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BOUNDED l-INDEX AND l-M-INDEX AND COMPOSITIONS OF ANALYTIC FUNCTIONS

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We partially proved a conjecture from Mat. Stud. **47** (2017), no.2, 207–210: for an entire function f the function $H(z)=f(1/(1-z)^n)$, $n\in\mathbb{N}$, is of bounded l-index in $\mathbb{C}\setminus\{0\}$ with $l(|z|)=\beta/(1-|z|)^{n+1}$, $\beta>1$, if and only if f is of bounded index. Also the boundedness of l-M-index of the function H is investigated. For arbitrary entire functions f and g the boundedness of the l-M-index of the function $F(z)=f(g(z))$ is studied with respect to boundedness of the M-index of a function f with $l(r)=M_g'(r)$, $M_g(r)=\max\{|g(z)|\colon |z|=r\}$.

1. Introduction. In the paper, we consider compositions of such types $H(z)=f((1-z)^{-n})$, $n\in\mathbb{N}$, and $F(z)=f(g(z))$, where f and g are entire functions. There are presented investigations of boundedness of l-index and l-M-index for these composite functions, where l is a positive continuous function in some domain $G\subseteq\mathbb{C}$. We need some definitions and notations.

Let G be an arbitrary domain in $\mathbb{C}$ and l be a positive and continuous function in G such that for all $z\in G$

$$l(z)>\frac{\beta}{\operatorname{dist}\{z,\partial G\}},\quad \beta=\mathrm{const}>1. \tag{1}$$

An analytic function f in G is said ([17, 21]) to be *of bounded l-index* if there exists $N\in\mathbb{Z}_+$ such that for all $n\in\mathbb{Z}_+$ and $z\in G$

$$\frac{|f^{(n)}(z)|}{n!l^n(z)}\le\max\left\{\frac{|f^{(k)}(z)|}{k!l^k(z)}\colon 0\le k\le N\right\}. \tag{2}$$

The least such integer is called the *l-index of f* and is denoted by $N(f;l,G)$.

Let $0<R\le+\infty$, $\mathbb{D}_R=\{z\colon |z|<R\}$ and l be a positive continuous function on $[0,\,R)$, which satisfies

$$l(r)>\frac{\beta}{R-r},\quad \beta=\mathrm{const}>1. \tag{3}$$

If $G=\mathbb{D}_R$ then an analytic in $\mathbb{D}_R$ function f [21] is of bounded l-index if there exists $N\in\mathbb{Z}_+$ such that for all $n\in\mathbb{Z}_+$ and $z\in\mathbb{D}_R$ inequality (2) holds with $l(|z|)$ instead of $l(z)$.

If $R=+\infty$ (i. e. f is an entire function) then condition (3) is unnecessary. We remark also that if f is an entire function and $l(|z|)\equiv 1$ then f is said [18] to be of bounded index. If $R=1$ then $\mathbb{D}=\mathbb{D}_1$.

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For $r \in [0, \beta]$ we put $\lambda_1(r) = \inf\{\frac{l(z)}{l(z_0)} \colon |z - z_0| \le \frac{r}{l(z_0)},\ z_0 \in G\}$ and $\lambda_2(r) = \sup\{\frac{l(z)}{l(z_0)} \colon |z - z_0| \le \frac{r}{l(z_0)},\ z_0 \in G\}$. By $Q_\beta(G)$ we denote the class of positive coninuous functions l satisfying (1) and for some $r_0 \in [0, \beta]$

$$0 < \lambda_1(r_0) \le \lambda_2(r_0) < +\infty. \tag{4}$$

We remark that if (4) holds for some $r_0 \in [0, \beta]$ then the inequality is valid for all $r \in [0, \beta]$.

Recently, there was suggested the following conjecture.

