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ANALYTIC FUNCTIONS IN THE UNIT BALL OF BOUNDED L-INDEX: ASYMPTOTIC AND LOCAL PROPERTIES

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We have generalized some criteria of boundedness of **L**-index in joint variables for analytic functions in the unit ball, where $\mathbf{L}\colon \mathbb{B}^n \to \mathbb{R}^n_+$ is a continuous vector-function, $\mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$. One of propositions gives an estimate of the coefficients of power series expansions by a dominating homogeneous polynomial for analytic functions in the unit ball. Also we provide growth estimates of these functions. They describe the behavior of maximum modulus of analytic function on a skeleton in a polydisc by behavior of the function **L**.

Most of our results are based on polydisc exhaustion of the unit ball. Nevertheless, we have generalized criteria of boundedness of **L**-index in joint variables which describe local behavior of partial derivatives on sphere in $\mathbb{C}^n$. The proposition uses a ball exhaustion.

An analog of Hayman's theorem is applied to investigation of boundedness of **L**-index in joint variables for analytic solutions in the unit ball of some linear higher-order systems of PDE's. There were found sufficient conditions providing the boundedness. Growth estimates of analytic solutions in the unit ball are also obtained.

1. Introduction. The most important classes of analytic functions of several variables are analytic functions in the polydisc and analytic functions in the unit ball. In our investigations, we develop theory of functions of bounded index for these classes ([1, 2, 12–14]). A concept of bounded index is very flexible to investigate properties of analytic solutions of ordinary and partial differential equations and its systems ([7]).

The paper is a continuation of our investigations from [1, 2]. There was introduced a concept of analytic function of bounded **L**-index in joint variables in a ball in $\mathbb{C}^n$. We obtained some criteria of **L**-index boundedness in joint variables. They describe local behavior of partial derivatives and maximum modulus of analytic functions in the unit ball. They are generalizations of corresponding theorems which are known for entire functions of several variables ([9, 10, 29]).

In [2], we announced a possibility of application of Hayman's theorem to linear higher-order system of PDE's whose coefficients are analytic functions in the unit ball. There was presented an application scheme to a special system. Now, we consider a more general system of PDE's. Besides, some asymptotic estimates for the class are deduced. They describe growth of the logarithm of the maximum modulus of an analytic function on the skeleton of a polydisc via the behavior of some continuous vector function $\mathbf{L}\colon \mathbb{C}^n \to \mathbb{R}^n_+$.

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The main method of our investigations uses a polydisc exhaustion of the unit ball. It is very convenient for the concept of bounded **L**-index in joint variables and functions of several variables. Nevertheless, a ball exhaustion seems to be a more natural approach for analytic function in the unit ball than a polydisc exhaustion. Thus, we also have established some results about local behavior of partial derivatives for functions of this class which are based on Cauchy's integral formula for a ball.

There is another approach to introduce a concept of bounded index in $\mathbb{C}^n$ and $\mathbb{B}^n$. It uses a slice function. These functions are called functions of bounded L-index in direction. They are considered in [3, 5, 6, 8, 16].

In view of results from [18, 19] it is not difficult to prove that for every analytic function F in the unit ball with bounded multiplicities there exists a continuous function $\mathbf{L}\colon \mathbb{C}^n \to \mathbb{R}_+$ such that F is of bounded **L**-index in joint variables. Thus, the class of analytic function in the unit ball of bounded **L**-index in joint variables is very wide.

2. Main definitions and notations. We need some standard notations. Denote

$$\mathbb{R}_+ = [0, +\infty),\ \mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n_+,\ \mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n_+,$$
$$\mathbf{1}_j = (0, \dots, 0, \underbrace{1}_{j\text{-th place}}, 0, \dots, 0) \in \mathbb{R}^n_+,\ R = (r_1, \dots, r_n) \in \mathbb{R}^n_+,$$

$$z = (z_1, \dots, z_n) \in \mathbb{C}^n,\ |z| = \sqrt{\sum_{j=1}^n |z_j|^2}.$$

For $A = (a_1, \dots, a_n) \in \mathbb{R}^n$, $B = (b_1, \dots, b_n) \in \mathbb{R}^n$ we will use formal notations without violation of the existence of these expressions $AB = (a_1b_1, \cdots, a_nb_n)$, $A/B = (a_1/b_1, \dots, a_n/b_n)$, $A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$, $\|A\| = a_1 + \cdots + a_n$, and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, \dots, n\}$; the relation $A \le B$ is defined similarly. For $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ denote $K! = k_1! \cdot \ldots \cdot k_n!$. Addition, scalar multiplication, and conjugation are defined on $\mathbb{C}^n$ componentwise. For $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$ we define

$$\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n,$$

where w_k is the complex conjugate of w_k. The polydisc $\{z \in \mathbb{C}^n\colon |z_j - z_j^0| < r_j,\ j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n(z^0, R)$, its skeleton $\{z \in \mathbb{C}^n\colon |z_j - z_j^0| = r_j,\ j = 1, \dots, n\}$ is denoted by $\mathbb{T}^n(z^0, R)$, and the closed polydisc $\{z \in \mathbb{C}^n\colon |z_j - z_j^0| \le r_j,\ j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n[z^0, R]$, $\mathbb{D}^n = \mathbb{D}^n(\mathbf{0}, \mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$. The open ball $\{z \in \mathbb{C}^n\colon |z - z^0| < r\}$ is denoted by $\mathbb{B}^n(z^0, r)$, its boundary is a sphere $\mathbb{S}^n(z^0, r) = \{z \in \mathbb{C}^n\colon |z - z^0| = r\}$, the closed ball $\{z \in \mathbb{C}^n\colon |z - z^0| \le r\}$ is denoted by $\mathbb{B}^n[z^0, r]$, $\mathbb{B}^n = \mathbb{B}^n(\mathbf{0}, 1)$, $\mathbb{D} = \mathbb{B}^1 = \{z \in \mathbb{C}\colon |z| < 1\}$.

For $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ and the partial derivatives of an analytic function $F(z) = F(z_1, \dots, z_n)$ in $\mathbb{B}^n$ we use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1+\cdots+k_n} f}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}.$$

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z)\colon \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function such that

$$(\forall z \in \mathbb{B}^n)\colon\ l_j(z) > \beta/(1 - |z|),\ j \in \{1, \dots, n\}, \tag{1}$$

where $\beta > \sqrt{n}$ is a some constant.

S. N. Strochyk, M. M. Sheremeta, V. O. Kushnir ([25,35,36]) imposed a similar condition for a function $l\colon \mathbb{D} \to \mathbb{R}_+$ and $l\colon G \to \mathbb{R}_+$, where G is an arbitrary domain in $\mathbb{C}$.

An analytic function $F\colon \mathbb{B}^n \to \mathbb{C}$ is said ([1, 2]) to be of *bounded* **L**-*index (in joint variables),* if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{B}^n$ and for all $J \in \mathbb{Z}^n_+$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon\ K \in \mathbb{Z}^n_+,\ \|K\| \le n_0\right\}. \tag{2}$$

The least such integer n_0 is called the **L**-*index in joint variables of the function F* and is denoted by $N(F, \mathbf{L}, \mathbb{B}^n)$. There are many papers about entire functions of several variables of bounded index ([21,22,24,27–29]) and of bounded **L**-index in joint variables ([5,9–11,17]).

By $Q(\mathbb{B}^n)$ we denote the class of functions **L**, satisfying (1) and the following condition

$$(\forall R \in \mathbb{R}^n_+, |R| \le \beta,\ j \in \{1, \dots, n\})\colon\ 0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty, \tag{3}$$

where

$$\lambda_{1,j}(R) = \inf_{z^0\in\mathbb{B}^n} \inf\left\{\frac{l_j(z)}{l_j(z^0)}\colon z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}, \tag{4}$$

$$\lambda_{2,j}(R) = \sup_{z^0\in\mathbb{B}^n} \sup\left\{\frac{l_j(z)}{l_j(z^0)}\colon z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}. \tag{5}$$

$$\Lambda_1(R) = (\lambda_{1,1}(R), \dots, \lambda_{1,n}(R)),\ \Lambda_2(R) = (\lambda_{2,1}(R), \dots, \lambda_{2,n}(R)). \tag{6}$$

It is not difficult to verify that the class $Q(\mathbb{B}^n)$ can be defined as following

$$\text{for every } j\in\{1, \dots, n\}\ \sup_{z,w\in\mathbb{B}^n}\left\{\frac{l_j(z)}{l_j(w)}\colon |z_k - w_k| \le \frac{r_k}{\min\{l_k(z), l_k(w)\}}, k \in \{1, \dots, n\}\right\}<\infty, \tag{7}$$

i. e. conditions (3) and (7) are equivalent (see a definition of a similar class for $\mathbb{C}^n$ in [6]).

We also need the following assertions. They are generalizations of corresponding propositions for entire functions of bounded L-index in direction [3, 16] and of bounded **L**-index in joint variables ([10, 17]) and of bounded index ([23]).

Theorem 1 ([2])**.** *Let* $\mathbf{L} \in Q(\mathbb{B}^n)$*. An analytic function F in $\mathbb{B}^n$ has bounded* **L**-*index in joint variables if and only if there exist $p \in \mathbb{Z}_+$ and $c \in \mathbb{R}_+$ such that for each $z \in \mathbb{B}^n$*

$$\max\left\{\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)}\colon \|J\| = p + 1\right\} \le c\cdot\max\left\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)}\colon \|K\| \le p\right\}. \tag{8}$$

Theorem 2 ([2])**.** *Let* $\mathbf{L} \in Q^n$*, $F\colon \mathbb{B}^n \to \mathbb{C}$ be an analytic function. If there exist R', $R'' \in \mathbb{R}^n_+$, $\mathbf{0} < R' < R''$, $|R''| < \beta$ and $p_1 = p_1(R', R'') \ge 1$ such that for every $z^0 \in \mathbb{C}^n$ inequality*

$$\max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R''}{\mathbf{L}(z^0)}\right)\right\} \le p_1 \max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R'}{\mathbf{L}(z^0)}\right)\right\} \tag{9}$$

holds then the function F is of bounded **L**-*index in joint variables.*

3. Properties of power series expansion of analytic functions in the unit ball. Let $z^0 \in \mathbb{B}^n$. We develop an analytic function $F\colon \mathbb{B}^n \to \mathbb{C}$ in the power series written in a diagonal form

$$F(z) = \sum_{k=0}^{\infty} p_k(z - z^0) = \sum_{k=0}^{\infty} \sum_{\|J\|=k} b_J (z - z^0)^J, \tag{10}$$

where p_k are homogeneous polynomials of k-th degree, $b_J = \frac{F^{(J)}(z^0)}{J!}$. A polynomial $p_{k_0}, k_0 \in \mathbb{Z}_+$, is called a dominating polynomial in the power series expansion (10) on $\mathbb{T}^n(z^0, R)$ if for every $z \in \mathbb{T}^n(z^0, R)$ the next inequality holds:

$$\left| \sum_{k \neq k^0} p_k(z - z^0) \right| \le \frac{1}{2} \max\{|b_J| R^J \colon \|J\| = k^0\}.$$

Recently, Theorems 3 and 4 were obtained for entire functions ([11]) and for analytic function in a bidisc ([14]). Now we deduce these propositions for analytic functions in the unit ball.

