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PROGRESS IN THE OPEN PROBLEMS IN THEORY OF FUNCTIONS OF BOUNDED INDEX

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We give overview of solved problems in theory of functions of bounded index from the paper Bandura, A.I., Skaskiv, O.B.: Open problems for entire functions of bounded index in direction. Mat. Stud. **43**(1), 103–109 (2015). Mostly problems concern with entire functions of several variables and analytic theory of differential equations. We pose also new problems.

We present overview of solved problems from [10] and give some remarks.

We need some notation and definitions. Let $L\colon \mathbb{C}^n \to \mathbb{R}_+$ be any fixed continuous function. An entire function $F(z)$, $z \in \mathbb{C}^n$, is called ([11, 12]) a *function of bounded L-index in a direction* $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\},$$

where $\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} := F(z)$, $\frac{\partial F(z)}{\partial \mathbf{b}} := \sum\limits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j = \langle \mathbf{grad} F, \overline{\mathbf{b}}\rangle$, $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} := \frac{\partial}{\partial \mathbf{b}}\big(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\big)$, $k \ge 2$.

For $\eta > 0$, $z \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and a positive continuous function $L:\ \mathbb{C}^n \to \mathbb{R}_+$ we define

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{C}^n} \sup_{t_1,t_2\in\mathbb{C}} \left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} \colon |t_1-t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}}\right\}.$$

By $Q^n_{\mathbf{b}}$ we denote the class of functions L such that $\lambda(\eta)$ is finite for any $\eta > 0$. We also use the notation $Q = Q^1_1$.

Let us consider the following partial differential equation

$$P_1(z_1, z_2)\frac{\partial F}{\partial z_1} + P_2(z_1, z_2)\frac{\partial F}{\partial z_2} = h(z_1, z_2). \tag{1}$$

Problem 1 ([10, Problem 3])**.** *Let $P_1(z_1, z_2)$, $P_2(z_1, z_2)$ be entire functions of bounded L-index in directions $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ respectively. What are a direction $\mathbf{b}$ and additional assumptions such that every entire solution $F(z)$ of equation* (1) *has a bounded L-index in the direction $\mathbf{b}$?*

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A. A. Goldberg observed [13] that the following algebraic partial differential equation

$$z_1\frac{\partial F}{\partial z_1}-z_2\frac{\partial F}{\partial z_2}=0$$

is satisfied by the function $w=f(z_1z_2)$, where $f(u)$ is an arbitrary entire function. But there are known sufficient conditions of L-index boundedness for composition of entire functions [9]. In view of these results, we conclude that if function $f(u)$ has bounded index then the function $f(z_1z_2)$ has bounded $L_{\mathbf{b}}$-index in the direction $\mathbf{b}=(b_1,b_2)\in\mathbb{C}^2\setminus\{0\}$ with $L_{\mathbf{b}}(z_1,z_2)=|b_1z_2+b_2z_1|+1$. In other words, we need some additional assumptions (for example, initial conditions) to certainly describe behavior of entire solutions of (1). Of course, this is also true for more general equation [10, Problem 1]

$$f_1(z)\frac{\partial F}{\partial \mathbf{b}_1}+f_2(z)\frac{\partial F}{\partial \mathbf{b}_2}=h(z).$$

The similar conclusion is valid for the following problem.

Problem 2 ([10, Problem 2]). *Let $g(z_1,z_2)$ be an entire function of bounded L-index in the directions $\mathbf{b}_1$ and $\mathbf{b}_2$. What are a function L^* and a direction $\mathbf{b}^*$ that an entire solution of equation $\frac{\partial^2 F}{\partial\mathbf{b}_1\partial\mathbf{b}_2}=g(z_1,z_2)$ has a bounded L^*-index in the direction $\mathbf{b}^*$?*

Indeed, for $\mathbf{b}_1=(1,0)$, $\mathbf{b}_2=(0,1)$ and $g(z_1,z_2)\equiv 0$ we have that any entire function $F(z_1,z_2)=f(z_1)+g(z_2)$ is a solution of $\frac{\partial^2 F}{\partial z_1\partial z_2}=0$, where f and g are arbitrary entire functions. Thus, if we choose entire functions with unbounded multiplicities of zeros as f and g and such that $f(0)=0$ and $g(0)=0$ then the function $F(z_1,z_2)$ is of unbounded L-index in the direction $(1,0)$ and $(0,1)$ for any positive continuous function $L\colon\mathbb{C}^n\to\mathbb{R}_+$. This means that the problem also requires initial conditions to certainly describe behavior of entire solutions.

