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D. Blackmore[a], A. Prykarpatski[b], M. Vovk[c], P. Pukach[d], Ya. Prykarpatsky[e]

THE PFEIFFER-LAX-SATO TYPE VECTOR FIELD EQUATIONS AND THE RELATED INTEGRABLE VERSAL DEFORMATIONS

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We study versal deformatiions of the Pfeiffer-Lax-Sato type vector field equations, related with a centrally extended metrized Lie algebra as the direct sum of vector fields and differential forms on torus.

1. The Pfeiffer-Lax-Sato vector field equations. Consider for simplicity a vector field $X\colon \mathbb{R}\times\mathbb{T}^n \to T(\mathbb{R}\times\mathbb{T}^n)$ on the $(n+1)$-dimensional toroidal cylinder $\mathbb{R}\times\mathbb{T}^n$ for arbitrary $n\in\mathbb{Z}_+$, which we will write in the slightly special form

$$A = \frac{\partial}{\partial t} + \Big\langle a(t,x), \frac{\partial}{\partial x}\Big\rangle, \tag{1}$$

where $(t,x)\in\mathbb{R}\times\mathbb{T}^n, a(t,x)\in\mathbb{E}^n$, $\frac{\partial}{\partial x} := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})^\intercal$ and $\langle\cdot,\cdot\rangle$ is the standard scalar product on the Euclidean space $\mathbb{E}^n$. With the vector field (1) one can associate the linear equation

$$A\psi = 0 \tag{2}$$

for some function $\psi\in C^2(\mathbb{R}\times\mathbb{T}^n;\mathbb{R})$, which we will call an "invariant" of the vector field.

Next, we study the existence and number of such functionally-independent invariants to the equation (2). For this let us pose the following Cauchy problem for equation (2): Find a function $\psi\in C^2(\mathbb{R}\times\mathbb{T}^n;\mathbb{R})$, which at point $t^{(0)}\in\mathbb{R}$ satisfies the condition $\psi(t,x)|_{t=t^{(0)}} = \psi^{(0)}(x)$, $x\in\mathbb{T}^n$, for a given function $\psi^{(0)}\in C^2(\mathbb{T}^n;\mathbb{R})$. For the equation (2) there is a naturally related parametric vector field on the torus $\mathbb{T}^n$ in the form of the ordinary vector differential equation

$$dx/dt = a(t,x), \tag{3}$$

to which there corresponds the following Cauchy problem: find a function $x\colon \mathbb{R}\to\mathbb{T}^n$ satisfying

$$x(t)|_{t=t^{(0)}} = z \tag{4}$$

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for an arbitrary constant vector $z \in \mathbb{T}^n$. Assuming that the vector-function $a \in C^1(\mathbb{R} \times \mathbb{T}^n; \mathbb{E}^n)$, it follows from the classical Cauchy theorem [11] on the existence and unicity of the solution to (3) and (4) that we can obtain a unique solution to the vector equation (3) as some function $\Phi \in C^1(\mathbb{R} \times \mathbb{T}^n; \mathbb{T}^n), x = \Phi(t, z)$, such that the matrix $\partial\Phi(t, z)/\partial z$ is nondegenerate for all $t \in \mathbb{R}$ sufficiently close to $t^{(0)} \in \mathbb{R}$. Hence, the Implicit Function Theorem ([11, 12]) implies that there exists a mapping $\Psi \colon \mathbb{R} \times \mathbb{T}^n \to \mathbb{T}^n$, such that

$$\Psi(t, x) = z \tag{5}$$

for every $z \in \mathbb{T}^n$ and all $t \in \mathbb{R}$ sufficiently enough to $t^{(0)} \in \mathbb{R}$. Supposing now that the functional vector $\Psi(t, x) = (\psi^{(1)}(t, x), \psi^{(2)}(t, x), ..., \psi^{(n)}(t, x))^\intercal$, $(t, x) \in \mathbb{R} \times \mathbb{T}^n$, is constructed, from the arbitrariness of the parameter $z \in \mathbb{T}^n$ one can deduce that all functions $\psi^{(j)} \colon \mathbb{R} \times \mathbb{T}^n \to \mathbb{T}^1$, $j = \overline{1, n}$, are functionally independent invariants of the vector field equation (2), that is $A\psi^{(j)} = 0, j = \overline{1, n}$. Thus, the vector field equation (2) has exactly $n \in \mathbb{Z}_+$ functionally independent invariants, which make it possible, in particular, to solve the Cauchy problem posed above. Namely, let a mapping $\alpha \colon \mathbb{T}^n \to \mathbb{R}$ be chosen such that $\alpha(\Psi(t, x))|_{t=t^{(0)}} = \psi^{(0)}(x)$ for all $x \in \mathbb{T}^n$ and a fixed $t^{(0)} \in \mathbb{R}$. Inasmuch as the superposition of functions $\alpha \circ \Psi \colon \mathbb{R} \times \mathbb{T}^n \to \mathbb{T}^1$ is, evidently, also an invariant for the equation (2), it provides the solution to this Cauchy problem, which we can formulate as the following result.