Theorem 3. *Let $\mathbf{L} \in Q(\mathbb{B}^n)$. If an analytic function F in $\mathbb{B}^n$ has bounded $\mathbf{L}$-index in joint variables then there exists $p \in \mathbb{Z}_+$ that for all $d \in (0; \frac{\beta}{\sqrt{n}}]$ there exists $\eta(d) \in (0; d)$ such that for each $z^0 \in \mathbb{B}^n$ and some $r = r(d, z^0) \in (\eta(d), d)$, $k^0 = k^0(d, z^0) \le p$ the polynomial p_{k^0} is a dominating polynomial in the series* (10) *on $\mathbb{T}^n(z^0, \frac{r\mathbf{1}}{\mathbf{L}(z^0)})$.*

Proof. Let F be an analytic function of bounded $\mathbf{L}$-index in joint variables with $N = N(F, \mathbf{L}, \mathbb{B}^n) < +\infty$ and n_0 be the $\mathbf{L}$-index in joint variables at a point $z^0 \in \mathbb{D}^2$, i.e. n_0 is the least number, for which inequality (2) holds at the point z^0. Then for each $z^0 \in \mathbb{B}^n$ $n_0 \le N$.

We put

$$a_J^* = \frac{|b_J|}{\mathbf{L}^J(z^0)} = \frac{|F^{(J)}(z^0)|}{J! \mathbf{L}^J(z^0)},$$
$$a_k = \max\{a_J^* \colon \|J\| = k\}, \ c = 2\{(N + n + 1)!(n + 1)! + (N + 1)C_{n+N-1}^N\}.$$

Let $d \in (0; \frac{\beta}{\sqrt{n}}]$ be an arbitrary number. We also denote $r_m = \frac{d}{(d+1)c^m}$, $\mu_m = \max\{a_k r_m^k \colon k \in \mathbb{Z}_+\}$, $s_m = \min\{k \colon a_k r_m^k = \mu_m\}$ for $m \in \mathbb{Z}_+$.

Since $z^0 \in \mathbb{B}^n$ is a fixed point the inequality $a_K^* \le \max\{a_J^* \colon \ \|J\| \le n_0\}$ is valid for all $K \in \mathbb{Z}_+^n$. Then $a_k \le a_{n_0}$ for all $k \in \mathbb{Z}_+$. Hence, for all $k > n_0$, in view of $r_0 < 1$, we have $a_k r_0^k < a_{n_0} r_0^{n_0}$. This implies $s_0 \le n_0$. Since $c r_m = r_{m-1}$, we obtain that for each $k > s_{m-1}$ $(r_{m-1} < 1)$

$$a_{s_{m-1}} r_m^{s_{m-1}} = a_{s_{m-1}} r_{m-1}^{s_{m-1}} c^{-s_{m-1}} \ge a_k r_{m-1}^k c^{-s_{m-1}} = a_k r_m^k c^{k - s_{m-1}} \ge c a_k r_m^k. \tag{11}$$

It yields that $s_m \le s_{m-1}$ for all $m \in \mathbb{N}$. Thus, we can rewrite

$$\mu_0 = \max\{a_k r_0^k \colon k \le n_0\}, \ \mu_m = \max\{a_k r_m^k \colon k \le s_{m-1}\}, \ m \in \mathbb{N}.$$

Let us introduce additional notations for $m \in \mathbb{N}$

$$\mu_0^* = \max\left\{a_k r_0^k \colon s_0 \ne k \le n_0\right\}, \ s_0^* = \min\{k \colon k \ne s_0, a_k r_0^k = \mu_0^*\},$$

$$\mu_m^* = \max\{a_k r_m^k \colon s_m \neq k \leq s_{m-1}\},\ s_m^* = \min\{k \colon k \neq s_m, a_k r_m^k = \mu_m^*\}.$$

We will show that there exists $m_0 \in \mathbb{Z}_+$ such that

$$\frac{\mu_{m_0}^*}{\mu_{m_0}} \leq \frac{1}{c}. \tag{12}$$

Suppose that for all $m \in \mathbb{Z}_+$ the next inequality holds

$$\frac{\mu_m^*}{\mu_m} > \frac{1}{c}. \tag{13}$$

If $s_m^* < s_m$ ($s_m^* \neq s_m$ in view of definition) then we have

$$a_{s_m^*} r_{m+1}^{s_m^*} = \frac{a_{s_m^*} r_m^{s_m^*}}{c^{s_m^*}} = \frac{\mu_m^*}{c^{s_m^*}} > \frac{\mu_m}{c^{s_m^*+1}} = \frac{a_{s_m} r_m^{s_m}}{c^{s_m^*+1}} = \frac{a_{s_m} r_{m+1}^{s_m}}{c^{s_m^*+1-s_m}} \geq a_{s_m} r_{m+1}^{s_m}.$$

Besides, for every $k > s_m^*$, $k \neq s_m$, (i. e., $k-1 \geq s_m^*$) it can be deduced similarly that

$$a_{s_m^*} r_{m+1}^{s_m^*} = \frac{a_{s_m^*} r_m^{s_m^*}}{c^{s_m^*}} \geq \frac{a_k r_m^k}{c^{s_m^*}} \geq \frac{a_k r_m^k}{c^{k-1}} = c a_k r_{m+1}^k.$$

Hence, $a_{s_m^*} r_{m+1}^{s_m^*} > a_k r_{m+1}^k$ for all $k > s_m^*$. Then

$$s_{m+1} \leq s_m^* \leq s_m - 1. \tag{14}$$

On the contrary, if $s_m < s_m^* \leq s_{m-1}$, then the equality $s_{m+1} = s_m$ may holds. Indeed, by definition $s_{m+1} \leq s_m$. It means that the specified equality is possible. But if $s_{m+1} < s_m$ then $s_{m+1} \leq s_m - 1$ (they are natural numbers!). Hence, we obtain (14).

Thus, the inequalities $s_{m+1}^* \leq s_m$ and $s_m^* \neq s_{m+1}$ imply that $s_{m+1}^* < s_{m+1}$. As above instead of (14) we have

$$s_{m+2} \leq s_{m+1}^* \leq s_{m+1} - 1 = s_m - 1.$$

Therefore, if for all $m \in \mathbb{Z}_+$ (13) holds, then for every $m \in \mathbb{Z}_+$ either $s_{m+2} \leq s_{m+1} \leq s_m - 1$ or $s_{m+2} \leq s_m - 1$ holds, that is $s_{m+2} \leq s_m - 1$, because $s_{m+2} \leq s_{m+1}$. It follows that

$$s_m \leq s_{m-2} - 1 \leq \ldots \leq s_{m-2[m/2]} - [m/2] \leq s_0 - [m/2] \leq n_0 - [m/2] \leq N - [m/2].$$

In other words, $s_m < 0$ for $m > 2N+1$, which is impossible. Therefore, there exists $m_0 \leq 2N+1$ such that (12) holds. We put $r = r_{m_0}$, $\eta(d) = \frac{d}{(d+1)c^{2(N+1)}}$, $p = N$ and $k_0 = s_{m_0}$. Then for all $\|J\| \neq k_0 = s_{m_0}$ in $\mathbb{T}^n(z^0, \frac{r_{\mathbf{1}}}{\mathbf{L}(z^0)})$, in view (11) and (12) we obtain

$$|b_J||(z-z^0)^J| = a_J^* r^{\|J\|} \leq a_{\|J\|} r^{\|J\|} \leq \frac{1}{c} a_{s_{m_0}} r_{m_0}^{s_{m_0}} = \frac{1}{c} a_{k_0} r^{k_0}.$$

Thus, for $z \in \mathbb{T}^n(z^0, \frac{r\nVdash}{\mathbf{L}(z^0)})$

$$\left|\sum_{\|J\| \neq k_0} b_J (z-z^0)^J\right| \leq \sum_{\|J\| \neq k_0} a_j^* r^{\|J\|} \leq \sum_{\substack{k=0,\\ k \neq k_0}}^{\infty} a_k C_{n+k-1}^k r^k =$$

$$= \sum_{\substack{k=0,\\ k\neq s_{m_0}}}^{s_{m_0-1}} a_k C^k_{n+k-1} r^k + \sum_{k=s_{m_0-1}+1}^{\infty} a_k C^k_{n+k-1} r^k. \tag{15}$$

We will estimate two sums in (15). From (12) it follows that $\mu^*_{m_0} \leq \frac{1}{c}\mu_{m_0}$ or $\max\{a_k r^k_{m_0} \colon k \neq s_{m_0},\ k \leq s_{m_0-1}\} \leq \frac{1}{c}\max\{a_k r^k_{m_0} \colon k \neq s_{m_0}, k \leq s_{m_0-1}\}$, i. e. $a_k r^k \leq \frac{1}{c} a_{k_0} r^{k_0}$. Taking into account (14), it can be deduced that

$$\sum_{\substack{k=0,\\ k\neq s_{m_0}}}^{s_{m_0-1}} a_k C^k_{n+k-1} r^k \leq \frac{a_{k_0} r^{k_0}}{c} \sum_{k=0}^{N} C^k_{n+k-1} \leq \frac{a_{k_0} r^{k_0}}{c} (N+1) C^N_{n+N-1}. \tag{16}$$