Remind Problem 4 from [10]. Let us consider the ordinary differential equation

$$w'=f(z,w). \tag{2}$$

Problem 3 ([10, Problem 4]). *Let $f(z,w)$ be a function of bounded L-index in directions $\mathbf{e}_1=(1,0)$ and $\mathbf{e}_2=(0,1)$. What is a function l such that an entire solution $w=w(z)$ of equation (2) has a bounded l-index?*

The problem is completely solved in [8]. Moreover, there was considered more general equation

$$w^{(p)}=f(z,w). \tag{3}$$

We denote $\mathbf{e}_1=(1,0)$, $\mathbf{e}_2=(0,1)$. Let us write

$$G_r(F):=G_r^{\mathbf{b}}(F)=\bigcup_{z\colon F(z)=0}\{z+t\mathbf{b}\colon |t|<r/L(z)\},$$

and let a_k^0 be zeros of the function $F(z^0+t\mathbf{b})$ for a given $z^0\in\mathbb{C}^n$. By $n_{z^0}(r,F)=n_{\mathbf{b}}\big(r,z^0,1/F\big):=\sum_{|a_k^0|\le r}1$ we denote the counting function of the zeros a_k^0 of the slice function $F(z^0+t\mathbf{b})$ in the disc $\{t\in\mathbb{C}\colon |t|\le r\}$. If for a given $z^0\in\mathbb{C}^n$ and for all $t\in\mathbb{C}$ $F(z^0+t\mathbf{b})\equiv 0$, then we put $n_{z^0}(r)=-1$. Denote $n(r)=\sup_{z\in\mathbb{C}^n}n_z(r/L(z))$. Our solution is provided by the following statement.

Proposition 1. *Let $l_j\in Q^2_{\mathbf{e}_j}$, and $f(z,w)$ be an entire function of bounded l_j-index in the directions $\mathbf{e}_j$ for every $j\in\{1,2\}$. If there exist $C>0$ and $l\in Q$ such that for all $(z,w)\in\mathbb{C}^2\setminus(G_r^{\mathbf{e}_1}(f)\cup G_r^{\mathbf{e}_2}(f))$*

$$l_1(z,w)+l_2(z,w)|f(z,w)|\le Cl(z), \tag{4}$$

then every entire function satisfying (3) *has bounded l-index.*

Besides, we conjectured possibility of replacing universal quantifier by existential quantifier (Conjectures 1 and 2 in [10]) in some criteria of L-index boundedness in direction.

Problem 4 ([10, Problem 6])**.** *Is Conjecture 1 true?*

Conjecture 1. *Let $L \in Q^n_{\mathbf{b}}$. An entire function $F(z)$ is of bounded L-index in direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exist $R > 0$, $P_2(R) \geq 1$ and $\eta(R) \in (0, R)$ such that for all $z^0 \in \mathbb{C}^n$ and some $r = r(z^0) \in [\eta(R), R]$ inequality* $\max\Big\{|F(z^0 + t\mathbf{b})|\colon |t| = \dfrac{r}{L(z^0)}\Big\} \leq P_2 \min\Big\{|F(z^0 + t\mathbf{b})|\colon |t| = \dfrac{r}{L(z^0)}\Big\}$ *holds.*

Conjecture 1 is completely proved in [7,11].

