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A NOTE ON BORNOLOGIES

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A bornology on a set X is a family $\mathcal{B}$ of subsets of X closed under taking subsets, finite unions and such that $\bigcup \mathcal{B} = X$.

We prove that, for a bornology $\mathcal{B}$ on X, the following statements are equivalent:

(1) there exists a vector topology τ on the vector space $\mathbb{V}(X)$ over $\mathbb{R}$ such that $\mathcal{B}$ is the family of all subsets of X bounded in τ;

(2) there exists a uniformity $\mathcal{U}$ on X such that $\mathcal{B}$ is the family of all subsets of X totally bounded in $\mathcal{U}$;

(3) for every $Y \subseteq X$, $Y \notin \mathcal{B}$, there exists a metric d on X such that $\mathcal{B} \subseteq \mathcal{B}_d$, $Y \notin \mathcal{B}_d$, where $\mathcal{B}_d$ is the family of all closed discrete subsets of (X, d);

(4) for every $Y \subseteq X$, $Y \notin \mathcal{B}$, there exists $Z \subseteq Y$ such that $Z' \notin \mathcal{B}$ for each infinite subset Z' of Z.

A bornology $\mathcal{B}$ satisfying (4) is called antitall. We give topological and functional characterizations of antitall bornologies.

To Taras Banakh on 50th birthday

A family $\mathcal{I}$ of subsets of a set X is called an *ideal* (in the Boolean algebra $\mathcal{P}_X$ of all subsets of X) if $\mathcal{I}$ is closed under formations of finite unions and subsets. If $\bigcup \mathcal{I} = X$ then $\mathcal{I}$ is called a *bornology*, so a bornology is an ideal containing the ideal $\mathcal{F}_X$ of all finite subsets of X.

For an ideal $\mathcal{I}$, a family $\mathcal{F} \subseteq \mathcal{I}$ is called a *base* of $\mathcal{I}$ if, for any $A \in \mathcal{I}$, there exists $B \in \mathcal{F}$ such that $A \subseteq B$.

An ideal $\mathcal{I}$ on X is called *tall* if any infinite subset Y of X contains an infinite subset Z such that $Z \in \mathcal{I}$.

We say that an ideal $\mathcal{I}$ on X is *antitall* if, for any $Y \subseteq X$, $Y \notin \mathcal{I}$ there exists $Z \subseteq Y$ such that $Z' \notin \mathcal{I}$ for each infinite subset Z' of Z. Clearly, every bornology with a countable base is antitall. In particular, a bornology of all bounded subsets of a metric space is antitall. On the other hand, for every bornology $\mathcal{B}$ with countable base, there is a metric d on X such that $\mathcal{B}$ is the bornology of bounded subsets of (X, d).

Every bornology is the meet of some tall and antitall bornologies, see Proposition 1.

Given a bornology $\mathcal{B}$ on X and a set $\mathcal{S}$ of bornologies on X, we say that $\mathcal{B}$ is *approximated* by $\mathcal{S}$ if, for every $Y \subseteq X$, $Y \notin \mathcal{B}$, there exists $\mathcal{B}' \in \mathcal{S}$ such that $\mathcal{B} \subseteq \mathcal{B}'$ and $Y \notin \mathcal{B}'$. If $\mathcal{B}$ is approximated by $\mathcal{S}$ and $\cap \mathfrak{F} \in \mathcal{S}$ for any $\mathfrak{F} \subseteq \mathcal{S}$ then $\mathcal{B} \in \mathcal{S}$.

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We use the standard topological terminology, see [1]. For the history of bornology, see [2].

1. Vector bornologies. For a set X, $\mathbb{V}(X)$ denotes the vector space over $\mathbb{R}$ with the basis X. Under a *vector topology* on $\mathbb{V}(X)$, we mean a topology τ such that $(\mathbb{V}(X), \tau)$ is a topological vector space.