For all $k \geq s_{m_0-1}+1$ $a_k r^k_{m_0-1} \leq \mu_{m_0-1}$ holds. Then $a_k r^k_{m_0} = \frac{a_k r^k_{m_0-1}}{c^k} \leq \frac{\mu_{m_0-1}}{c^k}$. In view of (12) we deduce

$$\sum_{k=s_{m_0-1}+1}^{\infty} a_k C^k_{n+k-1} r^k \leq \mu_{m_0-1} \sum_{k=s_{m_0-1}+1}^{\infty} C^k_{n+k-1} \frac{1}{c^k} \leq$$

$$\leq a_{s_{m_0-1}} r_{m_0}^{s_{m_0-1}} c^{s_{m_0-1}} \sum_{k=s_{m_0-1}+1} (k+1)(k+2)\dots(k+n) \frac{1}{c^k} \leq$$

$$\leq \frac{a_{s_{m_0}} r^{s_{m_0}}}{c} c^{s_{m_0-1}} \Bigg(\sum_{k=s_{m_0-1}+1}^{\infty} x^{k+n} \Bigg)^{(n)} \Bigg|_{x=\frac{1}{c}} = \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} \Bigg\{ \frac{x^{s_{m_0-1}+n+1}}{1-x} \Bigg\}^{(n)} \Bigg|_{x=\frac{1}{c}} =$$

$$= \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} \sum_{j=0}^{n} C^j_n (n-j)! (s_{m_0-1}+n+1)\dots(s_{m_0-1}+n-j+2) \times$$

$$\times \frac{x^{s_{m_0-1}+1+n-j}}{(1-x)^{n-j+1}} \Bigg|_{x=\frac{1}{c}} \leq \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0}-1} n! (N+n+1)! \sum_{j=0}^{n} \frac{(1/c)^{s_{m_0-1}+1+n-j}}{(1-1/c)^{n-j+1}} =$$

$$= n!(N+n+1)! \frac{a_{k_0} r^{k_0}}{c} \sum_{j=0}^{n} \frac{1}{(c-1)^{n-j+1}} \leq (n+1)!(N+n+1)! \frac{a_{k_0} r^{k_0}}{c}, \tag{17}$$

because $c \geq 2$. Hence, from (15)–(17) it follows that

$$\Bigg| \sum_{\|J\|\neq k_0} b_J (z-z^0)^J \Bigg| \leq \frac{((N+1)C^N_{n+N-1} + (n+1)!(N+n+1)!) a_{k_0} r^{k_0}}{c} \leq \frac{1}{2} a_{k_0} r^{k_0}.$$

It means that the polynomial P_{k_0} is the dominating polynomial in the series (10) on skeleton $\mathbb{T}^n(z^0, \frac{r\mathbf{1}}{\mathbf{L}(z^0)})$. □

Theorem 4. *Let* $\mathbf{L} \in Q(\mathbb{B}^n)$. *If there exist* $p \in \mathbb{Z}_+$, $d \in (0;1]$, $\eta \in (0;d)$ *such that for each* $z^0 \in \mathbb{B}^n$ *and some* $R = (r_1, \dots, r_n)$ *with* $r_j = r_j(d, z^0) \in (\eta, d)$, $j \in \{1, \dots, n\}$, *and certain* $k^0 = k^0(d, z^0) \leq p$ *the polynomial* p_{k^0} *is the dominating polynomial in the series* (10) *on* $\mathbb{T}^2(z^0, R/\mathbf{L}(z^0))$ *then the analytic in* $\mathbb{B}^n$ *function* F *has bounded* $\mathbf{L}$*-index in joint variables.*

Proof. Suppose that there exist $p \in \mathbb{Z}_+$, $d \leq 1$ and $\eta \in (0;d)$ such that for each $z^0 \in \mathbb{B}^n$ and some $R = (r_1, \dots, r_n)$ with $r_j = r_j(d, z^0) \in (\eta, d)$, $j \in \{1, \dots, n\}$, and $k_0 = k_0(1, z^0) \leq p$ the polynomial P_{k_0} is a dominating polynomial in the series (10) on $\mathbb{T}^n(z^0, \frac{R}{\mathbf{L}(z^0)})$. Let us denote $r_0 = \max_{1\le j\le n} r_j$. Then

$$\left|\sum_{\|J\|\neq k_0} b_J(z-z^0)^J\right| = \left|F(z) - \sum_{\|J\|=k_0} b_J(z-z^0)^J\right| \le \frac{a_{k_0} r_0^{k_0}}{2}.$$

Using Cauchy's inequality we have $|b_J(z-z^0)^J| = a_j^* R^J \le \frac{a_{k_0} r_0^{k_0}}{2}$ for all $J \in \mathbb{Z}_+^n$, $\|J\| \neq k_0$, that is for all $\|J\| = k \neq k_0$

$$a_k R^J \le \frac{a_{k_0} r_0^{k_0}}{2}. \tag{18}$$

Suppose that F is not a function of bounded **L**-index in joint variables. Then in view of Theorem 1 for all $p_1 \in \mathbb{Z}_+$ and $c \ge 1$ there exists $z^0 \in \mathbb{B}^n$ such that the next inequality

$$\max\left\{\frac{|F^{(J)}(z^0)|}{\mathbf{L}^J(z^0)} \colon \|J\| = p_1 + 1\right\} > c \max\left\{\frac{|F^{(K)}(z^0)|}{\mathbf{L}^K(z^0)} \colon \|K\| \le p_1\right\}$$

holds. We put $p_1 = p$ and $c = \left(\frac{(p+1)!}{\eta^{p+1}}\right)^n$. Then for this $z^0(p_1, c)$

$$\max\left\{\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(|z^0|)} \colon \|J\| = p+1\right\} > \frac{1}{\eta^{p+1}} \max\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^K(|z^0|)} \colon \|K\| \le p\},$$

that is $a_{p+1} > \frac{a_{k_0}}{\eta^{p+1}}$. Hence, $a_{p+1} r_0^{p+1} > \frac{a_{k_0} r_0^{p+1}}{\eta^{p+1}} \ge a_{k_0} r^{k_0}$. The last inequality contradicts (18). Therefore, F is of bounded **L**-index in joint variables. □

4. Properties of $Q(\mathbb{B}^n)$. Here we study some properties of the auxiliary class $Q(\mathbb{B}^n)$. Similar propositions for $\mathbb{C}^n$ are established in [15].

Theorem 5. *Let* $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, $l_j \colon \mathbb{B}^n \to \mathbb{C}$ *and* $\frac{\partial l_j}{\partial z_m}$ *be continuous functions in* $\mathbb{B}^n$, *for all* j, $m \in \{1, 2, \ldots, n\}$. *If for every* $j \in \{1, 2, \ldots, n\}$ $|l_j(z)|$ *satisfies* (1) *and there exist* $P > 0$ *and* $c > 0$ *such that for all* $z \in \mathbb{B}^n$ *and every* $j, m \in \{1, 2, \ldots, n\}$

$$\frac{1}{c + |l_j(z)|}\left|\frac{\partial l_j(z)}{\partial z_m}\right| \le P \tag{19}$$

then $\mathbf{L}^* \in Q(\mathbb{B}^n)$, *where* $\mathbf{L}^*(z) = (c + |l_1(z)|, \ldots, c + |l_n(z)|)$.

Proof. Clearly, the function $\mathbf{L}^*(z)$ is positive and continuous. For given $z \in \mathbb{B}^n$, $z^0 \in \mathbb{B}^n$ we define an analytic curve $\varphi \colon [0, 1] \to \mathbb{B}^n$

$$\varphi_j(\tau) = z_j^0 + \tau(z_j - z_j^0),\ j \in \{1, 2, \ldots, n\},$$

where $\tau \in [0, 1]$. It is known that for every continuously differentiable function g of real variable τ the inequality $\frac{d}{dt}|g(\tau)| \le |g'(\tau)|$ holds except the points where $g(\tau) = 0$. Using assymptions of this lemma, we establish the upper estimate of $\lambda_{2,j}(z_0, R)$:

$$\lambda_{2,j}(z_0, R) = \sup\left\{\frac{c + |l_j(z)|}{c + |l_j(z^0)|} \colon z \in \mathbb{D}^n\left[z^0, \frac{R}{\mathbf{L}^*(z^0)}\right]\right\} =$$

$$= \sup_{z\in\mathbb{D}^n\left[z^0,\frac{R}{\mathbf{L}^*(z^0)}\right]} \left\{\exp\left\{\ln(c+|l_j(z)|)-\ln(c+|l_j(z^0)|)\right\}\right\} =$$

$$= \sup\left\{\exp\left\{\int_0^1 \frac{d(c+|l_j(\varphi(\tau))|)}{c+|l_j(\varphi(\tau))|}\right\} : z\in\mathbb{D}^n\left[z^0,\frac{R}{\mathbf{L}^*(z^0)}\right]\right\} \le$$

$$\le \sup_{z\in\mathbb{D}^n\left[z^0,\frac{R}{\mathbf{L}^*(z^0)}\right]} \left\{\exp\left\{\int_0^1 \sum_{m=1}^n \frac{|\varphi_m'(\tau)|}{c+|l_j(\varphi(\tau))|}\left|\frac{\partial l_j(\varphi(\tau))}{\partial z_m}\right| d\tau\right\}\right\} \le$$

$$\le \sup_{z\in\mathbb{D}^n\left[z^0,\frac{R}{\mathbf{L}^*(z^0)}\right]} \left\{\exp\left\{\int_0^1 \sum_{m=1}^n P|z_m-z_m^0|d\tau\right\}\right\} \le$$

$$\le \sup_{z\in\mathbb{D}^n\left[z^0,\frac{R}{\mathbf{L}^*(z^0)}\right]} \left\{\exp\left\{\sum_{m=1}^n \frac{Pr_j}{c+|l_m(z^0)|}\right\}\right\} \le \exp\left(\frac{P}{c}\sum_{m=1}^n r_j\right).$$

Hence, for all

$$R \ge \mathbf{0}\ \lambda_{2,j}(R) = \sup_{z^0\in\mathbb{B}^n} \lambda_{2,j}(z^0,\eta) \le \exp\left(\frac{P}{c}\sum_{m=1}^n r_j\right) < \infty.$$

Using the inequality $\frac{d}{dt}|g(t)| \ge -|g'(t)|$ it can be proved that for every $\eta \ge 0$ one has $\lambda_{1,j}(R) \ge \exp\left(-\frac{P}{c}\sum_{m=1}^n r_j\right) > 0$. Therefore, $\mathbf{L}^* \in Q(\mathbb{B}^n)$. □

Particularly, if $\mathbf{L}(z) = (l_1(R),\ldots,l_n(R))$, $R = (|z_1|,\ldots,|z_n|)$, for every $j\in\{1,\ldots,n\}$ the function $l_j(R)$ is positive continuously differentiable and $|\nabla \ln l_j(R)| \le P$ for all $|R|<1$ then $\mathbf{L}\in Q(\mathbb{B}^n)$, where

$$\nabla l_j(R) = \Big(\frac{\partial l_j(R)}{\partial r_1},\ldots,\frac{\partial l_j(R)}{r_n}\Big).$$

At first we prove the following lemma.

Lemma 1. *If* $\mathbf{L}\in Q(\mathbb{B}^n)$ *then for every* $j\in\{1,\ldots,n\}$ *and for every fixed* $z^*\in\mathbb{B}^n$ $|z_j|l_j(z^*+z_j\mathbf{1}_j)\to\infty$ *as* $|z^*+z_j\mathbf{1}_j|\to 1-0$.

