$$= p_{\alpha_1}\ldots p_{\alpha_n}p_{\beta_m}^{-1}\ldots(p_{\beta_i}^{-1}\cdot p_{\gamma_i})\ldots p_{\gamma_k}\cdot(p_{\delta_1}\ldots p_{\delta_t})^{-1} = 0 = f(ab^{-1}\cdot cd^{-1}).$$

Hence map f is an isomorphism.

Proof of statement 2 is similar to that of the statement 3. □

The following Theorem extends Theorem 3 from [23] and Proposition 3.1 from [8] over the case of semitopological graph inverse semigroups.

Theorem D. *Let $G(E)$ be a semitopological GIS. Then each non-zero element of $G(E)$ is an isolated point in $G(E)$.*

Proof. First we prove that each vertex a of the graph E is an isolated point in $G(E)$. There are two cases to consider:

1) there exists an edge x such that $s(x) = a$;

2) the set $\{x \in E^1 \mid s(x) = a\}$ is empty.

First consider the case 1. Fix an arbitrary edge x such that $s(x) = a$. Observe that both sets $xx^{-1}\cdot G(E)$ and $G(E)\cdot xx^{-1}$ are retracts of $G(E)$ and do not contain point a. Then $U(a) = G(E)\setminus(xx^{-1}G(E)\cup G(E)\cdot xx^{-1})$ is an open neighborhood of a. Fix an arbitrary open neighborhood $U(xx^{-1})$ which does not contain 0. Since $xx^{-1}\cdot a\cdot xx^{-1} = xx^{-1}$ the continuity of left and right shifts in $G(E)$ yields an open neighborhood $V(a) \subset U(a)$ such that $xx^{-1} \cdot V(a) \cdot xx^{-1} \subset U(xx^{-1})$. Fix an arbitrary element $bc^{-1} \in V(a)$. Observe that the choice of $U(a)$ implies that x is neither a prefix of b nor c (in the other case $bc^{-1} = xx^{-1} \cdot bc^{-1} \in xx^{-1} \cdot G(E)$ or $bc^{-1} = bc^{-1} \cdot xx^{-1} \in G(E) \cdot xx^{-1}$). Since the set $U(xx^{-1})$ does not contain 0 we obtain that $xx^{-1} \cdot bc^{-1} \cdot xx^{-1} \neq 0$ and, as a consequence, b and c are prefixes of x. Hence $b = c = a$ which implies that $V(a) = \{a\}$.

Next consider the case 2. Since $a \cdot a \cdot a = a$, the continuity of left and right shifts in $G(E)$ yields an open neighborhood $V(a)$ such that $a \cdot V(a) \cdot a \subset G(E) \setminus \{0\}$. Fix an arbitrary element $bc^{-1} \in V(a)$. Since $s(b) \neq a$ and $s(c) \neq a$ we obtain that $a \cdot bc^{-1} \cdot a \neq 0$ iff $b = c = a$ which implies that $V(a) = \{a\}$.

Hence each vertex a is an isolated point in $G(E)$. Fix an arbitrary non-zero element $uv^{-1} \in G(E)$. Since $u^{-1} \cdot uv^{-1} \cdot v = v^{-1} \cdot v = r(v)$, the continuity of left and right shifts in $G(E)$ yields an open neighborhood V of uv^{-1} such that $u^{-1} \cdot V \cdot v \subseteq \{r(v)\}$. By Lemma 1, the set $u^{-1} \cdot V$ is finite. Repeating our arguments, by Lemma 1, the set V is finite which implies that point uv^{-1} is isolated in $G(E)$. □

Theorem D implies the following:

Corollary 1. *Let $G(E)$ be a locally compact non-discrete semitopological GIS. Then for each compact neighborhoods U, V of 0 the set $U \setminus V$ is finite.*

Lemma 3. *Each infinite GIS $G(E)$ admits a unique compact non-discrete shift-continuous topology τ. Moreover, the inversion is continuous in $(G(E), \tau)$.*