Let τ be a vector topology on X, U be a neighbourhood of 0 in τ. A subset $Y \subseteq X$ is called *U-bounded* if there exists $n \in \mathbb{N}$ such that $Y \subseteq nU$. If Y is U-bounded for every neighbourhood U of 0 in τ then Y is called τ-bounded. We denote by $\mathcal{B}_\tau$ the family of all τ-bounded subsets of X and observe that $\mathcal{B}_\tau$ is a bornology.

Theorem 1. *For a bornology $\mathcal{B}$ on a set X, the following statements are equivalent:*

(i) $\mathcal{B} = \mathcal{B}_\mu$ for some vector topology μ on $\mathbb{V}(X)$;

(ii) $\mathcal{B}$ is approximated by the set $\{\mathcal{B}_\tau : \tau$ is a vector topology on $\mathbb{V}(X)\}$;

(iii) $\mathcal{B}$ is antitall.

Proof. The implication $(i) \Longrightarrow (ii)$ is evident. To see $(ii) \Longrightarrow (i)$, we denote by μ the strongest vector topology such that each subset $Y \in \mathcal{B}$ is μ-bounded.
$(ii) \Longrightarrow (iii)$ Let $Y \subseteq X$ and $Y \notin \mathcal{B}$. We choose a vector topology τ such that $\mathcal{B} \subseteq \mathcal{B}_\tau$, $Y \notin \mathcal{B}_\tau$. Then there exists a neighbourhood U of 0 in τ such that $Y \setminus n\,U \neq \varnothing$ for each $n \in \mathbb{N}$. We take a subset $Z = \{z_n : n \in \mathbb{N}\}$ of Y such that $z_n \notin n\,U$. Then $Z' \notin \mathcal{B}_\tau$ for each infinite subset Z' of Z. Since $\mathcal{B} \subseteq \mathcal{B}_\tau$, we conclude that $\mathcal{B}$ is antitall.
$(iii) \Longrightarrow (ii)$. We write $\mathbb{V}(X)$ as $\oplus \mathbb{R}_x, x \in X$, $\mathbb{R}_x$ is a copy of $\mathbb{R}$, so each vector $v \in \mathbb{V}(X)$ is of the form $(\lambda_x)_{x \in X}$, $\lambda_x \in \mathbb{R}_x$ and $\lambda_x = 0$ for all but finitely many $x \in X$.

Let $Y \subseteq X$ and $Y \notin \mathcal{B}$. We take a countable subset Z of Y such that $Z' \notin \mathcal{B}$ for each infinite subset Z' of Z. We denote by $\mathcal{P}$ the partition of X into subsets $X \setminus Z$, $\{z\}$, $z \in Z$, and by $\Lambda_{\mathcal{P}}$ the family of all functions $\lambda : \mathcal{P} \longrightarrow \{\frac{1}{n} : n \in \mathbb{N}\}$. For each $\lambda \in \Lambda_{\mathcal{P}}$, we put

$$U(\lambda) = \{v \in \mathbb{V}(X) : v = (\lambda_x)_{x \in X}, \ \ |\lambda_x| < \lambda(P) \ for\ each\ x \in P, \ P \in \mathcal{P}\},$$

and take $\{U(\lambda) : \lambda \in \Lambda_{\mathcal{P}}\}$ as a base at 0 for some (uniquely determined) vector topology τ on $\mathbb{V}(X)$. By the construction, a subset A of X is τ-bounded if and only if $A = \cup \mathfrak{F}$ for some finite subset $\mathfrak{F}$ of $\mathcal{P}$. It follows that $\mathcal{B} \subseteq \mathcal{B}_\tau$ and $Z \notin \mathcal{B}_\tau$ so $Y \notin \mathcal{B}_\tau$. □

2. Totally bounded bornologies. For a uniformity $\mathcal{U}$ on a set X, $x \in X$ and an entourage $\varepsilon \in \mathcal{U}$, the set $B(x, \varepsilon) = \{y \in X : (x, y) \in \varepsilon\}$ is called the *ε-ball* centered at x. A subset $Y \subseteq X$ is called *totally bounded* in the uniform space $(X, \mathcal{U})$ if, for each $\varepsilon \in \mathcal{U}$, Y can be covered by finite number of ε-balls. If Y is not totally bounded then there exists an ε-discrete subset $\{y_n : n \in \omega\}$ of X, i.e. $B(y_n, \varepsilon) \cap B(y_m, \varepsilon) = \varnothing$ for all distinct $n, m \in \omega$.