Proposition 1. *The linear equation (2), generated by the vector field (3) on the torus $\mathbb{T}^n$, has exactly $n \in \mathbb{Z}_+$ functionally independent invariants.*

Consider now a Plucker type ([16]) differential form $\chi^{(n)} \in \Lambda^n(\mathbb{T}^n)$ on the torus $\mathbb{T}^n$ as

$$\chi^{(n)} := d\psi^{(1)} \wedge d\psi^{(2)} \wedge ... \wedge d\psi^{(n)}, \tag{6}$$

generated by the vector $\Psi \colon \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{T}^n$ of independent invariants (5), depending additionally on $n \in \mathbb{Z}_+$ parameters $t \in \mathbb{R}^n$, where by definition, for any $k = \overline{1, n}$

$$d\psi^{(k)} := \Big\langle \frac{\partial\psi^{(k)}}{\partial x}, dx \Big\rangle + \sum_{j=1}^{n} \frac{\partial\psi^{(k)}}{\partial t_j} dt_j \tag{7}$$

on the manifold $\mathbb{R}^n \times \mathbb{T}^n$. As follows from the Frobenius theorem [11, 13, 16], the Plucker type differential form (6) is for all fixed parameters $t \in \mathbb{R}^n$ nonzero on the manifold $\mathbb{T}^n$ owing to the functional independence of the invariants (5). It is easy to see that at the fixed parameters $t \in \mathbb{R}^n$ the following [25] Jacobi-Mayer type relationship

$$\begin{aligned} \left|\frac{\partial\Psi}{\partial x}\right|^{-1} d\psi^{(1)} \wedge d\psi^{(2)} \wedge ... \wedge d\psi^{(n)} = \Big(dx_1 - \sum_{j=\overline{1,n}} a_j^{(1)}(t, x) dt_j\Big) \wedge \\ \wedge \Big(dx_2 - \sum_{j=\overline{1,n}} a_j^{(2)}(t, x) dt_j\Big) \wedge ... \wedge \Big(dx_n - \sum_{j=\overline{1,n}} a_j^{(m)}(t, x) dt_j\Big) \end{aligned} \tag{8}$$

holds for $k = \overline{1, n}$ on the manifold $\mathbb{R}^n \times \mathbb{T}^n$, where $\left|\frac{\partial\Psi}{\partial x}\right|$ is the determinant of the Jacobi mapping $\frac{\partial\Psi}{\partial x} \colon T(\mathbb{T}^n) \to T(\mathbb{T}^n)$ of the mapping (5) subject to the torus variables $x \in \mathbb{T}^n$. On the right-hand side of (8) one has the volume measure on the torus $\mathbb{T}^n$, which is naturally dependent on $t \in \mathbb{R}^n$ owing to the general vector field relationships (3). Taking into account that for all $k = \overline{1, n}$ the full differentials $d\psi^{(k)} = 0$, that is vanishing on $\mathbb{R}^n \times \mathbb{T}^n$, the corresponding substitution of the reduced differentials $d\psi^{(k)} \in C^2(\mathbb{R}^n \times \mathbb{T}^n; \Lambda^1(\mathbb{T}^n)), k = \overline{1,n}$, into (8) easily gives rise, in particular, to the following set of the compatible vector field relationships

$$\frac{\partial \Psi}{\partial t_s} + \sum_{j,k=\overline{1,n}} \left[\left(\frac{\partial \Psi}{\partial x} \right)^{-1}_{jk} \frac{\partial \psi^{(k)}}{\partial t_s} \right] \frac{\partial \Psi}{\partial x_j} = 0, \tag{9}$$

for all $s = \overline{1,n}$. The latter property, as it was demonstrated by M. G. Pfeiffer in [20–25], makes it possible to solve effectively the classical M. A. Buhl problem [8–10] and has interesting applications [6, 16] in the theory of completely integrable dynamical systems of heavenly type [15] and versally deformed operator structures [4] on toroidal manifilds.

1.1. Vector field hierarchies on the torus with "*spectral*" parameter and the Lax–Sato integrable heavenly dynamical systems. Consider some naturally ordered infinite set of parametric vector fields (1) on the infinite dimensional toroidal manifold $\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n$ in the form

$$A^{(k)} = \frac{\partial}{\partial t_k} + \left\langle a^{(k)}(t,x;\lambda), \frac{\partial}{\partial x} \right\rangle + a_0^{(k)}(t,x;\lambda)\frac{\partial}{\partial \lambda} := \frac{\partial}{\partial t_k} + \mathrm{A}^{(k)}, \tag{10}$$

where $t_k \in \mathbb{R}, k \in \mathbb{Z}_+, (t,x;\lambda) \in (\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n) \times \mathbb{C}$ are the evolution parameters, and the dependence of smooth vectors $(a_0^{(k)}, a^{(k)})^\intercal \in \mathbb{E} \times \mathbb{E}^n, k \in \mathbb{Z}_+$, on the *"spectral"* parameter $\lambda \in \mathbb{C}$ is assumed to be holomorphic. Suppose now that the infinite hierarchy of linear equations

$$A^{(k)}\psi = 0 \tag{11}$$

for $k \in \mathbb{Z}_+$ has exactly $n+1 \in \mathbb{Z}_+$ common functionally independent invariants $\psi^{(j)}(\lambda) \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{C}), j = \overline{0,n}$ on the torus $\mathbb{T}^n$, suitably depending on the parameter $\lambda \in \mathbb{C}$. Then, owing to the existence theory ([11, 12]) for ordinary differential equations depending on the *"spectral"* parameter $\lambda \in \mathbb{C}$, these invariants may be assumed to be such that allow analytical continuation in the parameter $\lambda \in \mathbb{C}$ both inside the disc $\mathbb{D}^1_+ \subset \mathbb{C}$ of some disc $\mathbb{D}^1 \subset \mathbb{C}$ and subject to the parameter $\lambda^{-1} \in \mathbb{C}, |\lambda| \to \infty$, outside $\mathbb{D}^1_- \subset \mathbb{C}$ of this disc $\mathbb{D}^1 \subset \mathbb{C}$. This means that as $|\lambda| \to \infty$ we have the following expansions:

$$\begin{aligned} \psi^{(0)}(\lambda) &\sim \lambda + \sum_{k=0}^{\infty} \psi_k^{(0)}(\tau,x)\lambda^{-k}, \\ \psi^{(1)}(\lambda) &\sim \sum_{k=0}^{\infty} \tau_k^{(1)}(t,x)\psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi_k^{(1)}(\tau,x)\psi_0(\lambda)^{-k}, \\ \psi^{(2)}(\lambda) &\sim \sum_{k=0}^{\infty} \tau_k^{(2)}(t,x)\psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi_k^{(2)}(\tau,x)\psi_0(\lambda)^{-k}, \dots, \\ \psi^{(n)}(\lambda) &\sim \sum_{k=0}^{\infty} \tau_k^{(n)}(t,x)\psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi_k^{(n)}(\tau,x)\psi_0(\lambda)^{-k}, \end{aligned} \tag{12}$$

where we took into account that $\psi^{(0)}(\lambda) \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{C})$, $\lambda \in \mathbb{C}$, is the basic invariant solution to the equations (11), the functions $\tau \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R}^{n \times \mathbb{Z}_+})$for all $s = \overline{1,n}, l \in \mathbb{Z}_+$, are assumed to be independent and $\psi_k^{(j)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$ for all $k \in \mathbb{N}, j = \overline{0,n}$, are arbitrary invariants. Write down now the condition (8) on the manifold $\mathbb{C} \times \mathbb{T}^n$ in the form $\lambda \in \mathbb{C}$

$$\left| \frac{\partial \Psi}{\partial \mathrm{x}} \right|^{-1} d\psi^{(0)} \wedge d\psi^{(1)} \wedge d\psi^{(2)} \wedge \ldots \wedge d\psi^{(n)} = \Big(d\lambda - \sum_{j=\overline{1,n}} a_j^{(0)}(t,\mathrm{x})dt_j \Big) \wedge$$

$$\wedge\Big(dx_1-\sum_{j=\overline{1,n}}a_j^{(1)}(t,\mathrm{x})dt_j\Big)\wedge\Big(dx_2-\sum_{j=\overline{1,n}}a_j^{(2)}(t,\mathrm{x})dt_j\Big)\wedge\ldots\wedge\Big(dx_n-\sum_{j=\overline{1,n}}a_j^{(m)}(t,\mathrm{x})dt_j\Big),\quad (13)$$

where $\mathrm{x}:=(\lambda,x)\in\mathbb{C}\times\mathbb{T}^n$, $|\frac{\partial\Psi}{\partial\mathrm{x}}|$ is the Jacobi determinant of the mapping $\Psi:=(\psi^{(0)},\psi^{(1)},\psi^{(2)},\ldots,\psi^{(n)})^\intercal\in C^2(\mathbb{C}\times(\mathbb{R}^{\mathbb{Z}_+}\times\mathbb{T}^n);\mathbb{C}^{n+1})$ on the manifold $\mathbb{C}\times\mathbb{T}^n$. Inasmuch this mapping subject to the parameter $\lambda\in\mathbb{C}$ has analytical continuation [12] inside $\mathbb{D}^1_+\subset\mathbb{C}$ of the circle $\mathbb{D}^1\subset\mathbb{C}$ and subject to the parameter $\lambda^{-1}\in\mathbb{C}$ as $|\lambda|\to\infty$ outside $\mathbb{D}^1_-\subset\mathbb{C}$ of this disc $\mathbb{D}^1\subset\mathbb{C}$, one can easily obtain from the vanishing differential expressions

$$d\psi^{(j)}=\Big\langle\frac{\partial\psi^{(j)}}{\partial\mathrm{x}},d\mathrm{x}\Big\rangle+\sum_{k=0}^{\infty}\frac{\partial\psi^{(j)}}{\partial\tau_k^{(j)}}d\tau_k^{(j)}=0 \quad (14)$$

for all $j=\overline{0,n}$ and the relationship (13) on the extended manifold $\mathbb{C}\times\mathbb{T}^n\times\mathbb{T}^\infty$ of the independent variables $\mathrm{x}\in\mathbb{C}\times\mathbb{T}^n$, evolving analytically with respect to the parameters $\tau_k^{(j)}\in\mathbb{R}$, $j=\overline{1,n},k\in\mathbb{Z}_+$, the following Lax-Sato criterion:

$$\Big(\Big|\frac{\partial\Psi}{\partial\mathrm{x}}\Big|^{-1}d\psi^{(0)}\wedge d\psi^{(1)}\wedge d\psi^{(2)}\wedge\ldots\wedge d\psi^{(n)}\Big)_-=0, \quad (15)$$

where $(\ldots)_-$ means the asymptotic part of an expression in the bracket, depending on the parameter $\lambda^{-1}\in\mathbb{C}$ as $|\lambda|\to\infty$. The substitution of expressions (14) into (15) easily yields

$$-\frac{\partial\Psi}{\partial\tau_k^{(j)}}=\left[\Big(\frac{\partial\Psi}{\partial\mathrm{x}}\Big)^{-1}_{0j}\psi^{(0)}(\lambda)^k\right]_+\frac{\partial\Psi}{\partial\lambda}+\sum_{s=1}^{n}\left[\Big(\frac{\partial\Psi}{\partial\mathrm{x}}\Big)^{-1}_{sj}\psi^{(0)}(\lambda)^k\right]_+\frac{\partial\Psi}{\partial x_s} \quad (16)$$