Proof. In view of (1) we have $l_j(z^*+z_j\mathbf{1}_j) \ge \frac{\beta}{1-|z^*+z_j\mathbf{1}_j|} \to +\infty$ as $|z^*+z_j\mathbf{1}_j|\to 1-0$. □

5. Estimates of growth of analytic functions in ball. Denote

$$[0,2\pi]^n = \underbrace{[0,2\pi]\times\cdots\times[0,2\pi]}_{n-\text{th times}}.$$

For $R=(r_1,\ldots,r_n)\in\mathbb{R}_+^n$, $\Theta=(\theta_1,\ldots,\theta_n)\in[0,2\pi]^n$, $A=(a_1,\ldots,a_n)\in\mathbb{C}^n$ we write

$$Re^{i\Theta} = (r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n}),\ \arg A = (\arg a_1,\ldots,\arg a_n).$$

By $K(\mathbb{B}^n)$ we denote the class of positive continuous functions $\mathbf{L}=(l_1,\ldots,l_n)$, where $l_j\colon \mathbb{B}^n\to\mathbb{R}_+$ satisfy (1) and there exists $c\ge 1$ such that for every $R\in\mathbb{R}_+^n$ with $|R|<1$ and $j\in\{1,\ldots,n\}$

$$\max_{\Theta_1,\Theta_2\in[0,2\pi]^n} \frac{l_j(Re^{i\Theta_2})}{l_j(Re^{i\Theta_1})} \le c.$$

If $\mathbf{L}(z) = (l_1(|z_1|,\dots,|z_n|),\dots,l_n(|z_1|,\dots,|z_n|))$ then $\mathbf{L}\in K(\mathbb{B}^n)$. It is easy to prove that $\frac{|e^z|+1}{1-|z|}\in Q(\mathbb{D})\setminus K(\mathbb{D})$, but $\frac{e^{e^{|z|}}}{1-|z|}\in K(\mathbb{D})\setminus Q(\mathbb{D})$. Besides, if $\mathbf{L}_1,\mathbf{L}_2\in K(\mathbb{B}^n)$ then $\mathbf{L}_1+\mathbf{L}_2\in K(\mathbb{B}^n)$ and $\mathbf{L}_1\mathbf{L}_2\in K(\mathbb{B}^n)$. For simplicity, let us to write $M(F,R)=\max\{|F(z)|\colon z\in \mathbb{T}^n(\mathbf{0},R)\}$, where $|R|<1$. Denote $\boldsymbol{\beta}=(\frac{\beta}{c\sqrt{n}},\dots,\frac{\beta}{c\sqrt{n}})$.

Theorem 6. *Let $\mathbf{L}\in Q(\mathbb{B}^n)\cap K(\mathbb{B}^n)$, $\beta>c\sqrt{n}$. If an analytic function F in $\mathbb{B}^n$ has bounded $\mathbf{L}$-index in joint variables, then*

$$\ln M(F,R)=O\left(\min_{\sigma_n\in\mathcal{S}_n}\min_{\Theta\in[0,2\pi]^n}\sum_{j=1}^n\int_0^{r_j} l_j(R(j,\sigma_n,t)e^{i\Theta})dt\right)\ \text{as } |R|\to 1-0, \tag{20}$$

where σ_n is a permutation of $\{1,\dots,n\}$, $\mathcal{S}_n$ is a set of all permutations of $\{1,\dots,n\}$,

$$R(j,\sigma_n,t)=(r_1',\dots,r_n'),\quad r_k'=\begin{cases} r_k^0, \text{ if } \sigma_n(k)<j,\\ t, \text{ if } k=j,\\ r_k, \text{ if } \sigma_n(k)>j,\end{cases}\quad k\in\{1,\dots,n\},$$

$R^0=(r_1^0,\dots,r_n^0)$ is a fixed radius.

Proof. Let $R>\mathbf{0}$, $|R|<1$, $\Theta\in[0,2\pi]^n$ and the point $z^*\in\mathbb{T}^n(\mathbf{0},R+\frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})})$ be a such that

$$|F(z^*)|=\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(\mathbf{0},R+\frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}.$$

Denote $z^0=\frac{z^*R}{R+\boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})}$. Then

$$|z_j^0-z_j^*|=\left|\frac{z_j^*r_j}{r_j+\frac{\beta}{c\sqrt{n}l_j(Re^{i\Theta})}}-z_j^*\right|=\left|\frac{z_j^*\beta/(c\sqrt{n}l_j(Re^{i\Theta}))}{r_j+\frac{\beta}{c\sqrt{n}l_j(Re^{i\Theta})}}\right|=\frac{\beta}{c\sqrt{n}l_j(Re^{i\Theta})},$$

$$\mathbf{L}(z^0)=\mathbf{L}\left(\frac{z^*R}{R+\boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})}\right)=\mathbf{L}\left(\frac{(R+\boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta}))e^{i\arg z^*}R}{R+\boldsymbol{\beta}/\mathbf{L}(Re^{i\Theta})}\right)=\mathbf{L}(Re^{i\arg z^*}).$$

Since $\mathbf{L}\in K(\mathbb{B}^n)$ we have that $c\mathbf{L}(z^0)=c\mathbf{L}(Re^{i\arg z^*})\ge \mathbf{L}(Re^{i\Theta})\ge\frac{1}{c}\mathbf{L}(z^0)$. We consider two skeletons $\mathbb{T}^n(z^0,\frac{\mathbf{1}}{\mathbf{L}(z^0)})$ and $\mathbb{T}^n(z^0,\frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)})$. By Theorem 2 there exists $p_1=p_1(\frac{\mathbf{1}}{c},c\boldsymbol{\beta})\ge 1$ such that (9) holds with $R'=\frac{\mathbf{1}}{c}$, $R''=c\boldsymbol{\beta}$, i.e.

$$\begin{gathered}\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(\mathbf{0},R+\frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}=|F(z^*)|\le\\ \le\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(z^0,\frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}\le\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(z^0,\frac{c\boldsymbol{\beta}}{\mathbf{L}(z^0)}\right)\right\}\le\\ \le p_1\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(z^0,\frac{\mathbf{1}}{c\mathbf{L}(z^0)}\right)\right\}\le p_1\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(\mathbf{0},R+\frac{\mathbf{1}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}.\end{gathered}\tag{21}$$

The function $\ln^+\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}$ is a convex function of the variables $\ln r_1,\dots,\ln r_n$ (see [30, p. 84]). Hence, the function admits a representation

$$\ln^+\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}-\ln^+\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R+(r_j^0-r_j)\mathbf{1}_j)\}=$$

$$= \int\limits_{r_j^0}^{r_j} \frac{A_j(r_1,\ldots,r_{j-1},t,r_{j+1},\ldots,r_n)}{t}dt \tag{22}$$

for arbitrary $0 < r_j^0 \leq r_j$, where the functions $A_j(r_1,\ldots,r_{j-1},t,r_{j+1}, \ldots,r_n)$ are positive non-decreasing in variable t, $j \in \{1,\ldots,n\}$.

Using (21) we deduce

$$\ln p_1 \geq \ln\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(\mathbf{0},R+\frac{\boldsymbol{\beta}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}-$$
$$-\ln\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(\mathbf{0},R+\frac{\mathbf{1}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}=$$
$$=\sum_{j=1}^n \ln\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(\mathbf{0},R+\frac{\mathbf{1}+\sum_{k=j}^n(\frac{\beta}{c\sqrt{n}}-1)\mathbf{1}_k}{\mathbf{L}(Re^{i\Theta})}\right)\right\}-$$
$$-\ln\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(\mathbf{0},R+\frac{\mathbf{1}+\sum_{k=j+1}^n(\frac{\beta}{c\sqrt{n}}-1)\mathbf{1}_k}{\mathbf{L}(Re^{i\Theta})}\right)\right\}=$$
$$=\sum_{j=1}^n \int\limits_{r_j+1/l_j(Re^{i\Theta})}^{r_j+\beta/(c\sqrt{n}l_j(Re^{i\Theta}))} \frac{1}{t}A_j\left(r_1+\frac{1}{l_1(Re^{i\Theta})},\ldots,r_{j-1}+\frac{1}{l_{j-1}(Re^{i\Theta})},t,\right.$$
$$\left. r_{j+1}+\frac{\beta}{c\sqrt{n}l_{j+1}(Re^{i\Theta})},\ldots,r_n+\frac{\beta}{c\sqrt{n}l_n(Re^{i\Theta})}\right)dt\geq$$
$$\geq\sum_{j=1}^n \ln\left(1+\frac{\frac{\beta}{c\sqrt{n}}-1}{r_jl_j(Re^{i\Theta})+1}\right)A_j\left(r_1+\frac{1}{l_1(Re^{i\Theta})},\ldots,r_{j-1}+\frac{1}{l_{j-1}(Re^{i\Theta})},r_j,\right.$$
$$\left. r_{j+1}+\frac{\beta}{c\sqrt{n}l_{j+1}(Re^{i\Theta})},\ldots,r_n+\frac{\beta}{c\sqrt{n}l_n(Re^{i\Theta})}\right). \tag{23}$$

By Lemma 1 the function $r_jl_j(Re^{i\Theta})\to+\infty$ as $|R|\to 1-0$. Hence, for $j\in\{1,\ldots,n\}$ and $r_i\geq r_i^0$

$$\ln\left(1+\frac{\frac{\beta}{c\sqrt{n}}-1}{r_jl_j(Re^{i\Theta})+1}\right)\sim\frac{\frac{\beta}{c\sqrt{n}}-1}{r_jl_j(Re^{i\Theta})+1}\geq\frac{\frac{\beta}{c\sqrt{n}}-1}{2r_jl_j(Re^{i\Theta})},\ |R|\to 1-0.$$

Thus, for every $j\in\{1,\ldots,n\}$ inequality (23) implies that

$$A_j\left(r_1+\frac{1}{l_1(Re^{i\Theta})},\ldots,r_{j-1}+\frac{1}{l_{j-1}(Re^{i\Theta})},r_j,r_{j+1}+\frac{\beta}{c\sqrt{n}l_{i+1}(Re^{i\Theta})},\ldots,\right.$$
$$\left. r_n+\frac{\beta}{c\sqrt{n}l_n(Re^{i\Theta})}\right)\leq\frac{2\ln p_1}{\frac{\beta}{c\sqrt{n}}-1}\,r_jl_j(Re^{i\Theta}).$$