Given a uniformity $\mathcal{U}$ on X, we denote by $\mathcal{B}_{\mathcal{U}}$ the bornology of all totally bounded subsets of $(X, \mathcal{U})$.

Theorem 2. *For a bornology $\mathcal{U}$ on a set X, the following statements are equivalent:*

(i) $\mathcal{B} = \mathcal{B}_{\mathcal{M}}$ for some uniformity $\mathcal{M}$ on X;

(ii) $\mathcal{B}$ is approximated by the set $\{\mathcal{B}_{\mathcal{U}} : \mathcal{U}$ is a uniformity on $X\}$;

(iii) $\mathcal{B}$ *is antitall.*

Proof. The implication $(i) \Longrightarrow (ii)$ is evident. To see $(ii) \Longrightarrow (i)$, we take the strongest uniformity $\mathcal{M}$ such that each subset $Y \in \mathcal{B}$ is totally bounded in $(X, \mathcal{M})$.
$(ii) \Longrightarrow (iii)$. Let $Y \subseteq X$ and $Y \notin \mathcal{B}$. We choose a uniformity $\mathcal{U}$ on X such that $\mathcal{B} \subseteq \mathcal{B}_{\mathcal{U}}$ and $Y \notin \mathcal{B}_{\mathcal{U}}$. Then there exists $\varepsilon \in \mathcal{U}$ such that some countable subset Z of Y is ε-discrete. Clearly, $Z' \notin \mathcal{B}_{\mathcal{U}}$ for each infinite subset Z' of Z. Since $\mathcal{B} \subseteq \mathcal{B}_{\mathcal{U}}$, we conclude that $\mathcal{B}$ is antitall.
$(iii) \Longrightarrow (i)$. Let $Y \subseteq X$ and $Y \notin \mathcal{B}$. We take a countable subset Z of Y such that $Z' \notin \mathcal{B}$ for each countable subset Z' of Z. Then we define a uniformity $\mathcal{U}$ on X such that $X \setminus Z$ is totally bounded in $\mathcal{U}$ and Z is ε-discrete for some $\varepsilon \in \mathcal{U}$. Clearly, $\mathcal{B} \subseteq \mathcal{B}_{\mathcal{U}}$ and $Y \notin \mathcal{B}_{\mathcal{U}}$. □

3. Closed discrete bornologies. For a metric space (X, d), we denote by $\mathcal{B}_d$ the bornology of all closed discrete subsets of X. Clearly, $Y \notin \mathcal{B}_d$ if and only if there exists an injective sequence in Y converging in X.

Theorem 3. *For every bornology* $\mathcal{B}$ *on a set* X*, the following statements are equivalent:*

(i) $\mathcal{B}$ *is approximated by the set* $\{\mathcal{B}_d : d$ *is a metric on* $X\}$*;*

(ii) $\mathcal{B}$ *is antitall.*

Proof. $(i) \Longrightarrow (ii)$. Let $Y \subseteq X$, $Y \notin \mathcal{B}$. We take a metric d such that $\mathcal{B} \subseteq \mathcal{B}_d$, $Y \notin \mathcal{B}_d$. We choose a convergent sequence $(z_n)_{n \in \omega}$ in Y and put $Z = \{z_n : n \in \omega\}$. Then $Z' \in \mathcal{B}_d$ for each infinite subset Z' of Z. Since $\mathcal{B} \subseteq \mathcal{B}_d$, we see that $\mathcal{B}$ is antitall.
$(ii) \Longrightarrow (i)$. Let $Y \subseteq X$, $Y \notin \mathcal{B}$. We choose a countable subset Z of Y such that $Z' \notin \mathcal{B}$ for each countable subset Z' of Z. Then we endow X with a metric d such that $d(x, y) = 1$ for all distinct $x, y \in X \setminus Z$ and d induces a topology of convergent sequence on Z. Clearly, $\mathcal{B} \subseteq \mathcal{B}_d$ and $Y \notin \mathcal{B}_d$. □