for all $k\in\mathbb{Z}_+,j=\overline{1,n}$. These relationships (16) comprise an infinite hierarchy of Lax–Sato compatible [29,30] linear equations, where $(\ldots)_+$ denotes the asymptotic part of an expression in the bracket, depending on nonnegative powers of the complex parameter $\lambda\in\mathbb{C}$. As for the independent functional parameters $\tau_k^{(j)}\in C^2(\mathbb{R}^{\mathbb{Z}_+}\times\mathbb{T}^n;\mathbb{R})$ for all $k\in\mathbb{Z}_+,j=\overline{1,n}$, one can state their functional independence by taking into account their *a priori* linear dependence on the independent evolution parameters $t_k\in\mathbb{R}$, $k\in\mathbb{Z}_+$. On the other hand, taking into account the explicit form of the hierarchy of equations (16), following [6], it is not hard to show that the corresponding vector fields

$$\mathrm{A}_k^{(j)}:=\left[\Big(\frac{\partial\Psi}{\partial\mathrm{x}}\Big)^{-1}_{0j}\psi^{(0)}(\lambda)^k\right]_+\frac{\partial}{\partial\lambda}+\sum_{s=1}^{n}\left[\Big(\frac{\partial\Psi}{\partial\mathrm{x}}\Big)^{-1}_{sj}\psi^{(0)}(\lambda)^k\right]_+\frac{\partial}{\partial x_s} \quad (17)$$

on the manifold $\mathbb{C}\times\mathbb{T}^n$ satisfy for all $k,m\in\mathbb{Z}_+,j,l=\overline{1,n}$, the Lax-Sato compatibility conditions

$$\frac{\partial\mathrm{A}_m^{(l)}}{\partial\tau_k^{(j)}}-\frac{\partial\mathrm{A}_k^{(j)}}{\partial\tau_m^{(l)}}=[\mathrm{A}_k^{(j)},\mathrm{A}_m^{(l)}], \quad (18)$$

which are equivalent to the independence of the all functional parameters $\tau_k^{(j)}\in C^1(\mathbb{R}^{\mathbb{Z}_+}\times\mathbb{T}^n;\mathbb{R})$, $k\in\mathbb{Z}_+,j=\overline{1,n}$. As a corollary of the analysis above, one can show that the infinite hierarchy of vector fields (10) is a linear combination of the basic vector fields (17) and also satisfies the Lax type compatibility condition (18). Inasmuch the coefficients of vector fields (17) are suitably smooth functions on the manifold $\mathbb{R}^{\mathbb{Z}_+}\times\mathbb{T}^n$, the compatibility conditions

(18) yield the corresponding sets of differential-algebraic relationships on their coefficients, which have the common infinite set of invariants, thereby comprising an infinite hierarchy of completely integrable so called *heavenly* nonlinear dynamical systems on the corresponding multidimensional functional manifolds. That is, all of the above can be considered as an introduction to a recently devised [6,7,15,29,30] constructive algorithm for generating infinite hierarchies of commuting to each other vector fields on the toroidal manifolds $\mathbb{R}^{\mathbb{Z}_+}\times\mathbb{T}^n, n\in\mathbb{Z}_+$, analytically depending on the complex parameter $\lambda\in\mathbb{C}$.

The important problem consisting in effective description of the corresponding integrable versal deformations [2,3] of these infinite hierarchies of vector fields on the toroidal manifolds $\mathbb{R}^{\mathbb{Z}_+}\times\mathbb{T}^n, n\in\mathbb{Z}_+$, analytically depending on the complex parameter $\lambda\in\mathbb{C}$, is a main topic of next Sections following below.

2. Versal deformations of vector fields and differential forms.

2.1. Deformations. We consider a smooth vector field $A\in\Gamma(T(\mathbb{T}^n))$ on the n-dimensional torus $\mathbb{T}^n$. A deformation of the vector field $A\in\Gamma(T(\mathbb{T}^n))$ we will call a vector field $A(\tau)\in\Gamma(T(\mathbb{T}^n))$, which depends analytically on the parameter $\tau\in\mathbb{C}^k, k\in\mathbb{Z}_+$, in some vicinity of the point $\tau=0\in\mathbb{C}^k$, and such that $A(0)=A$. The space of parameters $\Upsilon\{\tau\in\mathbb{C}^k\}$ is often called a base of the deformation. Similarly will consider a differential 1-form $l\in\Lambda^1(\mathbb{T}^n)$ on the n-dimensional torus $\mathbb{T}^n$ and its related deformation $l(\tau)\in\Lambda^1(\mathbb{T}^n)$, which depends analytically on the parameter $\tau\in\mathbb{C}^k, k\in\mathbb{Z}_+$, in some vicinity of the point $\tau=0\in\mathbb{C}^k$and such that $l(0)=l$.