Proof. The topology τ is defined in the following way: each non-zero element is isolated in $(G(E), \tau)$ and an open neighborhood base of 0 consists of cofinite subsets of $G(E)$ which contain 0. Since for each open neighborhood V of 0, the set V^{-1} is cofinite in $G(E)$ and contains 0 we obtain that the inversion is continuous in $(G(E), \tau)$. To prove the continuity of left and right shifts in $(G(E), \tau)$ we need to check it at the unique non-isolated point 0. Fix an arbitrary non-zero element $uv^{-1} \in G(E)$ and an open neighborhood U of 0. By the definition of topology τ the set $A = G(E) \setminus U$ is finite. By Lemma 1, the set $B = \{ab^{-1} \in G(E) \mid uv^{-1} \cdot ab^{-1} \in A\}$ is finite and, obviously, does not contain 0. Then $V = G(E) \setminus B$ is an open neighborhood of 0 such that $uv^{-1} \cdot V \subseteq U$. Hence left shifts are continuous in $(G(E), \tau)$. Continuity of right shifts in $G(E)$ can be proved similarly. □

Let $G(E)$ be an arbitrary GIS and $\mathcal{L}, \mathcal{R}, \mathcal{D}$ be the Green relations on $G(E)$. By Lemma 3.1.13 from [16] for any two non-zero elements ab^{-1} and cd^{-1} of $G(E)$ the following conditions hold:

(1) $ab^{-1}\mathcal{L}cd^{-1}$ iff $b = d$;

(2) $ab^{-1}\mathcal{R}cd^{-1}$ iff $a = c$;

(3) $ab^{-1}\mathcal{D}cd^{-1}$ iff $r(a) = r(b) = r(c) = r(d)$.

Further, for a path $u \in \text{Path}(E)$ by L_u (resp. R_u) we denote an $\mathcal{L}$-class (resp. $\mathcal{R}$-class) which contains the element uu^{-1}. For a vertex $e \in E^0$ by D_e denote the $\mathcal{D}$-class containing e. The condition (3) implies that each non-zero $\mathcal{D}$-class contains exactly one vertex.

Recall that GIS $G(E)$ satisfies the condition $(\star)$ if for each infinite subset $A \subset \text{Path}(E)$ there exists an infinite subset $B \subset A$ and an element $\mu \in G(E)$ such that for each $x \in B$, $\mu \cdot x \in \text{Path}(E)$ and $|\mu \cdot x| > |x|$.

Lemma 4. *Let $G(E)$ be a locally compact non-discrete semitopological GIS satisfying the condition $(\star)$. Then there exists an element $v \in \mathrm{Path}(E)$ such that for each open compact neighborhood U of 0 the set $L_v \cap U$ is infinite.*

Proof. To derive a contradiction, suppose that for each element $v \in \mathrm{Path}(E)$ there exists an open compact neighborhood W_v of 0 such that the set $L_v \cap W_v$ is finite. Fix an arbitrary open compact neighborhood U of 0. By Corollary 1, the set $U \setminus W_v$ is finite for each element $v \in \mathrm{Path}(E)$. Hence the set $U \cap L_v$ is finite for each path v. Let $T = \{v \in \mathrm{Path}(E) \mid L_v \cap U \neq \varnothing\}$. Since the set U is infinite we obtain that the set T is infinite as well. For each $v \in T$ fix an element $u_v v^{-1} \in L_v \cap U$ such that $|u_v| \geq |y|$ for every element $yv^{-1} \in L_v \cap U$. Since $G(E)$ satisfies the condition $(\star)$, there exists an infinite subset $A \subset \{u_v\}_{v \in T}$ and an element $\mu \in G(E)$ such that $\mu \cdot y \in \mathrm{Path}(E)$ and $|\mu \cdot y| > |y|$ for each element $y \in A$. Since $\mu \cdot 0 = 0$, the continuity of left shifts in $G(E)$ yields an open neighborhood V of 0 such that $\mu \cdot V \subset U$. Since the set $U \setminus V$ is finite (see Corollary 1), we obtain that there exists an element $v \in T$ such that $u_v v^{-1} \in V \cap U$. Observe that $\mu \cdot u_v v^{-1} \neq 0$, because $\mu \cdot u_v \in \mathrm{Path}(E)$ and $r(\mu \cdot u_v) = r(u_v) = r(v)$. Hence $\mu \cdot u_v v^{-1} \in L_v \cap U$ and $|\mu \cdot u_v| > |u_v|$ which contradicts the choice of the element $u_v v^{-1}$. □