Conjecture 1 ([1]). *For an entire function f the function $F(z) = f\left((1 - z)^{-n}\right)$, $n \in \mathbb{N}$, is of bounded l-index in $\mathbb{D}$ with $l(|z|) = \beta(1 - |z|)^{-n-1}$, $\beta > 1$, if and only if f is of bounded index.*

Now we consider a more general hypothesis:

Conjecture 2 (M. M. Sheremeta). *Let f be an entire function and*

$$g(z) = \frac{q_1}{(1 - z)^p} + \frac{q_2}{(1 - z)^{p-1}} + \ldots + \frac{q_p}{1 - z} + q_{p+1},\ p \in \mathbb{N},\ q_j \in \mathbb{C},\ q_1 \ne 0.$$

A composite function $F = f \circ g$ in $\mathbb{D}$ is of bounded l-index in $\mathbb{C} \backslash \{0\}$ with $l(|z|) = \beta(1 - |z|)^{-p-1}$ if and only if f is of bounded index.

In the paper, Conjecture 2 is partially proved (Theorems 5, 6, 7). The similar problem we also consider for bounded l-M-index (Theorem 9 and 10).

Note that there are few papers about boundedness of l-index for compositions of entire functions of one variable ([20, 21]). The growth of a composition of entire functions of finite order is examined in [22]. The L-index in direction of some compositions of entire functions of several variables and its properties are studied in [4–6, 13]. But the l-M-index of composite functions are not investigated yet.

2. Auxiliary propositions. We need the following assertions.

Theorem 1 (Theorem 1.5, [21]). *Let $\beta > 1$ and $l \in Q_\beta(G)$. An analytic function f in the domain G is of bounded l-index if and only if there exist numbers $m \in \mathbb{Z}_+$ and $C > 0$ such that for each $z \in G$*

$$\frac{|f^{(m+1)}(z)|}{l^{m+1}(z)} \le C \max\left\{\frac{|f^{(k)}(z)|}{l^k(z)} \colon 0 \le k \le m\right\}. \tag{5}$$

Theorem 1 is an analog of known Hayman's Theorem [16] for analytic functions.

Theorem 2 (Theorem 2.2, [21]). *Let G be an arbitrary domain in $\mathbb{C}$ and a domain $D \subset G$ be such that* $\operatorname{dist}\{\partial D, \partial G\} > 0$. *Let $\beta > 1$ and l be a positive continuous functions in G such that $l(z) \ge \beta/d$ for all $z \in G$. If a function f is analytic in G then f is of bounded l-index in D.*

Theorem 3 (Theorem 1.6, [21]). *Let $\beta > 1$, $\frac{1}{\beta} < \theta_1 \le \theta_2 < +\infty$, $l \in Q_\beta(G)$ and $\theta_1 l(z) \le l_*(z) \le \theta_2 l(z)$ for all $z \in G$. An analytic in the domain G function f is of bounded l_*-index if and only if it is of bounded l-index.*

Remark 1. Actually, in the proof of Theorem 3 there was proved that if l_1, l_2 are positive continuous functions in G, $l_1(z) \le l_2(z)$ for all $z \in G$, f is analytic function in G then $N(f; l_2, G) \le N(f; l_1, G)$.

Theorem 4 (Theorem 3.3, [21]). *Let $0 < R \le +\infty$, $l \in Q_\beta(\mathbb{D}_R)$ and an analytic in $\mathbb{D}_R$ function f is of bounded l-index. Then $\ln M(r,f) = O(\int_0^r l(t)dt)$, $r \to R$, where $M(r,f) = \max\{|f(z)|\colon |z| = r\}$.*

Some analogs of Theorems 1–4 are also obtained for analytic functions in a polydisc [11, 12], in the unit ball [7, 8]. The similar results are known for entire functions in $\mathbb{C}^n$ ([3, 9, 10]).