Definition 1. Two vector fields deformations $A(\tau)$ and $B(\tau)\in\Gamma(T(\mathbb{T}^n))$ are called *equivalent*, if there exists such a deformation $g(\tau)\in\mathrm{Diff}(\mathbb{T}^n)$ of the identity $Id\in\mathrm{Diff}(\mathbb{T}^n)$, that $Ad_{g(\tau)}A(\tau)=B(\tau)$, where $ad\colon\ \mathrm{Diff}(\mathbb{T}^n)\times\Gamma(T(\mathbb{T}^n))\to\Gamma(T(\mathbb{T}^n))$ is the usual [2, 3, 5, 14] adjoint mapping of the space $\mathrm{Diff}(\mathbb{T}^n)$ on $\Gamma(T(\mathbb{T}^n))$. Similarly, two 1-form deformations $l(\tau)$ and $p(\tau)\in\Lambda^1(\mathbb{T}^n)$ are called equivalent, if there exists such a deformation $g(\tau)\in\mathrm{Diff}(\mathbb{T}^n)$ of the identity $I\in\mathrm{Diff}(\mathbb{T}^n)$, that $Ad^*_{g(\tau)}l(\tau)=p(\tau)$, where $Ad^*\colon\ \mathrm{Diff}(\mathbb{T}^n)\times\Lambda^1(\mathbb{T}^n)\to\Lambda^1(\mathbb{T}^n)$ is the usual adjoint mapping of the space $\mathrm{Diff}(\mathbb{T}^n)$ on $\Lambda^1(\mathbb{T}^n)$.

Let φ be a germ of a holomorphic at zero mapping $\mathbb{C}^m\to\mathbb{C}^k$, that is a set of converging at $0\in\mathbb{C}^m$ degree series of complex variables, and assume that $\varphi(0)=0$. The mapping $\varphi\colon\Upsilon\{\sigma\in\mathbb{C}^m\}\to\Upsilon\{\tau\in\mathbb{C}^m\}$ defines evidently a new deformation $\check{\varphi}l(\sigma)\in\Lambda^1(\mathbb{T}^n)$ of the 1-form $l\in\Lambda^1(\mathbb{T}^n)$ and a new deformation $\hat{\varphi}A(\sigma)$ of the vector field $A\in\Gamma(T(\mathbb{T}^n))$ via the expressions

$$(\check{\varphi}l)(\sigma)=l(\varphi(\sigma)),\qquad(\hat{\varphi}A)(\sigma)=A(\varphi(\sigma))\tag{19}$$

on the deformation base $\Upsilon\{\sigma\in\mathbb{C}^k\}$.

Definition 2. The deformation $(\check{\varphi}l)(\sigma)\in\Lambda^1(\mathbb{T}^n)$ is called *induced from the deformation* $l(\tau)\in\Lambda^1(\mathbb{T}^n)$ *under the mapping* $\varphi\colon\Upsilon\{\sigma\in\mathbb{C}^m\}\to\Upsilon\{\tau\in\mathbb{C}^m\}$. Similarly, the deformation $(\hat{\varphi}A)(\sigma)\in\Gamma(T(\mathbb{T}^n))$ is called induced from the deformation $A(\tau)\in\Gamma(T(\mathbb{T}^n))$ under the mapping $\varphi\colon\Upsilon\{\sigma\in\mathbb{C}^k\}\to\Upsilon\{\tau\in\mathbb{C}^m\}$.

2.2. Versal deformations.

Definition 3. A vector field deformation $A(\tau)\in\Gamma(T(\mathbb{T}^n))$ is called [2, 3, 14] *versal,* if it generates every other deformation $B(\sigma)\in\Gamma(T(\mathbb{T}^n))$ of the vector field $A\in\Gamma(T(\mathbb{T}^n))$, that is there exists such a mapping $\varphi\colon\Upsilon\{\sigma\in\mathbb{C}^k\}\to\Upsilon\{\tau\in\mathbb{C}^m\}$ and a deformation

$g(\tau) \in \mathrm{Diff}(\mathbb{T}^n)$ of the identity $Id \in \mathrm{Diff}(\mathbb{T}^n)$ that it is equivalent to the deformation obtained from the induced deformation $A(\varphi(\tau)) \in \Gamma(T(\mathbb{T}^n))$

$$B(\sigma) = Ad_{g(\sigma)}(\hat{\varphi}A)(\sigma) \tag{20}$$

on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$. Similarly, a 1-form deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ is called *versal,* if it generates every other 1-form deformation $p(\tau) \in \Lambda^1(\mathbb{T}^n)$ of the 1-form $l \in \Lambda^1(\mathbb{T}^n)$, that is there exists such a mapping $\varphi\colon \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$ and a deformation $g(\sigma) \in \mathrm{Diff}(\mathbb{T}^n)$ of the identity $I \in \mathrm{Diff}(\mathbb{T}^n)$ that it is equivalent to the deformation obtained from the induced deformation $p(\varphi(\sigma)) \in \Lambda^1(\mathbb{T}^n)$

$$p(\sigma) = Ad^*_{g(\sigma)}(\check{\varphi}l)(\sigma) \tag{21}$$

on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$.

3. Versality and transversality.

3.1. Transversality. Let $N \subset M$ be a smooth submanifold of a manifold M. Consider a smooth mapping $A\colon \Upsilon \to M$, and let a point $\tau \in \Upsilon$ for which $A(\tau) \in N$.

Definition 4. A mapping $A\colon \Upsilon \to M$ is called *transvesal* ([2, 3]) to the submanifold $N \subset M$, if

$$T_{A(\tau)}(M) = T_{A(\tau)}(N) + A_* T_\tau(\Upsilon). \tag{22}$$

As the diffeomorphism group $\mathrm{Diff}(\mathbb{T}^n)$naturally acts on a fixed vector field $A \in \Gamma(T(\mathbb{T}^n))$, its orbit $Or(A; \mathrm{Diff}(\mathbb{T}^n)) = Ad_{\mathrm{Diff}(\mathbb{T}^n)}A \subset \Gamma(T(\mathbb{T}^n))$. Thus, a deformation $A(\tau) \in \Gamma(T(\mathbb{T}^n))$ can be considered as a mapping $A\colon \Upsilon \to \Gamma(T(\mathbb{T}^n))$ of the deformation base $\Upsilon\{\sigma \in \mathbb{C}^m\}$ into the space of vector fields $\Gamma(T(\mathbb{T}^n))$on the torus $\mathbb{T}^n$. The following lemma ([3, 14]) holds.