Lemma 5. *Let $G(E)$ be a locally compact non-discrete semitopological GIS satisfying the condition $(\star)$. Then there exists a $\mathcal{D}$-class D_e such that the set $L \cap U$ is infinite for each open neighborhood U of 0 and $\mathcal{L}$-class $L \subset D_e$.*

Proof. By Lemma 4, there exists element $v \in \mathrm{Path}(E)$ such that the set $L_v \cap U$ is infinite for each open compact neighborhood U of 0. Recall that $D_{r(v)} = \{ab^{-1} \mid r(a) = r(b) = r(v)\}$. Fix an arbitrary element $u \in \mathrm{Path}(E) \cap D_{r(v)}$ and an open compact neighborhood U of 0. Observe that element $vu^{-1} \neq 0$, because $r(u) = r(v)$. Since $0 \cdot vu^{-1} = 0$ the continuity of right shifts in $G(E)$ yields an open neighborhood V of 0 such that $V \cdot vu^{-1} \subset U$. Observe that $L_v \cdot vu^{-1} = L_u$. By Corollary 1, the set $L_v \cap V$ is infinite. By Lemma 1, $(L_v \cap V) \cdot vu^{-1}$ is an infinite subset of $U \cap L_u$. □

Now our aim is to prove our main result which generalizes Theorem A and Theorem B.

Theorem. *Let E be a strongly connected graph which has finitely many vertices. Then any locally compact shift-continuous topology on GIS $G(E)$ is either compact or discrete.*

3. Proof of the Main Theorem. The proof of the Main Theorem is divided into a series of 5 lemmas. In the following lemmas 6–10 we assume that graph E is strongly connected and has finitely many vertices. As a consequence, the semigroup $G(E)$ satisfies the condition $(\star)$ (see Remark 2 from [5]). By Theorem B, the Main Theorem holds if the graph E contains only one vertex (in this case $G(E)$ is either finite or isomorphic to a λ-polycyclic monoid). Hence we can assume that the graph E contains at least two vertices. By e we denote an arbitrary vertex such that the set $L \cap U$ is infinite for any open neighborhood U of 0 and any $\mathcal{L}$-class $L \subset D_e$ (Lemma 5 implies that such vertex e exists). Recall that by $\langle C^e \rangle$ we denote the inverse subsemigroup of $G(E)$ which is generated by the set $C^e = \{u \in \mathrm{Path}(E) \mid d(u) = r(u) = e\}$.

Lemma 6. *Let $G(E)$ be a locally compact non-discrete semitopological GIS. Then the set $\langle C^e \rangle \setminus U$ is finite for each open neighborhood U of 0.*

Proof. By the assumption, there exists a vertex f and paths x, y such that $s(x) = r(y) = e$ and $r(x) = s(y) = f$. Since $xy \in C^e_1$, by Lemma 2, $\langle C^e \rangle$ is an infinite set. Fix an arbitrary compact open neighborhood U of 0. Recall that $L_e \cap U$ is infinite. Since the graph E contains finitely many vertices, there exists a vertex f such that the set $B = \{u \in L_e \cap U \mid s(u) = f\}$ is infinite. We claim that 0 is a limit point of $\langle C^e \rangle$. Indeed, if $f = e$ then $B \subset \langle C^e \rangle$ and hence 0 is a limit point of $\langle C^e \rangle$. Assume that $f \neq e$. Since graph E is strongly connected, there exists a path $v \in \operatorname{Path}(E)$ such that $s(v) = e$ and $r(v) = f$. Since $v \cdot 0 = 0$, the continuity of right shifts in $G(E)$ yields an open neighborhood V of 0 such that $v \cdot V \subset U$. By Corollary 1, the set $U \setminus V$ is finite which implies that the set $B \cap V$ is infinite. By Lemma 1, the set $v(V \cap B)$ is an infinite subset of U. Observe that for each element $u \in vB$, $s(u) = r(u) = e$. Hence 0 is a limit point of $\langle C^e \rangle$. Observe that $\langle C^e \rangle \cup \{0\}$ is a closed and hence locally compact subsemigroup of $G(E)$ which is isomorphic to the polycyclic monoid $\mathcal{P}_\lambda$ where $\lambda = |C^e_1 \setminus \{e\}|$ (see Lemma 2). By Theorem B, semigroup $\langle C^e \rangle$ is compact which implies that $\langle C^e \rangle \setminus U$ is finite for each open neighborhood U of 0. ☐