Problem 5 ([10, Problem 7])**.** *Is Conjecture 2 true?*

Conjecture 2. *Let $F(z)$ be an entire in $\mathbb{C}^n$ function, $L \in Q^n_{\mathbf{b}}$ and $\mathbb{C}^n \setminus G^{\mathbf{b}}_r(F) \neq \varnothing$. $F(z)$ is a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if: 1) there exist $r > 0$, $P = P(r) > 0$ such that for each $z \in \mathbb{C}^n \setminus G^{\mathbf{b}}_r(F)$ inequality $|\frac{1}{F(z)}\frac{\partial F(z)}{\partial \mathbf{b}}| \leq PL(z)$ holds;*
2) there exist $r > 0$, $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for every $z^0 \in \mathbb{C}^n$, for which $F(z^0 + t\mathbf{b}) \not\equiv 0$, and for all $t_0 \in \mathbb{C}$ inequality $n_{\mathbf{b}}(\frac{r}{L(z^0)}, z^0, \frac{1}{F}) \leq \widetilde{n}(r)$ holds.

Conjecture 2 is partially proved in [7,11]. Particularly, we proved the following statement.

Theorem. *Let $L \in Q^n_{\mathbf{b}}$, $F(z)$ be an entire function in $\mathbb{C}^n$. If the following conditions hold: 1) there exist $r_1 > 0$, $P > 0$ such that for each $z \in \mathbb{C}^n \setminus G_{r_1}(F)$ inequality $|\frac{1}{F(z)}\frac{\partial F(z)}{\partial \mathbf{b}}| \leq PL(z)$ holds; 2) there exists $r_2 > 0$ such that $n(r_2) \in [-1; \infty)$ and $2r_1 \cdot n(r_2) < r_2/\lambda_{\mathbf{b}}(r_2)$, where r_1 is chosen from previous condition, then the function F is of bounded L-index in the direction $\mathbf{b}$.*

Note that Conjecture 2 from [10] is proved under the additional restriction $2n(r_2)r_1 < r_2/\lambda_{\mathbf{b}}(r_2)$. It is currently unknown whether the condition is essential.

Problem 6 ([11, Problem 3])**.** *Denote $r_0 = \sup_{r>0} \frac{r}{2n(r)\lambda(r)}$. Are there an entire function F in $\mathbb{C}^n$ and a continuous function $L\colon \mathbb{C}^n \to \mathbb{R}_+$ with properties: 1) there exist $r_1 > r_0$ and $P > 0$ such that for each $z \in \mathbb{C}^n \setminus G^{\mathbf{b}}_{r_1}(F)$ $\frac{1}{|F(z)|}|\frac{\partial F(z)}{\partial \mathbf{b}}| \leq PL(z)$; 2) for any $r_2 < r_0$ $\sup\limits_{z \in \mathbb{C}^n \setminus G^{\mathbf{b}}_{r_2}(F)} \frac{1}{|F(z)|L(z)} \left|\frac{\partial F(z)}{\partial \mathbf{b}}\right| = +\infty$ and $n(r_2) \in (0; +\infty)$?*

Also there was formulated the following problem.

Problem 7 ([10, Problem 17])**.** *What are conditions on the zero set and growth of entire functions providing the index boundedness of $F(z^0_1 + b_1 t, z^0_2 + b_2 t)$ for every $(z^0_1, z^0_2) \in \mathbb{C}^2$ and the index unboundedness of $F(z_1, z_2)$ in the direction $\mathbf{b} = (b_1, b_2)$?*

We find some answer to the question in [6]. There was proved that if $f(t)$, $t \in \mathbb{C}$, is an even entire transcendental function of bounded index then 1) for each direction $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ and for every fixed z^0_1, $z^0_2 \in \mathbb{C}$; the function $g(t) = f(\sqrt{(z^0_1 + b_1 t)(z^0_2 + b_2 t)})$ is an entire function of bounded index ($t \in \mathbb{C}$); 2) $f(\sqrt{z_1 z_2})$ is of unbounded index in each direction $\mathbf{b}$.

Another interesting property of entire function o bounded index has discovered by W. K. Hayman([14]). An entire function $f(z)$ is said to have bounded value distribution (Turán, 1953) if there exists constant p, R such that equation $f(z) = \omega$ never has more than p roots in any disc of radius R. W. K. Hayman([14]) proved that every entire function is a function of bounded value distribution if and only if its derivative is a function of bunded index.

In view of this property, we pose the following problem.

Problem 8. *What are operators preserving index boundedness for analytic functions?*

Conjecture 3. *Differential operators preserving univalence also must preserve index boundedness for some additional assumptions.*

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