3. Bounded l-index of some compositions. Taking into account Conjecture 2 it is possible to prove such a theorem.

Theorem 5. *Let f be an entire function. A function $f(\frac{q_1}{z^p} + \frac{q_2}{z^{p-1}} + \ldots + q_{p+1})$ has bounded l-index in $\mathbb{C} \setminus \{0\}$ with $l(|z|) = \beta|z|^{-p-1}$ if and only if the function f has bounded index, where $p \in \mathbb{N}$, $q_j \in \mathbb{C}$, $q_1 \ne 0$.*

Proof. If $p = 1$ and $q_2 = 0$ then the same proposition is obtained in [21, p. 99]. We will deduce the statement for arbitrary $p \in \mathbb{N}$.

Let $F(z) = f(\frac{q_1}{z^p} + \frac{q_2}{z^{p-1}} + \ldots + q_{p+1})$. It is easy to check that

$$F'(z) = -f'\left(\frac{q_1}{z^p} + \frac{q_2}{z^{p-1}} + \ldots + q_{p+1}\right)\left(\frac{q_1 p}{z^{p+1}} + \frac{q_2(p-1)}{z^p} + \ldots + \frac{q_p}{z^2}\right),$$

$$F^{(k)}(z) = f^{(k)}\left(\frac{q_1}{z^p} + \frac{q_2}{z^{p-1}} + \ldots + q_{p+1}\right)\left(\frac{q_1 p}{z^{p+1}} + \frac{q_2(p-1)}{z^p} + \ldots + \frac{q_p}{z^2}\right)^k +$$

$$+\sum_{j=1}^{k-1} f^{(j)}\left(\frac{q_1}{z^p} + \frac{q_2}{z^{p-1}} + \ldots + q_{p+1}\right)\left(\frac{c_{j,pj+k}}{z^{pj+k}} + \ldots + \frac{c_{j,j+k}}{z^{j+k}}\right) \tag{6}$$

and

$$f'\left(\frac{q_1}{z^p} + \frac{q_2}{z^{p-1}} + \ldots + q_{p+1}\right) = -\frac{F'(z)z^{p+1}}{q_1 p + q_2(p-1)z + \ldots + q_p z^{p-1}},$$

$$f^{(k)}\left(\frac{q_1}{z^p} + \frac{q_2}{z^{p-1}} + \ldots + q_{p+1}\right) = (-1)^k F^{(k)}(z)\left(\frac{z^{p+1}}{q_1 p + q_2(p-1)z + \ldots + q_p z^{p-1}}\right)^k +$$

$$+\sum_{j=1}^{k-1} F^{(j)}(z)\frac{z^{pk+j}Q_{j,k}(z)}{(q_1 p + q_2(p-1)z + \ldots + q_p z^{p-1})^{2k-1}}, \tag{7}$$

where $Q_{j,k}(z)$ are some polynomials with degrees depending from j and k.

Equalities (6)–(7) can be proved by the method of mathematical induction (see a similar proof in [13]).

Let f be an entire function of bounded index. Then from (6) by Theorem 1 we have

$$|F^{(m+1)}(z)||z|^{(p+1)(m+1)} \le$$

$$\le |f^{(m+1)}\left(\frac{q_1}{z^p} + \frac{q_2}{z^{p-1}} + \ldots + q_{p+1}\right)|\left|q_1 p + q_2(p-1)z + \ldots + q_p z^{p-1}\right|^{m+1} +$$

$$+\sum_{j=1}^{m}\left|f^{(j)}\left(\frac{q_1}{z^p} + \frac{q_2}{z^{p-1}} + \ldots + q_{p+1}\right)\right|\left(|c_{j,pj+m+1}||z|^{p(m+1-j)} + \ldots + |c_{j,j+m+1}||z|^{(m+1)p-j}\right) \le$$

$$\le \left(C + \sum_{j=1}^{m}\left(|c_{j,pj+m+1}||z|^{p(m+1-j)} + \ldots + |c_{j,j+m+1}||z|^{(m+1)p-j}\right)\right) \times$$

$$\times \max\left\{\left|f^{(j)}\left(\frac{q_1}{z^p}+\frac{q_2}{z^{p-1}}+\ldots+q_{p+1}\right)\right|: 0\le j\le m\right\}\le$$
$$\le C_1\max\left\{\left|f^{(k)}\left(\frac{q_1}{z^p}+\frac{q_2}{z^{p-1}}+\ldots+q_{p+1}\right)\right|: 0\le k\le m\right\}$$