Lemma 7. *Let $G(E)$ be a locally compact non-discrete semitopological GIS. Then the set $L_e \setminus U$ is finite for each open neighborhood U of 0.*

Proof. Suppose that there exists an open compact neighborhood U of 0 such that the set $A = L_e \setminus U$ is infinite. Since the graph E has finitely many vertices, we can find a vertex f and an infinite subset $B \subset A$ such that $s(u) = f$ for each element $u \in B$. The strong connectedness of the graph E yields a path v such that $s(v) = e$ and $r(v) = f$. Observe that Lemma 6 implies that the set $vB \cap U$ is infinite, because vB is an infinite subset of $\langle C^e \rangle$. Since $v^{-1} \cdot 0 = 0$, the continuity of left shifts in $G(E)$ yields an open neighborhood V of 0 such that $v^{-1} \cdot V \subset U$. By Corollary 1, the set $U \setminus V$ is finite. Then there exists an element $b \in B$ such that $vb \in V$. Hence $v^{-1} \cdot vb = b \in U$ which contradicts the choice of U. ☐

Lemma 8. *Let $G(E)$ be a locally compact non-discrete semitopological GIS. Then the set $L \setminus U$ is finite for any open neighborhood U of 0 and any $\mathcal{L}$-class $L \subset D_e$.*

Proof. Fix an arbitrary $\mathcal{L}$-class $L \subset D_e$ and an open compact neighborhood U of 0. Clearly, $L = L_v$ for some path v such that $r(v) = e$. Since $0 \cdot v^{-1} = 0$, the continuity of right shifts in $G(E)$ yields an open neighborhood V of 0 such that $V \cdot v^{-1} \subset U$. Observe that $L_e \cdot v^{-1} = L_v$. By Lemma 7, the set $L_e \setminus V$ is finite. Hence the set $L_v \setminus U$ is finite as well. ☐

Lemma 9. *Let $G(E)$ be a locally compact non-discrete semitopological GIS. Then the set $D_e \setminus U$ is finite for each open neighborhood U of 0.*

Proof. To derive a contradiction, suppose that there exists an open neighborhood U of 0 such that the set $A = D_e \setminus U$ is infinite. Without loss of generality we can assume that U is compact. Put $T = \{v \in \operatorname{Path}(E) \cap D_e \mid L_v \setminus U \neq \varnothing\}$. By Lemma 8, the set $L_v \setminus U$ is finite for each path $v \in D_e$. Since the set U is infinite, we obtain that the set T is infinite as well. For each path $v \in T$ by u_v we denote an arbitrary path satisfying the following conditions:

- $u_v v^{-1} \notin U$;
- if $uv^{-1} \notin U$ for some path u then $|u_v| \geq |u|$.

Since the set T is infinite, the set $B = \{u_v v^{-1} \mid v \in T\}$ is infinite as well. Since graph E has finitely many vertices, there exists a vertex f and an infinite subset $C \subset T$ such that $s(u_v) = f$ for each path $v \in C$. Put $D = \{u_v v^{-1} \in B \mid v \in C\}$. Fix an arbitrary path a such that $|a| \geq 1$ and $r(a) = f$ (by the strong connectedness of graph E such path a always exists). The choice of elements u_v implies that $aD = \{a u_v v^{-1} \mid v \in C\}$ is an infinite subset of

U. Since $a^{-1} \cdot 0 = 0$, the continuity of left shifts in $G(E)$ yields an open neighborhood $V \subset U$ such that $a^{-1} \cdot V \subset U$. By Corollary 1, the set $U \setminus V$ is finite which implies that the set $V \cap aD$ is infinite. Fix an arbitrary element $au_v v^{-1} \in aD \cap V$. Then $a^{-1} \cdot au_v v^{-1} = u_v v^{-1} \in U \cap D$ which contradicts the choice of U. □

The following lemma completes the proof of the Main Theorem.