Lemma 1. *A deformation $A(\tau) \in \Gamma(T(\mathbb{T}^n))$ is versal iff the mapping $A\colon \{\tau \in \mathbb{C}^m\} \to \Gamma(T(\mathbb{T}^n))$ is transversal to the orbit of the corresponding element $A \in \Gamma(T(\mathbb{T}^n))$, that is any deformation $B(\sigma) = Ad^*_{g(\sigma)}(\hat{\varphi}A)(\sigma)$ on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$ for some mapping $\varphi\colon \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$.*

Proof. Really, owing to the versality condition (20), for any deformation $B(\tau) \in \Gamma(T(\mathbb{T}^n))$ of the vector field $A \in \Gamma(T(\mathbb{T}^n))$ one has

$$B(\tau) = Ad_{g(\tau)}(\hat{\varphi}A)(\tau) \tag{23}$$

on the deformation base $\Upsilon\{\tau \in \mathbb{C}^m\}$. Then, upon differentiating (23) with respect to $\tau \in \Upsilon$ one obtains that

$$B_*(0)\xi = A_*(0)\hat{\varphi}(0)\xi + [C_*(0)\xi, A] \tag{24}$$

for any $\xi \in T(\Upsilon)$,. where $[\cdot,\cdot]$ is the usual commutator of vector fields on $\mathbb{T}^n$ and

$$\nabla_\tau \varphi(\tau)|_{\tau=0} := \xi \in T_0(\Upsilon),\ \nabla_\tau g(\tau)|_{\tau=0} := C_*(0) \in \Gamma(T(\mathbb{T}^n)).$$

Now it is easy to see that (24) is equivalent to the transversality condition (22), if to put $M := \Gamma(T(\mathbb{T}^n)), N := Or(A; \mathrm{Diff}(\mathbb{T}^n) \subset \Gamma(T(\mathbb{T}^n))$.

Consider now a smooth mapping $\alpha\colon \mathrm{Diff}(\mathbb{T}^n) \to \Gamma(T(\mathbb{T}^n))$, where

$$\alpha(g) := Ad_g A, \tag{25}$$

which induces the tangent mapping $\alpha_*\colon \mathrm{diff}(\mathbb{T}^n) \to T_A(\ \Gamma(T(\mathbb{T}^n)))$, where $\mathrm{diff}(\mathbb{T}^n) := T_{Id}(\mathrm{Diff}(\mathbb{T}^n)$ is the Lie algebra of vector fields on the torus $\mathbb{T}^n$ and acts as

$$\alpha_* C = [C, A]. \tag{26}$$

The kernel $Ker\alpha_*$ is a Lie subalgebra of vector fields commuting with the vector field $A \in \Gamma(T(\mathbb{T}^n))$and is called its *centralizer.* It is also interesting to observe that the codimension $co\dim Or(A; \mathrm{Diff}(\mathbb{T}^n) = \dim Ker\alpha_*$. As a result from reasoning in [2,3,14] for small enough $\tau \in \Upsilon$ there exists an invertible mapping $\beta\colon V \times \Upsilon\{\tau \in \mathbb{C}^m\} \to \Gamma(T(\mathbb{T}^n))$ for V to be a submanifold of $\mathrm{Diff}(\mathbb{T}^n)$, transversal to the centralizer $\dim Ker\alpha_*$ and of maximal dimension $\dim V = \dim Or(A; \mathrm{Diff}(\mathbb{T}^n)$, allowing the representation

$$\beta(g, \tau) = Ad_g A(\tau) \tag{27}$$

on the deformation base $\Upsilon\{\tau \in \mathbb{C}^m\}$ for some $g \in V$. Let now $B(\sigma) \in \Gamma(T(\mathbb{T}^n))$ be an arbitrary transversal deformation. Then it can be represented as $B(\sigma) = \beta(v, \tau)$, giving rise to the following expression

$$B(\sigma) = Ad_{g(\sigma)} A(\varphi(\sigma)), \tag{28}$$

where $\varphi(\sigma) := \pi_2\beta^{-1}(B(\sigma)), g(\sigma) := \pi_1\beta^{-1}(B(\sigma))$ and π_1 and π_2 are projections of $V \times \Upsilon\{\tau \in \mathbb{C}^k\}$ on the first and the second factor, respectively. The obtained expression (28) exactly means that this arbitrary deformation $B(\sigma) \in \Gamma(T(\mathbb{T}^n))$ is versal, thus proving the lemma. □

Consider now a 1-form deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ on the deformation base $\Upsilon\{\tau \in \mathbb{C}^m\}$. The same way as above one can prove the following dual to Lemma (1) proposition.

Proposition 2. *A 1-form deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ is versal iff the mapping $l\colon \Upsilon\{\tau \in \mathbb{C}^m\} \to \Lambda^1(\mathbb{T}^n)$ is transversal to the orbit of the corresponding element $l \in \Lambda^1(\mathbb{T}^n)$, that is any deformation $p(\sigma) = Ad^*_{g(\sigma)}(\check{\varphi} l)(\sigma)$ on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$ for some mapping $\varphi\colon \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$.*

Being interested in describing versal deformations of pencils of differential forms, analytically depending on the *"spectral"* parameter $\lambda \in \mathbb{C}$, we will proceed below first to studying their orbits from the Marsden-Weinstein reduction theory point of view.