Example. We take a set X of cardinality $> 2^{\aleph_0}$ and denote by $\mathcal{B}$ the bornology of all finite subsets of X. Evidently, $\mathcal{B}$ is antitall. We assume that there is a metric ρ on X such that $\mathcal{B} = \mathcal{B}_\rho$. Since every closed dense subset of (X, ρ) is finite, we conclude that $|X| \leq 2^{\aleph_0}$. Hence, $\mathcal{B} \notin \{\mathcal{B}_d : d$ is a metric on $X\}$.

4. Tall and antitall bornologies. We endow a set X with the discrete topology, identify the Stone-Čech compactification βX of X with the set of all ultrafilters on X and denote $X^* = \beta X \setminus X$, so X^* is the set of all free ultrafilters on X. Then the family $\{\bar{A} : A \subseteq X\}$, where $\bar{A} = \{p \in \beta X : A \in p\}$, forms the base for the topology of βX. Given a filter φ on X, we denote $\bar{\varphi} = \bigcap\{\bar{A} \in \varphi\}$, so φ defines the closed subset $\bar{\varphi}$ of βX, and each non-empty closed subset K of βX can be obtained in this way: $K = \bar{\varphi}$, $\varphi = \{A \subseteq X : K \subseteq \bar{A}\}$.

For an ideal $\mathcal{I}$ in $\mathcal{P}_X$, we put

$$\mathcal{I}^\wedge = \{p \in \beta G : X \setminus A \in p \text{ for each } A \in \mathcal{I}\},$$

and note that $\mathcal{I}$ is a bornology if and only if $\mathcal{I}^\wedge \subseteq X^*$.

Using this correspondence between bornologies on X and closed subsets of X^*, we get

Proposition 1. *A bornology* $\mathcal{B}$ *on* X *is tall if and only if* $\mathcal{B}^\wedge$ *is nowhere dense in* X^**. A bornology* $\mathcal{B}$ *on* X *is antitall if and only if* $\mathcal{B}^\wedge$ *has a dense subset open in* X^**. Every bornology on* X *is the intersection of some tall and antitall bornologies.*

Proposition 2. *For a bornology $\mathcal{B}$ on X, the following statements are equivalent:*

(i) *$\mathcal{B}$ is antitall;*

(ii) *if $Y \subseteq X$ and $Y \notin \mathcal{B}$ then there exists a function $f\colon X \longrightarrow \omega$ such that f is bounded on each member of $\mathcal{B}$ but f is unbounded on Y.*

Remark. When this note was in the late embryonal state, Taras Banakh noticed that each tall ideal on X does not satisfy (ii) of Proposition 2. After that, the appearance of antitall ideals was inevitable.

Given a bornology $\mathfrak{I}$ on a set X, how to construct the smallest antitall bornology $\mathcal{B}$ such that $\mathfrak{I} \subseteq \mathcal{B}$? We make it in topological and functional ways.

Applying Proposition 1, we conclude that $\mathcal{B}^\wedge = \text{cl (int } \mathfrak{I}^\wedge)$. So we take the filter φ such that $\overline{\varphi} = \mathcal{B}^\wedge$. Then $\mathcal{B} = \{X \setminus A : A \in \varphi\}$.

By Proposition 2, to get $\mathcal{B}$ it suffices to join to $\mathfrak{I}$ each subset B of X such that every function $f : X \longrightarrow \omega$, bounded on every member of $\mathfrak{I}$, is bounded on B.