Lemma 10. *Any non-discrete locally compact shift-continuous topology on GIS $G(E)$ is compact.*

Proof. By Lemma 7, the set $L_e \setminus U$ is finite for each open neighborhood U of 0. Fix an arbitrary compact open neighborhood U of 0 and an arbitrary vertex $f \in E^0 \setminus \{e\}$. The strong connectedness of the graph E implies that there exists a path u such that $s(u) = e$ and $r(u) = f$. Since $0 \cdot u = 0$, the continuity of right shifts in $G(E)$ yields an open neighborhood $V \subset U$ of 0 such that $V \cdot u \subset U$. Observe that the set $V \cap L_e$ is infinite and, by Lemma 1, $(V \cap L_e) \cdot u$ is an infinite subset of $L_f \cap U$. Hence we can apply Lemmas 5–9 to the vertex f and obtain that the set $D_f \setminus U$ is finite. Since each non-zero $\mathcal{D}$-class contains a unique vertex and the graph E has finitely many vertices, we conclude that $G(E)$ has finitely many $\mathcal{D}$-classes. Hence the set $G(E) \setminus U$ is finite. □

4. A generalization of the Main Theorem. Observe that the Main Theorem remains true if the graph E is a disjoint union of two graphs E_1 and E_2 such that the graph E_1 satisfies conditions of the Main Theorem and the GIS $G(E_2)$ is finite. However, the Main Theorem can not be generalized over the case when the graph E is a disjoint union of two graphs E_1 and E_2 such that both semigroups $G(E_1)$ and $G(E_2)$ are infinite.

Proposition 1. *Let E be a graph which is a disjoint union of two graphs E_1 and E_2 such that both semigroups $G(E_1)$ and $G(E_2)$ are infinite. Then there exists a topology τ on $G(E)$ such that $(G(E), \tau)$ is a locally compact, non-compact, non-discrete quasi-topological semigroup.*

Proof. Assume that $E = E_1 \sqcup E_2$ and both semigroups $G(E_1)$ and $G(E_2)$ are infinite. We introduce a topology τ on $G(E)$ in the following way: each non-zero element uv^{-1} is isolated in $G(E)$. An open neighborhood base of the point 0 consists of cofinite subsets of $G(E_1)$ which contains point 0. Similar arguments as in Lemma 3 imply the continuity of the inversion in $G(E)$. To prove that $(G(E), \tau)$ is a semitopological semigroup we need to consider the following four cases:

1) $uv^{-1} \cdot 0 = 0$, where $uv^{-1} \in G(E_1)$;
2) $0 \cdot uv^{-1} = 0$, where $uv^{-1} \in G(E_1)$;
3) $uv^{-1} \cdot 0 = 0$, where $uv^{-1} \in G(E_2)$;
4) $0 \cdot uv^{-1} = 0$, where $uv^{-1} \in G(E_2)$.

The continuity of left (resp. right) shifts in the first (resp. second) case follows from Lemma 3. The continuity of left and right shifts in cases three and four can be derived from the following equation

$$uv^{-1} \cdot G(E_1) = G(E_1) \cdot uv^{-1} = 0, \text{ where } uv^{-1} \in G(E_2).$$

□

Theorem C and Proposition 1 imply the following:

Corollary 2. *Let $G(E)$ be a GIS which satisfies the dichotomy of the Main Theorem, i.e., a locally compact shift-continuous topology on $G(E)$ is either compact or discrete. Then $G(E)$ satisfies the condition $(\star)$ and the graph E cannot be represented as a union of two graphs E_1 and E_2 such that semigroups $G(E_1)$ and $G(E_2)$ are infinite.*

The above corollary leads us to the following question:

Question. *Is it true that a GIS $G(E)$ satisfies the dichotomy of the Main Theorem iff $G(E)$ satisfies the condition $(\star)$ and the graph E cannot be represented as a disjoint union of two graphs E_1 and E_2 such that semigroups $G(E_1)$ and $G(E_2)$ are infinite?*

Acknowledgements. The author acknowledges professor Taras Banakh for his comments and suggestions.