for all $|z|\le 1$. Using (7), we deduce

$$\left|f^{(k)}\left(\frac{q_1}{z^p}+\frac{q_2}{z^{p-1}}+\ldots+q_{p+1}\right)\right|\le |F^{(k)}(z)||z|^{(p+1)k}\frac{1}{|q_1p+q_2(p-1)z+\ldots+q_pz^{p-1}|^k}+$$
$$+\sum_{j=1}^{k-1}|F^{(j)}(z)||z|^{(p+1)j}\frac{|Q_{j,k}(z)||z|^{p(k-j)}}{|q_1p+q_2(p-1)z+\ldots+q_pz^{p-1}|^{2k-1}}\le$$
$$\le\left(1+\sum_{j=1}^{k-1}\frac{|Q_{j,k}(z)||z|^{p(k-j)}}{|q_1p+q_2(p-1)z+\ldots+q_pz^{p-1}|^{2k-1}}\right)\max\left\{|F^{(j)}(z)||z|^{(p+1)j}: 1\le j\le k\right\}\le$$
$$\le 2\max\left\{|F^{(j)}(z)||z|^{(p+1)j}: 1\le j\le k\right\}$$

for all $|z|<\varepsilon$, where $\varepsilon>0$ is sufficiently small and such that the disc $|z|\le\varepsilon$ does not contain zeros of the polynomial $q_1p+q_2(p-1)z+\ldots+q_pz^{p-1}$. It is possible because $q_1\ne 0$. Therefore,

$$|F^{(m+1)}(z)||z|^{(p+1)(m+1)}\le 2C_1\max\left\{|F^{(j)}(z)||z|^{(p+1)j}: 1\le j\le mk\right\}.$$

Using Theorem 1 and 2, we conclude that the function F has bounded l-index with $l(|z|)=\frac{\beta}{|z|^{p+1}}$.

On the contrary, let F be of bounded l-index with $l(|z|)=\frac{\beta}{|z|^{p+1}}$. As above, by Theorem 1 from (7) we obtain

$$\left|f^{(m+1)}\left(\frac{q_1}{z^p}+\frac{q_2}{z^{p-1}}+\ldots+q_{p+1}\right)\right|\le C_1\max\left\{|F^{(j)}(z)||z|^{(p+1)j}: 0\le j\le m\right\},$$

and (6) yields

$$|F^{(j)}(z)||z|^{(p+1)j}\le 2\max\left\{|f^{(k)}\left(\frac{q_1}{z^p}+\frac{q_2}{z^{p-1}}+\ldots+q_{p+1}\right)|:\ 0\le k\le j\right\}$$

for all $|z|<\varepsilon$, where $\varepsilon>0$ is sufficiently small. Hence,

$$\left|f^{(m+1)}\left(\frac{q_1}{z^p}+\frac{q_2}{z^{p-1}}+\ldots+q_{p+1}\right)\right|\le 2C_1\max\left\{|f^{(k)}\left(\frac{q_1}{z^p}+\frac{q_2}{z^{p-1}}+\ldots+q_{p+1}\right)|:\ 0\le k\le m\right\}$$

that is

$$\left|f^{(m+1)}(w)\right|\le 2C_1\max\left\{|f^{(k)}(w)|:\ 0\le k\le m\right\}$$

for all large $|w|$. Thus, by Theorem 1 and 2 the function f has bounded index. □

Let $q_2=q_3=\ldots=q_{p+1}=0$, i.e. $F(z)=f(\frac{q_1}{(1-z)^p})$. Replacement of z by $1-z$ in Theorems 1 and 5 gives us that there exist $m\in\mathbb{Z}_+$ and $C>0$ such that

$$|f^{(m+1)}(z)|\le C\max\{|f^{(j)}(z): 0\le j\le m\},$$

if and only if there exist $m\in\mathbb{Z}_+$ and $C>0$ such that

$$\frac{|F^{(m+1)}(z)|}{(\frac{\beta}{|1-z|^{p+1}})^{m+1}}\le C\max\left\{\frac{|F^{(j)}(z)|}{(\frac{\beta}{|1-z|^{p+1}})^j}: 0\le j\le m\right\}. \tag{8}$$