4. Torus diffeomorphism group and its orbits. Let us now consider the action of the diffeomorphism group $\mathrm{Diff}(\mathbb{T}^n)$ on the space $\mathcal{G} := \mathrm{diff}(\mathbb{T}^n) \ltimes \mathrm{diff}(\mathbb{T}^n)^*$, being the semidirect product $\Gamma(T(\mathbb{T}^n)) \ltimes \Lambda^1(\mathbb{T}^n) \simeq \mathrm{diff}(\mathbb{T}^n) \ltimes \mathrm{diff}(\mathbb{T}^n)^*$. It is well known ([18, 19]) that the semidirect sum $\mathcal{G} = \mathrm{diff}(\mathbb{T}^n) \ltimes \mathrm{diff}(\mathbb{T}^n)^*$ is a metrized Lie algebra with the Lie structure

$$[a_1 \ltimes l_1, a_2 \ltimes l_2] := [a_1, a_2] \ltimes (ad^*_{a_1} l_2 - ad^*_{a_2} l_1), \tag{29}$$

allowing to identify it with its adjoint space $\mathcal{G}^* \simeq \mathcal{G}$ via the nondegenerate and symmetric scalar product

$$(a_1 \ltimes l_1, a_2 \ltimes l_2) = (l_1, a_2) + (l_2, a_1) \tag{30}$$

for arbitrary $a_1 \ltimes l_1, a_2 \ltimes l_2 \in \mathcal{G}^* \simeq \mathcal{G}$, where $(\cdot,\cdot)\colon \Lambda^1(\mathbb{T}^n) \times \Gamma(T(\mathbb{T}^n)) \to \mathbb{C}$ is the standard pairing.

Consider now the point product $\check{\mathcal{G}} := \prod_{z\in\mathbb{S}^1}\tilde{\mathcal{G}}$ of Lie algebra $\mathcal{G}$ and endow it wit the central extension generated by a two-cocycle $\omega_2\colon \check{\mathcal{G}}\times\check{\mathcal{G}}\to\mathbb{C}$, where

$$\omega_2(a_1\ltimes l_1, a_2\ltimes l_2) := \int_{\mathbb{S}^1}[(l_1,\partial a_2/\partial z)-(l_2,\partial a_1/\partial z)]dz \tag{31}$$

for arbitrary $a_1\ltimes l_1, a_2\ltimes l_2\in\check{\mathcal{G}}$. Thus, the adjoint space $\check{\mathcal{G}}^*$ is a Poisson manifold [2,5,28,31] endowed with the canonical Lie-Poisson structure

$$\begin{aligned}\{f,h\}_0 &:= (a\ltimes l,[\nabla f(a\ltimes l),\nabla h(a\ltimes l)])+ \\ &+\int_{\mathbb{S}^1}\Big[\Big\langle\nabla f_a(a\ltimes l),\frac{\partial}{\partial z}\nabla h_l(a\ltimes l)\Big\rangle-\Big\langle\nabla h_a(a\ltimes l),\frac{\partial}{\partial z}\nabla f_l(a\ltimes l)\Big\rangle\Big]dz,\end{aligned} \tag{32}$$

where $f,h\in\mathcal{D}(\check{\mathcal{G}}^*)$, $\nabla f(a\ltimes l):=\nabla f_l(a\ltimes l)\ltimes\nabla f_a(a\ltimes l)\in\check{\mathcal{G}}$, $\nabla h(a\ltimes l):=\nabla h_l(a\ltimes l)\ltimes\nabla h_a(a\ltimes l)\in\check{\mathcal{G}}$ and $\nabla\colon\mathcal{D}(\check{\mathcal{G}}^*)\to\check{\mathcal{G}}$ is the usual functional gradient mapping. If to take now a constant vector field $d(a\ltimes l)/ds = J(\alpha) := \sum_{j,k=\overline{1,n}}\alpha_{jk}\partial/\partial x_j\ltimes dx_k\in\check{\mathcal{G}}^*$, depending on the constant parameters $\alpha_{kj}\in\mathbb{C}, j,k=\overline{1,n}$, one can construct ([17,27]) by means of the Lie differentiation $L_{J(\alpha,\beta)}$ of the bracket (32) a new Poisson bracket

$$\begin{aligned}\{f,h\}_1 &:= L_{J(\alpha,\beta)}\{f,h\}_0-\{L_{J(\alpha,\beta)}f,h\}_0-\{f,L_{J(\alpha,\beta)}h\}_0= \\ &= (J(\alpha),[\nabla f(a\ltimes l),\nabla h(a\ltimes l)]),\end{aligned} \tag{33}$$

defined for any $f,h\in\mathcal{D}(\check{\mathcal{G}}^*)$ and satisfying the Jacobi condition.

Consider now the infinitesimal $\mathrm{Diff}(\mathbb{T}^n)$-actions on the space $\check{\mathcal{G}}^*\simeq\check{\mathcal{G}}$ subject to the Poisson brackets (32) and (33)

$$d(a\ltimes l)/d\tau = \{h,a\ltimes l\}_0 = \Big(-[\nabla h_l,a]+\frac{\partial}{\partial z}\nabla h_l\Big)\ltimes\Big(ad^*_{\nabla h_l}l-ad^*_a\nabla h_a-\frac{\partial}{\partial z}\nabla h_a\Big) \tag{34}$$

subject to any function $h\in\mathcal{D}(\check{\mathcal{G}}^*)$ and

$$d(a\ltimes l)/d\xi = \{f,a\ltimes l\}_1 = -\sum_{j=\overline{1,n}}\alpha_{jk}[\nabla f_l,\partial/\partial x_j]\ltimes ad^*_{\nabla f_l}dx_k \tag{35}$$