Let $R^0=(r_1^0,\ldots,r_n^0)$, where every r_j^0 is chosen above. Applying (22) n-th times we obtain consequently

$$\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}=$$

$$= \ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R+(r_1^0-r_1)\mathbf{1}_1)\}+\int_{r_1^0}^{r_1}\frac{A_1(t,r_2,\ldots,r_n)}{t}dt=$$

$$= \ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R+(r_1^0-r_1)\mathbf{1}_1+(r_2^0-r_2)\mathbf{1}_2)\}+$$

$$+\int_{r_1^0}^{r_1}\frac{A_1(t,r_2,\ldots,r_n)}{t}dt+\int_{r_2^0}^{r_2}\frac{A_2(r_1^0,t,r_3\ldots,r_n)}{t}dt=$$

$$=\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R^0)\}+\sum_{j=1}^n\int_{r_j^0}^{r_j}\frac{A_j(r_1^0,\ldots,r_{j-1}^0,t,r_{j+1},\ldots,r_n)}{t}dt\le$$

$$\le\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R^0)\}+$$

$$+\frac{2\ln p_1}{\frac{\beta}{c\sqrt{n}}-1}\sum_{j=1}^n\int_{r_j^0}^{r_j}l_j(r_1^0e^{i\theta_1},\ldots,r_{j-1}^0e^{i\theta_{j-1}},te^{i\theta_j},r_{j+1}e^{i\theta_{j+1}},\ldots,r_ne^{i\theta_n})dt\le$$

$$\le\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R^0)\}+$$

$$+\frac{2\ln p_1}{\frac{\beta}{c\sqrt{n}}-1}\sum_{j=1}^n\int_{0}^{r_j}l_j(r_1^0e^{i\theta_1},\ldots,r_{j-1}^0e^{i\theta_{j-1}},te^{i\theta_j},r_{j+1}e^{i\theta_{j+1}},\ldots,r_ne^{i\theta_n})dt\le$$

$$\le(1+o(1))\frac{2\ln p_1}{\frac{\beta}{c\sqrt{n}}-1}\sum_{j=1}^n\int_{0}^{r_j}l_j(r_1^0e^{i\theta_1},\ldots,r_{j-1}^0e^{i\theta_{j-1}},te^{i\theta_j},r_{j+1}e^{i\theta_{j+1}},\ldots,r_ne^{i\theta_n})dt.$$

The function $\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}$ is independent of Θ. Thus, the following estimate

$$\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}=$$

$$=O\left(\min_{\Theta\in[0,2\pi]^n}\sum_{j=1}^n\int_0^{r_j}l_j(r_1^0e^{i\theta_1},\ldots,r_{j-1}^0e^{i\theta_{j-1}},te^{i\theta_j},r_{j+1}e^{i\theta_{j+1}},\ldots,r_ne^{i\theta_n})dt\right),$$

holds as $|R|\to 1-0$. Obviously, the similar equality can be proved for arbitrary permutation σ_n of the set $\{1,2,\ldots,n\}$. Thus, estimate (20) holds. Theorem 6 is proved. □

Corollary 1. *If* $\mathbf{L}\in Q(\mathbb{B}^n)\cap K(\mathbb{B}^n)$, $\min_{\Theta\in[0,2\pi]^n} l_j(Re^{i\Theta})$ *is non-decreasing in each variable* r_k, $k,j\in\{1,\ldots,n\}$, $k\neq j$, *analytic function* F *in* $\mathbb{B}^n$ *has bounded* **L**-*index in joint variables then*

$$\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}=O\left(\min_{\Theta\in[0,2\pi]^n}\sum_{j=1}^n\int_0^{r_j}l_j(R^{(j)}e^{i\Theta})dt\right)$$

as $|R|\to 1-0$, *where* $R^{(j)}=(r_1,\ldots,r_{j-1},t,r_{j+1},\ldots,r_n)$.

Note that Theorem 6 is new even for $n=1$ (see Theorem 3.3 in [35]) because we replace the condition $l=l(|z|)$ by the condition $l\in K(\mathbb{D})$, i.e. there exists $c>0$ such that for every $r\in(0,1)$ $\max_{\theta_1,\theta_2\in[0,2\pi]}\frac{l(re^{i\theta_2})}{l(re^{i\theta_1})}\le c$. Particularly, the following proposition is valid.

Corollary 2. *If* $l\in Q\cap K$ *and an analytic function* f *in* $\mathbb{D}$ *has bounded* l-*index then*

$$\ln\max\{|f(z)|\colon |z|=r\}=O\left(\min_{\theta\in[0,2\pi]}\int_0^r l(te^{i\Theta})dt\right)\quad\text{as } r\to 1-0.$$

Let us denote $a^+=\max\{a,0\}$, $u_j(t)=u_j(t,R,\Theta)=l_j(\frac{tR}{r^*}e^{i\Theta})$, where $a\in\mathbb{R}$, $t\in\mathbb{R}_+$, $j\in\{1,\ldots,n\}$, $r^*=\max_{1\le j\le n}r_j\neq 0$ and $\frac{t}{r^*}|R|<1$.

Theorem 7. *Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable r_k, $k \in \{1,\dots,n\}$, $|R| < 1$, $\Theta \in [0,2\pi]^n$. If the function $\mathbf{L}$ satisfies* (1) *and an analytic function F in $\mathbb{B}^n$ has bounded $\mathbf{L}$-index $N = N(F,\mathbf{L})$ in joint variables then for every $\Theta \in [0,2\pi]^n$ and for every $R \in \mathbb{R}^n_+$, $|R| < 1$, and $S \in \mathbb{Z}^n_+$*

$$\ln\max\left\{\frac{|F^{(S)}(Re^{i\Theta})|}{S!\mathbf{L}^S(Re^{i\Theta})}\colon\ \|S\| \le N\right\} \le \ln\max\left\{\frac{|F^{(S)}(\mathbf{0})|}{S!\mathbf{L}^S(\mathbf{0})}\colon\ \|S\| \le N\right\} +$$
$$+\int_0^{r^*}\left(\max_{\|S\|\le N}\left\{\sum_{j=1}^n \frac{r_j}{r^*}(k_j+1)l_j\Big(\frac{\tau}{r^*}Re^{i\Theta}\Big)\right\} + \max_{\|S\|\le N}\left\{\sum_{j=1}^n \frac{k_j(-u_j'(\tau))^+}{l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)}\right\}\right)d\tau. \tag{24}$$

Proof. Let $R \in \mathbb{R}\setminus\{\mathbf{0}\}$, $\Theta \in [0,2\pi]^n$. Denote $\alpha_j = \frac{r_j}{r^*}$, $j \in \{1,\dots,n\}$ and $A = (\alpha_1,\dots,\alpha_n)$. We consider the function

$$g(t) = \max\left\{\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\colon\ \|S\| \le N\right\}, \tag{25}$$

where $At = (\alpha_1 t,\dots,\alpha_n t)$, $Ate^{i\Theta} = (\alpha_1 te^{i\theta_1},\dots,\alpha_n te^{i\theta_n})$.

Since the function $\frac{|F^{(S)}(Ate^{i\Theta})|}{K!\mathbf{L}^K(Ate^{i\Theta})}$ is continuously differentiable by real $t \in [0,+\infty)$, outside the zero set of the function $|F^{(S)}(Ate^{i\Theta})|$, the function $g(t)$ is a continuously differentiable function on $[0,\frac{r^*}{|R|})$, except, perhaps, for a countable set of points.

Therefore, using the inequality $\frac{d}{dr}|g(r)| \le |g'(r)|$ which holds except for the points $r = t$ such that $g(t) = 0$, we deduce

$$\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\right) = \frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\frac{d}{dt}|F^{(S)}(Ate^{i\Theta})| +$$
$$+|F^{(S)}(Ate^{i\Theta})|\frac{d}{dt}\frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})} \le \frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\left|\sum_{j=1}^n F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})\alpha_j e^{i\theta_j}\right| -$$
$$-\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n \frac{k_j u_j'(t)}{l_j(Ate^{i\Theta})} \le \sum_{j=1}^n \frac{|F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})|}{(S+\mathbf{1}_j)!\mathbf{L}^{S+\mathbf{1}_j}(Ate^{i\Theta})}\alpha_j(k_j+1)l_j(Ate^{i\Theta}) +$$
$$+\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n \frac{k_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}. \tag{26}$$

For absolutely continuous functions h_1, h_2, $\dots$, h_k and $h(x) := \max\{h_j(z)\colon 1 \le j \le k\}$, $h'(x) \le \max\{h_j'(x)\colon 1 \le j \le k\}$, $x \in [a,b]$ (see [35, Lemma 4.1, p. 81]). The function g is absolutely continuous, therefore, from (26) it follows that

$$g'(t) \le \max\left\{\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\right) : \|S\| \le N\right\} \le$$
$$\le \max_{\|S\|\le N}\left\{\sum_{j=1}^n \frac{\alpha_j(s_j+1)l_j(Ate^{i\Theta})|F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})|}{(K+\mathbf{1}_j)!\mathbf{L}^{K+\mathbf{1}_j}(Ate^{i\Theta})} + \frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n \frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\} \le$$
$$\le g(t)\left(\max_{\|S\|\le N}\left\{\sum_{j=1}^n \alpha_j(s_j+1)l_j(Ate^{i\Theta})\right\} + \max_{\|S\|\le N}\left\{\sum_{j=1}^n \frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}\right) = g(t)(\beta(t)+\gamma(t)),$$

where

$$\beta(t)=\max_{\|S\|\le N}\left\{\sum_{j=1}^{n}\alpha_j(s_j+1)l_j(Ate^{i\Theta})\right\},\gamma(t)=\max_{\|S\|\le N}\left\{\sum_{j=1}^{n}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}.$$

Thus, $\frac{d}{dt}\ln g(t)\le\beta(t)+\gamma(t)$ and

$$g(t)\le g(0)\exp\int_0^t(\beta(\tau)+\gamma(\tau))d\tau, \tag{27}$$

because $g(0)\neq 0$. But $r^*A=R$. Substituting $t=r^*$ in (27) and taking into account (25), we deduce

$$\ln\max\left\{\frac{|F^{(S)}(Re^{i\Theta})|}{S!\mathbf{L}^S(Re^{i\Theta})}\colon\ \|S\|\le N\right\}\le\ln\max\left\{\frac{|F^{(S)}(\mathbf{0})|}{S!\mathbf{L}^S(\mathbf{0})}\colon\ \|S\|\le N\right\}+$$
$$+\int_0^{r^*}\left(\max_{\|S\|\le N}\left\{\sum_{j=1}^{n}\alpha_j(s_j+1)l_j(A\tau e^{i\Theta})\right\}+\max_{\|S\|\le N}\left\{\sum_{j=1}^{n}\frac{s_j(-u_j'(\tau))^+}{l_j(A\tau e^{i\Theta})}\right\}\right)d\tau,$$

i.e. (24) is proved. □

Theorem 8. *Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable r_k, $k\in\{1,\dots,n\}$, $|R|<1$, $\Theta\in[0,2\pi]^n$. If the function $\mathbf{L}$ satisfies (1) and an analytic function F in $\mathbb{B}^n$ has bounded $\mathbf{L}$-index $N=N(F,\mathbf{L})$ in joint variables and there exists $C>0$ such that the function $\mathbf{L}$ satisfies inequalities*

$$\sup_{|R|<1}\max_{t\in[0,r^*]}\max_{\Theta\in[0,2\pi]^n}\max_{1\le j\le n}\frac{(-(u_j(t,R,\Theta))_t')^+}{\frac{r_j}{r^*}l_j^2(\frac{t}{r^*}Re^{i\Theta})}\le C, \tag{28}$$

then

$$\varlimsup_{|R|\to 1-0}\frac{\ln\max\{|F(z)\colon\ z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max_{\Theta\in[0,2\pi]^n}\int_0^1\langle R,\mathbf{L}\left(\tau Re^{i\Theta}\right)\rangle d\tau}\le(C+1)N+1. \tag{29}$$