Since $1-|z|\le|1-z|$ inequality (8) yields the following inequality

$$\frac{|F^{(m+1)}(z)|}{(\frac{\beta}{(1-|z|)^{p+1}})^{m+1}}\le C\max\left\{\frac{|F^{(j)}(z)|}{(\frac{\beta}{(1-|z|)^{p+1}})^{j}} 0\le j\le m\right\},$$

in view of Remark 1 and Theorem 1.

Thus, if an entire function f is of bounded index then by Theorem 5 the function $F(z)=f(\frac{q_1}{(1-z)^p})$ is of bounded l-index with $l(|z|)=\frac{\beta}{(1-|z|)^{p+1}}$. And, on the contrary, if the function $F(z)=f(\frac{q_1}{(1-z)^p})$ has unbounded l-index then f has unbounded index.

Theorem 6. *Let f be an entire function of finite order, for which zero is a Picard exceptional value. If $F(z)=f\left((1-z)^{-p}\right)$ has bounded l-index in $\mathbb{D}$ with $l(r)=\beta(1-r)^{-p-1}$, then f has bounded index.*

Proof. Suppose that f is of unbounded index. Then the following estimate $\ln M_f(r)=O(r)$ as $r\to+\infty$ from Theorem 4 is not valid. Since f is an entire function of finite order with a Picard exceptional value 0, it admits the representation $f(z)=e^{P(z)}$, where $P(z)=a_0+a_1z+a_2z^2+\ldots+a_nz^n$, $a_n\ne0$, $n\ge2$. Hence,

$$F(z)=\exp\left\{(1+o(1))\frac{a_n}{(1-z)^{np}}\right\},\ z\to1.$$

But the function F is of bounded l-index with $l(r)=\frac{\beta}{(1-r)^{p+1}}$. By Theorem 4 F has property

$$\ln M_F(r)=O\left(\int\limits_0^r l(t)dt\right)=O\left(\frac{1}{(1-r)^p}\right),\ r\uparrow1.$$

On the other hand,

$$(1-r)^p\ln M_F(r)\ge(1-r)^p\ln|F(z)|=(1+o(1))\operatorname{Re}\left\{\frac{a_n}{(1-r)^{pn}}\right\}(1-r)^p=$$
$$=(1+o(1))\frac{\operatorname{Re}\{a_n\}}{(1-r)^{(n-1)p}}\to+\infty,$$

that is impossible. □

The next statement follows from Theorems 5 and 6.

Theorem 7. *In order that the function F be of bounded l-index with $l(r)=\beta(1-r)^{-p-1}$, it is sufficient, and if f is an entire function of finite order and 0 is its Picard exceptional value then it is necessary that f is of bounded index.*

4. Bounded l-M-index of some compositions. An analytic function f in $\mathbb{D}_R$ is said ([21, p.74]) to be of bounded l-M-index, if there exists $N\in\mathbb{Z}_+$ such that

$$\frac{M(r,f^{(n)})}{n!l^n(r)}\le\max\left\{\frac{M(r,f^{(k)})}{k!l^k(r)}:0\le k\le N\right\}$$

for all $n\in\mathbb{Z}_+$, $r\in[0,R)$.

A notion of an entire function f of bounded l-M-index is introduced by Sh. Abuarabi and M. Sheremeta ([2]). An entire function f of bounded l-M-index with $l(r) \equiv 1$ is called [15] a function of bounded M-index. There was proved [15] that f is a function of bounded M-index if and only if $\ln M(r, f) = O(r)$ as $r \to +\infty$. M. Sheremeta ([21]) generalized the result for bounded l-M-index.