Given an ideal $\mathfrak{I}$ on X, we can construct some antitall bornology $\mathcal{B}$ from member of $\mathfrak{I}$ in essentially different way. For a family $\mathfrak{F}$ of subsets of X, we denote

$$\mathfrak{F}^\neg = \{A \subseteq X : \text{ every infinite subset of A is not in } \mathfrak{F}\}.$$

If $\mathfrak{F}$ is inherited by subsets then $\mathfrak{F}^\neg$ is a bornology.

Proposition 3. *For every ideal $\mathfrak{I}$ on X, $\mathfrak{I}^\neg$ is an antitall bornology. If $\mathcal{B}$ is an antitall bornology then $(\mathcal{B}^\neg)^\neg = \mathcal{B}$.*

To prove Proposition 3, it suffices to understand a topological sense of the operation $\neg$. In fact,

$$(\mathfrak{I}^\neg)^\wedge = \text{cl}(X^* \setminus \text{cl (int } \mathfrak{I}^\wedge)).$$

We conclude with one more topological observation. For a bornology $\mathfrak{I}$ on X, we endow $\mathcal{P}_X$ with the topology of uniform convergence on subsets from $\mathfrak{I}$. For $Y \in \mathcal{P}_X$, the family $\{Z \in \mathcal{P}_X : Z \cap A = Y \cap A\}$, $A \in \mathfrak{I}$, is a base of $\tau_{\mathfrak{I}}$ at the point Y. We note that $(\mathcal{P}_X, \tau_{\mathfrak{I}})$ is a complete topological Boolean group (with the symmetric difference as a group operation). We consider X as the subspace $\{\{x\} : x \in X\}$ of $(\mathcal{P}_X, \tau_{\mathfrak{I}})$.

Proposition 4. *For a bornology $\mathfrak{I}$ on X, the following statements are equivalent:*

(i) *every infinite subset $Y \subseteq X$ has an infinite closed discrete subset in $\tau_{\mathfrak{I}}$;*

(ii) *$\mathfrak{F}_X$ is closed (and so complete) subgroup of $(\mathcal{P}_X, \tau_{\mathfrak{I}})$;*

(iii) *$\mathfrak{I}$ is tall.*

Proposition 5. *For a bornology $\mathfrak{I}$ on X, the following statements are equivalent:*

(i) *each non-closed in $\tau_{\mathfrak{I}}$ subset $Y \subseteq X$ has a convergent (to $\varnothing$) sequence;*

(ii) *$\mathfrak{I}$ is antitall.*

5. Classes of bornological spaces. 1. A set X endowed with a bornology $\mathcal{B}$ is called a *bornological space,* and is denoted by $(X, \mathcal{B})$. Each $A \in \mathcal{B}$ is called *bounded.*

A class of bornological spaces closed under subspaces, products and bornologous images is called a *variety.*

A mapping $f : (X, \mathcal{B}) \longrightarrow (X', \mathcal{B}')$ is *bornologous* if $f(A) \in \mathcal{B}'$ for each $A \in \mathcal{B}$.

The product of a family of bornological spaces is the Cartesian product of its supports endowed with the Cartesian product of its bornologies.

We denote by $\mathfrak{M}_{single}$ the variety of all singletons, $\mathfrak{M}_{bound}$ the variety of all bounded ($\mathcal{B} = \mathcal{P}_X$) bornological spaces, and $\mathfrak{M}_\kappa$ the variety of all κ-bounded bornological spaces. For an infinite cardinal κ, $(X, \mathcal{B})$ is called *κ-bounded* if $\mathcal{B} \supseteq [X]^{<\kappa}$.

Applying Theorem 2 from [3] and item 3 below, we conclude that each variety of bornological spaces lies in the chain

$$\mathfrak{M}_{single} \subset \mathfrak{M}_{bound} \subset \dots \mathfrak{M}_\kappa \subset \dots \mathfrak{M}_\omega.$$

We note that each variety in this chain, excepts $\mathfrak{M}_\omega$, consists of tall bornological spaces.