Proof. By Lemma 3 if $\mathbf{L}$ satisfies (1) then

$$\max_{\Theta\in[0,2\pi]^n}\int_0^1\left\langle R,\mathbf{L}\left(\tau Re^{i\Theta}\right)\right\rangle d\tau\to+\infty\text{ as }|R|\to 1-0. \tag{30}$$

Denote $\widetilde{\beta}(t)=\sum_{j=1}^n\alpha_jl_j(Ate^{i\Theta})$. If, in addition, (28) holds then for some S^*, $\|S^*\|\le N$ and $\widetilde{S}$, $\|\widetilde{S}\|\le N$,

$$\frac{\gamma(t)}{\widetilde{\beta}(t)}=\frac{\sum_{j=1}^n\frac{s_j^*(-u_j'(t))^+}{l_j(Ate^{i\Theta})}}{\sum_{j=1}^n\alpha_jl_j(Ate^{i\Theta})}\le\sum_{j=1}^{n}s_j^*\frac{(-u_j'(t))^+}{\alpha_jl_j^2(Ate^{i\Theta})}\le\sum_{j=1}^{n}s_j^*\cdot C\le NC,$$
$$\frac{\beta(t)}{\widetilde{\beta}(t)}=\frac{\sum_{j=1}^n\alpha_j(\tilde{s}_j+1)l_j(Ate^{i\Theta})}{\sum_{j=1}^n\alpha_jl_j(Ate^{i\Theta})}=1+\frac{\sum_{j=1}^n\alpha_j\tilde{s}_jl_j(Ate^{i\Theta})}{\sum_{j=1}^n\alpha_jl_j(Ate^{i\Theta})}\le 1+\sum_{j=1}^{n}\tilde{s}_j\le 1+N.$$

But $|F(Ate^{i\Theta})|\le g(t)\le g(0)\exp\int_0^t(\beta(\tau)+\gamma(\tau))d\tau$ and $r^*A=R$. Putting $t=r^*$ and taking into account (30), we obtain

$$\ln\max\{|F(z)\colon\ z\in\mathbb{T}^n(\mathbf{0},R)\}=\ln\max_{\Theta\in[0,2\pi]^n}|F(Re^{i\Theta})|\le\ln\max_{\Theta\in[0,2\pi]^n}g(r^*)\le$$

$$\leq \ln g(0) + \max_{\Theta\in[0,2\pi]^n} \int_0^{r^*} (\beta(\tau)+\gamma(\tau))d\tau \leq \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^n} \int_0^{r^*} \widetilde{\beta}(\tau)d\tau =$$

$$= \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^n} \int_0^{r^*} \sum_{j=1}^n \alpha_j l_j(A\tau e^{i\Theta})d\tau =$$

$$= \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^n} \int_0^{r^*} \sum_{j=1}^n \frac{r_j}{r^*} l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right) d\tau =$$

$$= \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^n} \int_0^{1} \sum_{j=1}^n r_j l_j(\tau Re^{i\Theta})d\tau.$$

Thus, we conclude that (29) holds. □

Theorem 9. *Let* $\mathbf{L}(Re^{i\Theta})$ *be a positive continuously differentiable function in each variable* r_k, $k \in \{1,\dots,n\}$, $|R|<1$, $\Theta\in[0,2\pi]^n$. *If the function* $\mathbf{L}$ *satisfies* (1) *and an analytic function* F *in* $\mathbb{B}^n$ *has bounded* $\mathbf{L}$*-index* $N=N(F,\mathbf{L})$ *in joint variables and*

$$r^*(-(u_j(t,R,\Theta))'_{t=r^*})^+/(r_j l_j^2(Re^{i\Theta})) \to 0 \tag{31}$$

uniformly in all $\Theta\in[0,2\pi]^n$, $j\in\{1,\dots,n\}$, *as* $|R|\to 1-0$ *then*

$$\overline{\lim_{|R|\to 1-0}} \frac{\ln\max\{|F(z)\colon\ z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau Re^{i\Theta})\rangle d\tau} \leq N+1. \tag{32}$$

Estimate (32) can be deduced by analogy to the proof of Theorem 8.

If $\mathbf{L}(z)=\mathbf{L}(R)$ then (31) can be written in a simplified form.

Corollary 3. *Let* $\mathbf{L}(R)$ *be a positive continuously differentiable function in each variable* r_k, $k\in\{1,\dots,n\}$, $|R|<1$. *If the function* $\mathbf{L}$ *satisfies* (1) *and an analytic function* F *in* $\mathbb{B}^n$ *has bounded* $\mathbf{L}$*-index* $N=N(F,\mathbf{L})$ *in joint variables and for every* $j\in\{1,\dots,n\}$

$$\frac{\langle R, \nabla l_j(R)\rangle}{r_j l_j^2(R)} \to 0, \text{ as } |R|\to 1-0$$

then

$$\overline{\lim_{|R|\to 1-0}} \frac{\ln\max\{|F(z)|\colon\ z\in T^n(\mathbf{0},R)\}}{\int_0^1 \langle R, \mathbf{L}(\tau R)\rangle d\tau} \leq N+1,$$

where $\nabla l_j(R) = (\frac{\partial l_j(R)}{\partial r_1},\dots,\frac{\partial l_j(R)}{r_n})$.

Our main result in this section is the following

Theorem 10. *Let* $\mathbf{L}(R)=(l_1(R),\dots,l_n(R))$, $l_j(R)$ *be a positive continuously differentiable non-decreasing function in each variable* r_k, $k\in\{1,\dots,n\}$, $|R|<1$. *If the function* $\mathbf{L}$ *satisfies* (1) *and an analytic function* F *in* $\mathbb{B}^n$ *has bounded* $\mathbf{L}$*-index* $N=N(F,\mathbf{L})$ *in joint variables then*

$$\overline{\lim_{|R|\to 1-0}} \frac{\ln\max\{|F(z)|\colon\ z\in T^n(\mathbf{0},R)\}}{\int_0^1 \langle R, \mathbf{L}(\tau R)\rangle d\tau} \leq N+1.$$

This statement is a consequence of Theorem 9, which is obtained for a more general function $\mathbf{L}$.

We will write $u(r,\theta)=l(re^{i\theta})$. Theorem 7 implies the following proposition for $n=1$.

Corollary 4. *Let $l(re^{i\theta})$ be a positive continuously differentiable function in variable $r\in[0,1)$ for every $\theta\in[0,2\pi]$. If an analytic function f in $\mathbb{D}$ has bounded l-index $N=N(f,l)$ and $\overline{\lim\limits_{r\to 1-0}}\max\limits_{\theta\in[0,2\pi]}\frac{(-u'_r(r,\theta))^+}{l^2(re^{i\Theta})}=C\geq 0$ then*

$$\overline{\lim_{r\to 1-0}}\frac{\ln\max\{|f(z)|\colon |z|=r\}}{\max\limits_{\theta\in[0,2\pi]}\int_0^r l\left(\tau e^{i\theta}\right)d\tau}\leq (C+1)N+1. \tag{33}$$

Estimate (32) is sharp. It is easy to check for these functions $F(z)=\exp\{\frac{1}{(1-z_1)(1-z_2)}\}$, $l_1(z_1,z_2)=\frac{1}{(1-|z_1|)^2(1-|z|)}$, $l_2(z_1,z_2)=\frac{1}{(1-|z|)(1-|z_2|)^2}$. Hence, we have $N(F,\mathbf{L},\mathbb{B}^n)=0$ and $\ln\max\{|F(z)|\colon z\in T^2(\mathbf{0},R)\}=\frac{1}{(1-r_1)(1-r_2)}$.

6. Bounded L-index in joint variables in a bounded domain. By $\overline{G}$ we denote the closure of a domain G. The following result is generalization of one-dimensional propositions from [26, 35].

Theorem 11. *Let $F(z)$ be an analytic function in $\mathbb{B}^n$, G be a bounded domain in $\mathbb{B}^n$, $d=\inf_{z\in\overline{G}}(1-|z|)>0$ and $\beta>\sqrt{n}$. If for every $j\in\{1,\ldots,n\}$ $l_j\colon\mathbb{B}^n\to\mathbb{R}_+$ is a continuous function satisfying $l_j(z)\geq\frac{\beta}{d}$ for all $z\in\mathbb{B}^n$ then there exists $m\in\mathbb{Z}_+$ such that for all $z\in\overline{G}$ and $J=(j_1,j_2,\ldots,j_n)\in\mathbb{Z}_+^n$*

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)}\leq\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon\ K\in\mathbb{Z}_+^n,\ \|K\|\leq m\right\}, \tag{34}$$

where $\mathbf{L}(z)=(l_1(z),\ldots,l_n(z))$.

Proof. If $F(z)\equiv 0$ then (34) is obvious. Let $F(z)\not\equiv 0$. For every fixed $z^0\in\overline{G}$ $\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(z^0)}$ is the modulus of a coefficient of the power series expansion of the function $F(z)$, $z\in\mathbb{T}^n(z^0,\frac{R_0}{\mathbf{L}(z^0)})$, where $|R_0|=\sqrt{n}$. Since $F(z)$ is analytic, for every $z^0\in\overline{G}$ $\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(z^0)}\to 0$ as $\|J\|\to\infty$, i. e. there exists $m_0=m(z^0)$, for which inequality (34) holds.