We need some notations from [21]. Let $-\infty < A \le +\infty$ and $\Omega(A)$ be a class of positive unbounded on $(-\infty, A)$ functions Φ such that the derivative Φ' is continuous, positive and increasing to $+\infty$ on $(-\infty, A)$. For $\Phi \in \Omega(A)$ by ϕ we denote the inverse function to Φ', and let $\Psi(\sigma) = \sigma - \Phi(\sigma)/\Phi'(\sigma)$ be the function associated with Φ in the sense of Newton.

As in [21, p. 84], an analytic function f in $\mathbb{D}_R$ is said to be of finite Φ-type, if $\Phi \in \Omega(\ln R)$, if $\ln M(r, f) = O(\Phi(\ln r))$ as $r \uparrow R$.

The next theorem is valid.

Theorem 8 (Theorem 4.5, [21]). *Suppose that a function $\Phi \in \Omega(\ln R)$, $0 < R \le +\infty$, satisfies the following conditions:*

1) Φ' *is continuously differentiable on* $(-\infty, \ln R)$;

2) $\overline{\lim}_{x\to\ln R} \frac{1}{\Phi'(x)}\Phi'\left(\Phi^{-1}\left(x + \frac{\ln\Phi'(\Phi^{-1}(x))}{\Phi'(\Phi^{-1}(x))}\right)\right) < +\infty$;

3) $\Phi'(x + \frac{\alpha}{\Phi'(x)}) = O(\Phi'(x))$ *as* $x \to \ln R$;

4) $\Phi'(\ln r) > \frac{\alpha}{\ln R - \ln r}$ $(0 \le r < R)$ *for some* $\alpha > 1$.

An analytic function f in $\mathbb{D}_R$ is of finite Φ-type if and only if it is of bounded l-M-index with $l(r) = \frac{\Phi'(\ln r)}{r}$ on $[r_0, R)$ and $l(r) = \frac{\Phi'(\ln r_0)}{r_0}$ on $[0, r_0]$, where $r_0 \in [0, R)$ is an arbitrary fixed number.

Theorem 8 for entire functions of bounded l-M-index is proved by Sh. Abuarabi and M. M. Sheremeta ([2]). Its analog for Dirichlet series is deduced in [19].

Using this theorem it is possible to prove the next assertion

Theorem 9. *Let f be an entire function and $F(z) = f\left(q(1-z)^{-p}\right)$, where $p \in \mathbb{N}$, $q \in \mathbb{R}\backslash\{0\}$. In order that the function F be of bounded l-M-index with $l(r) = \beta(1-r)^{-p-1}$, $\beta > 1$, it is sufficient, and if the power series coefficients of the function f are non-negative and $q > 0$ it is necessary that the function f is of bounded M-index.*

Proof. If the function f has bounded M-index then $\ln M(\rho, f) = O(\rho)$ as $\rho \to +\infty$. Since $M(r, F) \le M(\frac{|q|}{(1-r)^p}, f)$ we have $\ln M(r, F) = O(\frac{1}{(1-r)^p})$ as $r \uparrow 1$. But $1 - r = (1 + o(1))|\ln r|$ as $r \uparrow 1$. Hence, $\ln M(r, F) = O(\frac{1}{|\ln r|^p})$ as $r \uparrow 1$.

The function $\Phi(\sigma) = \frac{1}{|\sigma|^p}$, $\sigma < 0$, satisfies condtions of Theorem 8. Therefore, the function F is of bounded l-M-index with $l(r) = \frac{p}{r|\ln r|^{p+1}}$. This means that F is of bounded l-M-index with $l(r) = \frac{\beta}{(1-r)^{p+1}}$. The sufficiency is proved.