REFERENCES

1. G. Abrams, G. Aranda Pino, *The Leavitt path algebra of a graph*, J. Algebra, **293** (2005), 319–334.
2. A. Alali, N.D. Gilbert, *Closed inverse subsemigroups of graph inverse semigroups*, arXiv:1608.04538.
3. P. Ara, M.A. Moreno, E. Pardo, *Non-stable K-theory for graph algebras*, Algebr. Represent. Theory, **10** (2007), 157–178.
4. S. Bardyla, *On universal objects in the class of graph inverse semigroups*, preprint, arXiv:1709.01393v2.
5. S. Bardyla, *On locally compact topological graph inverse semigroups*, preprint, (2017), arXiv:1706.08594v2.
6. S. Bardyla, *On locally compact shift-continuous topologies on the α-bicyclic monoid*, Topological Algebra and its Applications, **6**, (2018), №1, 34–42.
7. S. Bardyla, *Classifying locally compact semitopological polycyclic monoids*, Math. Bulletin of the Shevchenko Scientific Society, **13**, (2016), 21–28.
8. S. Bardyla, O. Gutik, *On a semitopological polycyclic monoid*, Algebra Discr. Math., **21** (2016), №2, 163–183.
9. S. Bardyla, O. Gutik, *On a complete topological inverse polycyclic monoid*, Carpathian Math. Publ., **8** (2016), №2, 183–194.
10. M. Bertman, T. West, *Conditionally compact bicyclic semitopological semigroups*, Proc. Roy. Irish Acad., **A76**:21–23 (1976), 219–226.
11. A.H. Clifford, G.B. Preston, The Algebraic Theory of Semigroups, V.I and II, Amer. Math. Soc. Surveys, **7**, Providence, R.I., 1961 and 1967.
12. J. Cuntz, W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math., **56** (1980), 251–268.
13. C. Eberhart, J. Selden, *On the closure of the bicyclic semigroup*, Trans. Amer. Math. Soc., **144** (1969), 115–126.
14. R. Engelking, General Topology, 2nd ed., Heldermann, Berlin, 1989.
15. O. Gutik, *On the dichotomy of the locally compact semitopological bicyclic monoid with adjoined zero*, Visn. L'viv. Univ., Ser. Mekh.-Mat., **80** (2015), 33–41.
16. D. Jones, *Polycyclic monoids and their generalizations*, PhD Thesis, Heriot-Watt University, 2011.
17. D. Jones, M. Lawson, *Graph inverse semigroups: Their characterization and completion*, J. Algebra, **409** (2014), 444–473.
18. A. Kumjian, D. Pask, I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math., **184** (1998), 161–174.
19. M. Lawson, Inverse semigroups. The theory of partial symmetries, Singapore: World Scientific, 1998.
20. M. Lawson, *Primitive partial permutation representations of the polycyclic monoids and branching function systems*, Period. Math. Hungar., **58** (2009), 189–207.

21. J. Meakin, M. Sapir, *Congruences on free monoids and submonoids of polycyclic monoids*, J. Austral. Math. Soc. Ser. A, **54** (2009), 236–253.
22. Z. Mesyan, J.D. Mitchell, *The structure of a graph inverse semigroup*, Semigroup Forum, **93** (2016), 111–130.
23. Z. Mesyan, J.D. Mitchell, M. Morayne, Y.H. Péresse, *Topological graph inverse semigroups*, Topology and its Applications, **208** (2016), 106–126.
24. M. Nivat, J.-F. Perrot, *Une généralisation du monoide bicyclique*, C. R. Acad. Sci., Paris, Sér. A, **271** (1970), 824–827.
25. A.L. Paterson, Graph inverse semigroups, groupoids and their C^*-algebras, Birkhäuser, 1999.
26. W. Ruppert, Compact Semitopological Semigroups: An Intrinsic Theory, Lect. Notes Math., 1079, Springer, Berlin, 1984.

Ivan Franko National University of Lviv, Ukraine
sbardyla@yahoo.com

Received 5.01.2018