2. The class of all tall space is closed under subspaces and bornologous images (but not products). The class of all antitall spaces is closed under subspaces, products (but not bornologous images).

For a bornological space $(X, \mathcal{B})$, we define the *hyperbornology* $\exp \mathcal{B}$ on $\mathcal{B}$: the family $\{Y \in \mathcal{B} : Y \subseteq A\}$, $A \in \mathcal{B}$ is the base of $\exp \mathcal{B}$.

We show that if $\mathcal{B}$ is antitall then $\exp \mathcal{B}$ is antitall. Let $\mathcal{A}$ be a family of bounded subsets of $(X, \mathcal{B})$ such that $\mathcal{A} \notin \exp \mathcal{B}$. We denote $Y = \cup \mathcal{A}$. Since $Y \notin \mathcal{B}$ and $\mathcal{B}$ is antitall, there is a countable $Z \subseteq Y$ such that each countable subset of Z is unbounded in $(X, \mathcal{B})$. For each $z \in Z$, we pick $A_z \in \mathcal{A}$ such that $z \in A_z$. Then $\{A_z : z \in Z\}$ witnesses that $\exp \mathcal{B}$ is antitall.

On the tall hand, let $\mathcal{B}$ be a bornology on a countable set X such that $\mathcal{B} \neq \mathcal{P}_X$. We write X as the union of increasing chain $\{X_n : n \in \omega\}$ of finite subsets. Clearly, only finite subsets of $\{X_n : n \in \omega\}$ are in $\exp \mathcal{B}$. Hence, $\exp \mathcal{B}$ is not tall.

3. Following [5], we say that a family $\mathcal{E}$ of subsets of $X \times X$ is a coarse structure on a set X if

- each $\varepsilon \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x, x) : x \in X\}$;
- if $\varepsilon, \delta \in \mathcal{E}$ then $\varepsilon \circ \delta \in \mathcal{E}$ and $\varepsilon^{-1} \in \mathcal{E}$ where $\varepsilon \circ \delta = \{(x, y) : \exists z((x, z) \in \varepsilon, (z, y) \in \delta)\}$, $\varepsilon^{-1} = \{(y, x) : (x, y) \in \varepsilon\}$;
- if $\varepsilon \in \mathcal{E}$ and $\triangle_X \subseteq \varepsilon' \subseteq \mathcal{E}$ then $\varepsilon' \in \mathcal{E}$;
- for any $x, y \in X$, there is $\varepsilon \in \mathcal{E}$ such that $(x, y) \in \varepsilon$.

The pair $(X, \mathcal{E})$ is called a *coarse space.* For $x \in X$ and $\varepsilon \in \mathcal{E}$, we denote $B(x, \varepsilon) = \{y \in X : (x, y) \in \varepsilon\}$ and say that $B(x, \varepsilon)$ is the *ball of radius ε around x.* We note that a coarse space can be considered as an asymptotic counterpart of a uniform spaces and could be defined in terms of balls, see [4]. In this case a coarse space is called a *ballean.*

Let $(X, \mathcal{E})$ be a coarse space. A subset Y of X is called *bounded* if there exist $x \in X$ and $\varepsilon \in \mathcal{E}$ such that $Y \subseteq \mathcal{B}(x, \mathcal{E})$. The family of all bounded subsets of $(X, \mathcal{E})$ is a bornology. On the other side, for every bornology $\mathcal{B}$ on X, there is the smallest (by inclusion) coarse structure $\mathcal{E}_\mathcal{B}$ such that $\mathcal{B}$ is a bornology of all bounded subsets of $(X, \mathcal{E}_\mathcal{B})$. A coarse space $(X, \mathcal{E})$ is of the form $(X, \mathcal{E}_\mathcal{B})$ if and only if $(X, \mathcal{E})$ is thin: for every $\varepsilon \in \mathcal{E}$, there is a bounded subset A of $(X, \mathcal{E})$ such that $B(x, \varepsilon) = \{x\}$ for all $x \in X \setminus A$.

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