Assume on the contrary, that the set of m_0 is not uniformly bounded in $z^0:\sup_{z^0\in\overline{G}}m_0=+\infty$. Hence, for every $m\in\mathbb{Z}_+$ there exist $z_m\in\overline{G}$ and $J^m\in\mathbb{Z}_+^n$

$$\frac{|F^{(J^m)}(z^m)|}{J^m!\mathbf{L}^{J^m}(z_m)}>\max\left\{\frac{|F^{(K)}(z^m)|}{K!\mathbf{L}^K(z^m)}\colon\ K\in\mathbb{Z}_+^n,\ \|K\|\leq m\right\}. \tag{35}$$

Since $z^m\in\overline{G}$, there exists a subsequence $z'^m\to z'\in\overline{G}$ as $m\to+\infty$. By Cauchy's integral formula for any $J\in\mathbb{Z}_+^n$

$$\frac{F^{(J)}(z^0)}{J!}=\frac{1}{(2\pi i)^n}\int\limits_{z\in\mathbb{T}^n(z^0,R)}\frac{F(z)}{(z-z^0)^{J+\mathbf{1}}}dz.$$

We rewrite (35) in the form

$$\max\left\{\frac{|F^{(K)}(z^m)|}{K!\mathbf{L}^K(z^m)}\colon\ K\in\mathbb{Z}_+^n,\ \|K\|\leq m\right\}\leq$$

$$\leq \frac{1}{(2\pi)^n \mathbf{L}^{J^m}(z^m)} \int\limits_{z\in\mathbb{T}^n(z^0,\frac{R}{\mathbf{L}(z^m)})} \frac{|F(z)|}{|z-z^m|^{J^m+\mathbf{1}}}|dz| \leq \frac{1}{R^{J^m}} \max\{|F(z)|\colon z\in G_R\}, \tag{36}$$

where $G_R = \bigcup_{z^*\in\overline{G}} D^n[z^*, \frac{R}{\mathbf{L}(z^*)}]$, $|R|\leq\beta$. We choose R such that $r_j>1$ i. e. $|R|>\sqrt{n}$. Taking the limit in (36) as $m\to\infty$ we deduce

$$\forall K\in\mathbb{Z}^n_+ \quad \frac{|F^{(K)}(z')|}{K!\mathbf{L}^K(z')} \leq \lim_{m\to\infty} \frac{1}{R^{J^m}} \max\{|F(z)|\colon z\in G_R\} = 0.$$

as $m\to+\infty$. Thus, all partial derivatives of the function F at point z' equals 0. By uniqueness theorem $F(z)\equiv 0$. It is impossible. □

Remark 1. A similar proposition for analytic functions in $\mathbb{B}^n$ of bounded L-index in a direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ is valid under the additional assumption $\forall z\in\overline{G}$ $F(z+t\mathbf{b})\not\equiv 0$, where $t\in\mathbb{C}$ (see [4,5]).

7. Exhaustion of unit ball by balls of lesser radii. Denote $\ell(z)=\min_{1\leq j\leq n} l_j(z)$, $\mathcal{L}(z)=\max_{1\leq j\leq n} l_j(z)$. Obviously, that $\ell(z)\leq\mathcal{L}(z)$.

By $Q'(\mathbb{B}^n)$ we denote the class of functions $\mathbf{L}$, which satisfy the condition

$$(\forall r\in[0,\beta],\ j\in\{1,\dots,n\})\colon\ 0<\lambda_{1,j}(r)\leq\lambda_{2,j}(r)<\infty, \tag{37}$$

where

$$\lambda_{1,j}(r) = \inf_{z^0\in\mathbb{B}^n} \inf\left\{\frac{l_j(z)}{l_j(z^0)}\colon z\in\mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\right\}, \tag{38}$$

$$\lambda_{2,j}(r) = \sup_{z^0\in\mathbb{B}^n} \sup\left\{\frac{l_j(z)}{l_j(z^0)}\colon z\in\mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\right\}. \tag{39}$$

$$\Lambda_1(r)=(\lambda_{1,1}(r),\dots,\lambda_{1,n}(r)),\ \Lambda_2(r)=(\lambda_{2,1}(r),\dots,\lambda_{2,n}(r)). \tag{40}$$

These denotations of $\lambda_{1,j}(r)$, $\lambda_{2,j}(r)$, $\Lambda_1(r)$, $\Lambda_2(r)$ are valid in this section only. In other sections their meanings are defined in (4)–(5).

The following theorem is basic in the theory of functions of bounded index. It was necessary to prove more efficient criteria of index boundedness which describe a behavior of maximum modulus on a disc or a behavior of logarithmic derivative (see [3,5,25,29,32,35]). All cited papers used a polydisc exhaustion in $\mathbb{C}^n$ or a disc exhaustion in $\mathbb{C}$.

Theorem 12. *Let $\mathbf{L}\in Q'(\mathbb{B}^n)$. In order that an analytic function F in $\mathbb{B}^n$ be of bounded $\mathbf{L}$-index in joint variables it is necessary that for each $r\in(0,\beta]$ there exist $n_0\in\mathbb{Z}_+$, $p_0>0$ such that for every $z^0\in\mathbb{B}^n$ there exists $K^0\in\mathbb{Z}^n_+$, $\|K^0\|\leq n_0$, satisfying*

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\leq n_0,\ z\in\mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\} \leq p_0\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)} \tag{41}$$

and it is sufficient that for each $r\in(0,\beta]$ there exist $n_0\in\mathbb{Z}_+$, $p_0>0$ such that for every $z^0\in\mathbb{B}^n$ there exists $K^0\in\mathbb{Z}^n_+$, $\|K^0\|\leq n_0$, satisfying

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\leq n_0,\ z\in\mathbb{B}^n\left[z^0, r/\ell(z^0)|\right]\right\} \leq p_0\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \tag{42}$$

Proof. Let F be of bounded $\mathbf{L}$-index in joint variables with $N = N(F, \mathbf{L}, \mathbb{B}^n) < \infty$. For every $r \in (0, \beta]$ we put

$$q = q(r) = [2(N+1)r\sqrt{n}\prod_{j=1}^{n}(\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^{N+1}] + 1,$$

where $[x]$ is integer part of the real number x, i.e. the floor function. For $p \in \{0, \ldots, q\}$ and $z^0 \in \mathbb{B}^n$ we denote

$$S_p(z^0, r) = \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\},$$
$$S_p^*(z^0, r) = \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)} \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\}.$$

Using (4) and $\mathbb{B}^n\left[z^0, \frac{pr}{q\ell(z^0)}\right] \subset \mathbb{B}^n\left[z^0, \frac{r}{\mathcal{L}(z^0)}\right]$, we have

$$S_p(z^0, r) = \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\frac{\mathbf{L}^K(z^0)}{\mathbf{L}^K(z^0)} \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\} \le$$
$$\le S_p^*(z^0, r)\max\left\{\prod_{j=1}^{n}\frac{l_j^N(z^0)}{l_j^N(z)} \colon z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\} \le S_p^*(z^0, r)\prod_{j=1}^{n}(\lambda_{1,j}(r))^{-N}.$$

and, using (5), we obtain

$$S_p^*(z^0, r) = \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\frac{\mathbf{L}^K(z)}{\mathbf{L}^K(z^0)} \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\} \le$$
$$\le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}(\Lambda_2(r))^K \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\} \le S_p(z^0, r)\prod_{j=1}^{n}(\lambda_{2,j}(r))^N. \quad (43)$$

Let $K^{(p)}$ with $\|K^{(p)}\| \le N$ and $z^{(p)} \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]$ be such that

$$S_p^*(z^0, r) = \frac{|F^{(K^{(p)})}(z^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}. \quad (44)$$

Since by the maximum principle $z^{(p)} \in \mathbb{S}_n(z^0, \frac{pr}{q\mathcal{L}(z^0)})$, we have $z^{(p)} \ne z^0$. We choose

$$\widetilde{z}_j^{(p)} = z_j^0 + \frac{p-1}{p}(z_j^{(p)} - z_j^0), \ j \in \{1, \ldots, n\}.$$

Then we have

$$|\widetilde{z}^{(p)} - z^0| = \frac{p-1}{p}|z^{(p)} - z^0| = \frac{p-1}{p}\frac{pr}{q\mathcal{L}(z^0)}, \quad (45)$$
$$|\widetilde{z}^{(p)} - z^{(p)}| = |z^0 + \frac{p-1}{p}(z^{(p)} - z^0) - z^{(p)}| = \frac{1}{p}|z^0 - z^{(p)}| = \frac{1}{p}\frac{pr}{q\ell(z^0)} = \frac{r}{q\mathcal{L}(z^0)}. \quad (46)$$

From (45) we obtain $\widetilde{z}^{(p)} \in \mathbb{B}^n\left[z^0, \frac{(p-1)r}{q\mathcal{L}(z^0)}\right]$ and

$$S_{p-1}^*(z^0, r) \ge \frac{|F^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}.$$

From (44) it follows that

$$0 \le S_p^*(z^0,r) - S_{p-1}^*(z^0,r) \le \frac{|F^{(K^{(p)})}(z^{(p)})| - |F^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)} =$$
$$= \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)} \int_0^1 \frac{d}{dt} |F^{(K^{(p)})}(\widetilde{z}^{(p)} + t(z^{(p)} - \widetilde{z}^{(p)}))| dt \le$$
$$\le \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)} \int_0^1 \sum_{j=1}^n |z_j^{(p)} - \widetilde{z}_j^{(p)}| \left| F^{(K^{(p)}+\mathbf{1}_j)}(\widetilde{z}^{(p)} + t(z^{(p)} - \widetilde{z}^{(p)})) \right| dt =$$
$$= \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)} \sum_{j=1}^n |z_j^{(p)} - \widetilde{z}_j^{(p)}| \left| F^{(K^{(p)}+\mathbf{1}_j)}(\widetilde{z}^{(p)} + t^*(z^{(p)} - \widetilde{z}^{(p)})) \right|, \tag{47}$$

where $0 \le t^* \le 1, \widetilde{z}^{(p)} + t^*(z^{(p)} - \widetilde{z}^{(p)}) \in \mathbb{B}^n(z^0, \frac{pr}{q\mathcal{L}(z^0)})$. For $z \in \mathbb{B}^n(z^0, \frac{pr}{q\mathcal{L}(z^0)})$ and $J \in \mathbb{Z}_+^n$, $\|J\| \le N+1$ we have

$$\frac{|F^{(J)}(z)|\mathbf{L}^J(z)}{J!\mathbf{L}^J(z^0)\mathbf{L}^J(z)} \le (\Lambda_2(r))^J \max\left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le N \right\} \le$$
$$\le \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} \max\left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)} \colon \|K\| \le N \right\} \le$$
$$\le \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} S_p^*(z^0,r).$$