subject to a Casimir function $f\in\mathcal{D}(\check{\mathcal{G}}^*)$, respectively to the evolution parameters τ and $\xi\in\mathbb{C}$. Making use of the vector fields (34) and (35), one can construct the following integrable on the space $\check{\mathcal{G}}^*$ distributions:

$$\mathcal{D}_0=\Big\{\Big(-[\nabla h_l,a]+\frac{\partial}{\partial z}\nabla h_l\Big)\ltimes\Big(ad^*_{\nabla h_l}l-ad^*_a\nabla h_a-\frac{\partial}{\partial z}\nabla h_a\Big)\colon h\in I_1(\check{\mathcal{G}}^*)\Big\}, \tag{36}$$

where $I_1(\check{\mathcal{G}}^*)$ is the space of Casimir functions for the Poisson bracket (33), and

$$\mathcal{D}_1=\Big\{-\sum_{j,k=\overline{1,n}}\alpha_{jk}[\nabla f_l,\partial/\partial x_j]\ltimes ad^*_{\nabla f_l}dx_k\colon f\in\mathcal{D}(\check{\mathcal{G}}^*)\Big\}, \tag{37}$$

as $[\mathcal{D}_0,\,\mathcal{D}_0]\subset\mathcal{D}_0$ and $[\mathcal{D}_1,\,\mathcal{D}_1]\subset\mathcal{D}_1$. The set of maximal integral submanifolds of (37) generates the foliation $\check{\mathcal{G}}^*_J\backslash\mathcal{D}_0$, whose leaves are the intersections of fixed integral submanifolds $\check{\mathcal{G}}^*_J\subset\check{\mathcal{G}}^*$ of the distribution $\mathcal{D}_1$ passing through an element $a\ltimes l\in\mathcal{G}^*$ with the leaves of the distribution $\mathcal{D}_0$, generating a compatible system of the Pfeiffer-Lax-Sato type [15] vector field equations. If the foliation $\check{\mathcal{G}}^*_J\backslash\mathcal{D}_0$ is sufficiently smooth, one can define the quotient manifold $\check{\mathcal{G}}^*_{red}:=\check{\mathcal{G}}^*_J/(\check{\mathcal{G}}^*_J\backslash\mathcal{D}_0)$ with its associated projection mapping $\check{\mathcal{G}}^*_J\to\check{\mathcal{G}}^*_{red}$. The structure of the reduced manifold $\check{\mathcal{G}}^*_{red}$ is characterized by the following theorem.

Theorem 1. *On the manifold $\check{\mathcal{G}}^*_{red}$ the pair of Poisson structures $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$ are compatible, that is for any parameter $\lambda \in \mathbb{R}$ the algebraic sum $\{\cdot,\cdot\}_0 + \lambda\{\cdot,\cdot\}_1$ is Poisson too.*

A proof of Theorem 1 is strongly based on the classical differential-geometric Marsden-Weinstein reduced space construction.

As a consequence of Theorem 1 and reasonings, based on the structure of the distribution (36), one can describe its invariants on and a leave $\check{\mathcal{G}}^*_J$ and present the related coordinates on the reduced manifold $\check{\mathcal{G}}^*_{red} = \check{\mathcal{G}}^*_J/(\check{\mathcal{G}}^*_J \backslash \mathcal{D}_0)$. Thus, the related with (34) reduced flow on the manifold $\check{\mathcal{G}}^*_{red}$ will present the canonical representation of the studied versal deformation subject to a metric Lie algebra generated by the semidirect sum $\Gamma(T(\mathbb{T}^n)) \ltimes \Lambda^1(\mathbb{T}^n) \simeq \mathrm{diff}(\mathbb{T}^n) \ltimes \mathrm{diff}(\mathbb{T}^n)^*$ of the smooth affine vector fields $\Gamma(T(\mathbb{T}^n))$ on the torus $\mathbb{T}^n$ and its adjoint space $\Lambda^1(\mathbb{T}^n)$. As an example, for the one-dimensional torus $\mathbb{T}^1 \simeq \mathbb{S}^1$ one can take a seed element $a \ltimes l := v\partial/\partial x \ltimes udx \in \check{\mathcal{G}}^* = \Gamma(T(\mathbb{S}^1)) \ltimes \Lambda^1(\mathbb{S}^1)$, $x \in \mathbb{S}^1$, where $u, v \in C^2(\mathbb{S}^1 \times \mathbb{S}^1; \mathbb{R})$ are some smooth functions. If the constant element $J(\alpha) := \alpha\partial/\partial x \ltimes dx \in \check{\mathcal{G}}^* \simeq \check{\mathcal{G}}$, the space $I_2(\check{\mathcal{G}}^*)$ of the corresponding Casimir gradient easily ensue from (36), generating an interesting for applications compatible system of the Pfeiffer-Lax-Sato type vector field equations. Their detailed analytical structure is under preparation and will be presented in other place.

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[a] Department of Mathematical Sciences at NJIT
University Heights, Newark, NJ, USA
denblac@gmail.com

[b] The Department of Physics, Mathematics and Computer Science of
the Cracov University of Technology, Krakow, Poland
pryk.anat@cybergal.com

[c,d] Fundamental Sciences Institute
Lviv Polytechnical Universtity, Lviv, Ukraine
mira.vovk1@gmail.com

[e] The Department of Applied Mathematics
the University of Agriculture in Krakow, Poland
pryk.anat@cybergal.com

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