If $q > 0$ and the power series coefficients of the function f are non-negative, then maximum modulus of the functions f and F is attained on a positive ray, that is $M(r, F) = M(\frac{q}{(1-r)^p}, f)$. If the function f is of unbounded M-index then there exists a sequence $(\rho_k) \uparrow +\infty$ such that $\frac{\ln M(\rho_k,f)}{\rho_k} \to +\infty$ as $k \to \infty$. Put $r_k = 1-(q/\rho_k)^{1/p}$. We obtain $\frac{\ln M(r_k,F)}{\frac{q}{(1-r_k)^p}} \to +\infty$ as $k \to +\infty$. Therefore, $\frac{\ln M(r_k,F)}{\Phi(|\ln r_k|)} \to +\infty$ as $k \to \infty$ with $\Phi(\sigma) = \frac{1}{|\sigma|^p}$.

By Theorem 8 the function F is of unbounded l-M-index with $l(r) = \frac{\beta}{r|\ln r|^{p+1}}$. Hence, it is of unbounded l-M-index with $l(r) = \frac{\beta}{(1-r)^{p+1}}$. □

Remark 2. J. Clunie ([14]) has proved that for arbitrary entire functions f and g with $g(0)=0$
$$(\forall r>0)\colon\quad M_F(r)\geq M_f\left(c(\rho)M_g(\rho r)\right). \tag{9}$$
where $c(\rho)=\frac{(1-\rho)^2}{4\rho}$, $\rho\in(0;1)$. This leads to the following question: *Is it possible to deduce a similar estimate to* (9) *for the function* $F(z)=f((1-z)^{-p})$, $p\in\mathbb{N}$?

Proposition 1. *Let f be an entire function, $F(z)=f(\frac{1}{(1-z)^p})$, $p\in\mathbb{N}$. Then*
$$M_F(r)\geq\max\{|f(z+(1-r^2)^{-p})|\colon |z|=pr(1-r^2)^{-p}/4\}$$
for every $r\in(0;1)$.

Proof. We want to establish lower estimate of $M_F(r)=\max\{|f((1-z)^{-p})|\colon |z|=r\}$, where $r\in(0,1)$. Find the image of the circle $|z|=r$ under the mapping $w=\frac{1}{1-z}$. Clearly, the image is also a circle. From properties of the Möbius transformation it is known that the images of symmetric points with respect to a circle are symmetric points with respect to its image. The center of the circle and infinity are symmetric points. The point $z=r^2$ is symmetric to the point $z=1$ with respect to $|z|=r$. Since $w(1)=\infty$, we have that $w(r^2)=(1-r^2)^{-1}$ is the center of a new circle. Find the radius of a new circle $|w(r)-w(r^2)|=\left|\frac{1}{1-r}-\frac{1}{1-r^2}\right|=\frac{r}{1-r^2}$. Thus, image of the circle $|z|=r$ under the mapping $w=\frac{1}{1-z}$ is the circle $\left|w-\frac{1}{1-r^2}\right|=\frac{r}{1-r^2}$. Therefore,
$$M_F(r)=\max\left\{|f(w^p)|\colon \left|w-\frac{1}{1-r^2}\right|=\frac{r}{1-r^2}\right\}=$$
$$=\max\left\{\left|f\left(\left(s+\frac{1}{1-r^2}\right)^p\right)\right|\colon |s|=\frac{r}{1-r^2}\right\}$$
for $s=w-\frac{1}{1-r^2}$. Let $g(s)=(s+\frac{1}{1-r^2})^p$. Since $g(0)=\frac{1}{(1-r^2)^p}\neq0$, we put $g_1(s)=(s+\frac{1}{1-r^2})^p-\frac{1}{(1-r^2)^p}$. Then $f(g(s))=f(g_1(s)+\frac{1}{(1-r^2)^p})=f_1(g_1(s))$, where $f_1(z)=f(z+\frac{1}{(1-r^2)^p})$.

Then in view of (9) we obtain
$$M_F(r)=M_{f_1\circ g_1}\left(\frac{r}{1-r^2}\right)\geq M_{f_1}(c(\rho)M_{g_1}(\rho r))=M_{f_1}\left(c(\rho)\left(\frac{(\rho r+1)^p}{(1-r^2)^p}-\frac{1}{(1-r^2)^p}\right)\right)\geq$$
$$\geq M_{f_1}\left(\frac{(1-\rho)^2pr}{4(1-r^2)^p}\right).$$
For $\rho=0$ it follows that
$$M_F(r)\geq M_{f_1}\left(\frac{pr}{4(1-r^2)^p}\right)=\max\{|f(z+(1-r^2)^{-p})|\colon |z|=pr(1-r^2)^{-p}/4\}.$$
□

Using the result of J.Clunie [14] (see Remark 2) we will study the l-M-index of the function $F(z)=f(g(z))$ with respect to the l-M-indexes of the functions f and g.

Out main result is following.

Theorem 10. *Let f and g be entire functions. If f has bounded M-index and $M_g'(r)$ is continuously differentiable then the function $F(z)=f(g(z))$ has bounded l-M-index with $l(r)=M_g'(r)$. If the function $F(z)=f(g(z))$ is of bounded l-M-index with $l(r)=M_g'(r/2)$ and $g(z)$ is a polynomial then f is of bounded M-index.*

Proof. Suppose that f is of bounded M-index. Then $\ln M_f(r) = O(r)$, $r \uparrow \infty$. From (9) it follows that $\ln M_F(r) \le O(M_g(r))$, $r \to \infty$. Denote $\Phi(\ln r) = M_g(r)$, i.e. $\Phi(x) = M_g(e^x)$. Hence, $\Phi'(x) = M_g'(e^x)e^x$ and $\frac{\Phi'(\ln r)}{r} = \frac{M_g'(r)r}{r} = M_g'(r)$. Thus, by Theorem 8 we conclude that the function F is of bounded l-M-index with $l(r) = M_g'(r)$.

We will prove the second part of the theorem. On the contrary, suppose that f is of unbounded M-index. Then $\exists(\rho_k) \uparrow +\infty$ $(k \to +\infty)$ such that $\frac{\ln M_f(\rho_k)}{\rho_k} \to +\infty$. Inequality (9) can be rewritten as following $M_F(2r) \ge M_f(\frac{1}{8}M_g(r) - |g(0)|)$. Put $r_k = M_g^{-1}(8\rho_k)$. Hence, we obtain $\frac{1}{\rho_k}\ln M_F(2M_g^{-1}(8\rho_k)) \ge \frac{1}{\rho_k}\ln M_f(\rho_k - |g(0)|) \to +\infty$ $(k \to +\infty)$, that is

$$\frac{\ln M_F(2r_k)}{M_g(r_k)} \to +\infty \quad (k \to +\infty). \tag{10}$$

On the other hand, if the function F is of bounded l-M-index with $l(r) = M_g'(r)$, then $\ln M_F(r) = O(\ln M_g(r))$. To obtain contradiction with (10) we need $\ln M_F(2r) = O(\ln M_F(r))$, or $M_g(r) = O(M_g(r/2))$ as $r \to \infty$.

Remark that $M_g(2r) = O(M_g(r)) \Leftrightarrow g$ is polynomial. Indeed, $M_g(2r) \le KM_g(r)$. Substitute $r_n = 2^n$ and obtain $M_g(2^{n+1}) \le KM_g(2^n) \le K^n M_g(2)$. For $2^n \le r \le 2^{n+1}$ we have

$$\ln M_g(r) \le \ln M_g(2^{n+1}) \le n \ln K + \ln M_g(2) \le \frac{\ln K}{\ln 2}\ln r + \ln M_g(2) = K_1 \ln r + K_2.$$

Therefore, g is polynomial with $\deg g \le K_1$. □

The following question is arising.

Problem (M.M. Sheremeta)**.** *What are the entire functions f and g and constants c_1, c_2 such that $M_F(r) \ge M_f(c_1 M_g(r) - c_2)$ for $F(z) = f(g(z))$?*

Now, it is easy to see: if the Taylor series coefficients of the functions f and g are nonnegative then $M_F(r) = M_f(M_g(r))$.

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