From (47) and (46) we obtain

$$0 \le S_p^*(z^0,r) - S_{p-1}^*(z^0,r) \le$$
$$\le \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} S_p^*(z^0,r) \sum_{j=1}^n (k_j^{(p)}+1) l_j(z^0) |z_j^{(p)} - \widetilde{z}_j^{(p)}| =$$
$$= \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} S_p^*(z^0,r)(N+1) \sum_{j=1}^n l_j(z^0) |z_j^{(p)} - \widetilde{z}_j^{(p)}| \le$$
$$\le \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} (N+1) S_p^*(z^0,R) \sqrt{n} \mathcal{L}(z^0) |z^{(p)} - \widetilde{z}^{(p)}| =$$
$$= \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} \sqrt{n} \frac{(N+1)r}{q(r)} S_p^*(z^0,R) \le \frac{1}{2} S_p^*(z^0,R).$$

This inequality implies $S_p^*(z^0,r) \le 2S_{p-1}^*(z^0,r)$, and in view of inequalities (43) and (44) we have

$$S_p(z^0,r) \le 2\prod_{j=1}^n (\lambda_{1,j}(r))^{-N} S_{p-1}^*(z^0,r) \le 2\prod_{j=1}^n (\lambda_{1,j}(r))^{-N} (\lambda_{2,j}(r))^N S_{p-1}(z^0,r)$$

Therefore,

$$\max\left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right] \right\} = S_q(z^0,r) \le$$

$$\leq 2\prod_{j=1}^{n}(\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^{N}S_{q-1}(z^0,r)\leq\ldots\leq\Big(2\prod_{j=1}^{n}(\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^{N}\Big)^{q}S_0(z^0,r)=$$

$$=\Big(2\prod_{j=1}^{n}(\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^{N}\Big)^{q}\max\left\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^K(z^0)}:\|K\|\leq N\right\}. \qquad (48)$$

From (48) we obtain inequality (41) with $p_0=(2\prod_{j=1}^{n}(\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^{N})^q$ and some K^0 with $\|K^0\|\leq N$. The necessity of condition (41) is proved.

Now we prove the sufficiency. Suppose that for every $r\in(0,\beta]$ there exist $n_0\in\mathbb{Z}_+$, $p_0>1$ such that for all $z_0\in\mathbb{B}^n$ and some $K^0\in\mathbb{Z}_+^n$, $\|K^0\|\leq n_0$, the inequality (42) holds.

We write Cauchy's formula for a ball (see [37, p. 109] or [31, p. 349]) as following $\forall z^0\in\mathbb{B}^n$ $\forall K\in\mathbb{Z}_+^n$ $\forall S\in\mathbb{Z}_+^n$ $\forall z\in\mathbb{B}^n(z^0,r/\ell(z^0))$

$$F^{(K+S)}(z)=\frac{(n+\|S\|-1)!}{(n-1)!}\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|\xi-z^0|(\overline{\xi-z^0})^S F^{(K)}(\xi)}{(|\xi-z^0|^2-\langle z-z^0,\xi-z^0\rangle)^{n+\|S\|}}d\sigma(\xi),$$

where $d\sigma(\xi)$ is the normalized surface measure on $\mathbb{S}_n$, so that $\sigma(\mathbb{S}_n(\mathbf{0},1))=1$. Put $z=z^0:$

$$F^{(K+S)}(z^0)=\frac{(n+\|S\|-1)!}{(n-1)!}\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{(\overline{\xi-z^0})^S F^{(K)}(\xi)}{|\xi-z^0|^{2(n+\|S\|)-1}}d\sigma(\xi) \qquad (49)$$

Therefore, applying (42), we have

$$|F^{(K+S)}(z^0)|\leq\frac{(n+\|S\|-1)!}{(n-1)!}\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S||F^{(K)}(\xi)|}{|\xi-z^0|^{2(n+\|S\|)-1}}d\sigma(\xi)\leq$$

$$\leq\left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1}\frac{(n+\|S\|-1)!}{(n-1)!}\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S||F^{(K)}(\xi)|K!\mathbf{L}^K(\xi)}{K!\mathbf{L}^K(\xi)}d\sigma(\xi)\leq$$

$$\leq p_0\left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1}\frac{(n+\|S\|-1)!}{(n-1)!}\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S||F^{(K^0)}(z^0)|K!\mathbf{L}^K(z)}{K^0!\mathbf{L}^{K^0}(z^0)}d\sigma(\xi)\leq$$

$$\leq p_0\left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1}\frac{(n+\|S\|-1)!}{(n-1)!}\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^{n}\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\times$$

$$\times\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}|(\xi-z^0)^S|d\sigma(\xi)\leq$$

$$\leq p_0\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\frac{(n+\|S\|-1)!}{(n-1)!}\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^{n}\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\times$$

$$\times\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S|}{(r/\ell(z^0))^{\|S\|}}d\sigma\left(\frac{\xi-z^0}{r/\ell(z^0)}\right)\leq$$

$$\leq p_0\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\frac{(n+\|S\|-1)!}{(n-1)!}\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^{n}\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\times$$

$$\times \int\limits_{\mathbb{S}^n(\mathbf{0},1)} |\xi^S| d\sigma(\xi) = p_0 \left(\frac{\ell(z^0)}{r}\right)^{\|S\|} \frac{(n+\|S\|-1)!}{(n-1)!} \times$$

$$\times \frac{|F^{(K^0)}(z^0)| K! \prod_{j=1}^n \lambda_{2,j}^{n_0}(r) \mathbf{L}^K(z^0)}{K^0! \mathbf{L}^{K^0}(z^0)} \frac{\Gamma(n) \prod_{j=1}^n \Gamma(s_j/2+1)}{\Gamma(n+\|S\|/2)}. \quad (50)$$

This implies

$$\frac{|F^{(K+S)}(z^0)|}{(K+S)! \mathbf{L}^{K+S}(z^0)} \le$$

$$\le \frac{|F^{(K^0)}(z^0)|}{K^0! \mathbf{L}^{K^0}(z^0)} p_0 \left(\frac{\ell(z^0)}{r}\right)^{\|S\|} \frac{K! \prod_{j=1}^n \lambda_{2,j}^{n_0}(r)(n+\|S\|-1)! \prod_{j=1}^n \Gamma(s_j/2+1)}{(K+S)! \Gamma(n+\|S\|/2) \mathbf{L}^S(z^0)} \le$$

$$\le \frac{|F^{(K^0)}(z^0)|}{K^0! \mathbf{L}^{K^0}(z^0)} p_0 \frac{K! \prod_{j=1}^n \lambda_{2,j}^{n_0}(r)(n+\|S\|-1)! \prod_{j=1}^n \Gamma(s_j/2+1)}{(K+S)! \Gamma(n+\|S\|/2) r^{\|S\|}} \quad (51)$$

We choose $r > 1$. Since $\|K\| \le n_0$ the quantity $p_0 K! \prod_{j=1}^n \lambda_{2,j}^{n_0}(R)$ does not depend of S. Then there exists n_1 such that

$$\frac{p_0 K! \prod_{j=1}^n \lambda_{2,j}^{n_0}(r)}{r^{\|S\|}} \le 1 \text{ for all } \|S\| \ge n_1. \quad (52)$$

The asymptotic behavior of $\frac{(n+\|S\|-1)! \prod_{j=1}^n \Gamma(s_j/2+1)}{(K+S)! \Gamma(n+\|S\|/2) r^{\|S\|}}$ is more difficult as $\|S\| \to +\infty$. Using the Stirling formula $\Gamma(m+1) = \sqrt{2\pi m} \left(\frac{m}{e}\right)^m (1 + \frac{\theta}{12m})$, where $\theta = \theta(m) \in [0,1]$, we obtain

$$\frac{(n+\|S\|-1)! \prod_{j=1}^n \Gamma(s_j/2+1)}{(K+S)! \Gamma(n+\|S\|/2) r^{\|S\|}} \le \frac{(n+\|S\|-1)! \prod_{j=1}^n \Gamma(s_j/2+1)}{S! \Gamma(n+\|S\|/2) r^{\|S\|}}$$

$$= \frac{\sqrt{2\pi(n+\|S\|-1)} (\frac{n+\|S\|-1}{e})^{n+\|S\|-1} \prod_{j=1}^n \sqrt{2\pi s_j/2} (\frac{s_j}{2e})^{s_j/2}}{\prod_{j=1}^n \sqrt{2\pi s_j} (\frac{s_j}{e})^{s_j} \sqrt{2\pi(n+\|S\|/2-1)} (\frac{n+\|S\|/2-1}{e})^{n+\|S\|/2-1} r^{\|S\|}} \times$$

$$\times \frac{(1 + \frac{\theta(n+\|S\|-1)}{12(n+\|S\|-1)}) \prod_{j=1}^n (1 + \frac{\theta(s_j/2)}{12 s_j/2})}{(1 + \frac{\theta(n+\|S\|/2)}{12(n+\|S\|/2)}) \prod_{j=1}^n (1 + \frac{\theta(s_j)}{12 s_j})}.$$

Denoting

$$\Theta(S) = \frac{(1 + \frac{\theta(n+\|S\|-1)}{12(n+\|S\|-1)}) \prod_{j=1}^n (1 + \frac{\theta(s_j/2)}{12 s_j/2})}{(1 + \frac{\theta(n+\|S\|/2)}{12(n+\|S\|/2)}) \prod_{j=1}^n (1 + \frac{\theta(s_j)}{12 s_j})}$$

and simplifying the previous inequality we deduce

$$\frac{(n+\|S\|-1)! \prod_{j=1}^n \Gamma(s_j/2+1)}{(K+S)! \Gamma(n+\|S\|/2) r^{\|S\|}} \le$$

$$\le \Theta(S) \frac{2^{(1-n)/2} e^{-\|S\|/2}}{r^{\|S\|}} \left(\frac{n-1+\|S\|}{n-1+\|S\|/2}\right)^{n-1+\|S\|/2} \cdot (n-1+\|S\|)^{\|S\|/2} \times$$

$$\times \prod_{j=1}^n (\frac{e}{2s_j})^{s_j/2} \le \Theta(S) \frac{2^{(n-1+\|S\|)/2} e^{-\|S\|/2}}{r^{\|S\|}} (n-1+\|S\|)^{\|S\|/2} \prod_{j=1}^n (\frac{e}{2s_j})^{s_j/2} =$$

$$= \Theta(S)\frac{2^{(n-1)/2}}{r^{\|S\|}}\left(1+\frac{n-1}{\|S\|}\right)^{\frac{\|S\|}{n-1}\cdot\frac{n-1}{2}}\cdot\|S\|^{\|S\|/2}\prod_{j=1}^{n}\frac{1}{s_j^{s_j/2}}\le$$

$$\le \Theta(S)(2e)^{(n-1)/2}\left(\frac{1}{r}\prod_{j=1}^{n}\left(\frac{\|S\|}{s_j}\right)^{\frac{s_j}{2\|S\|}}\right)^{\|S\|}, \ s_j\to\infty. \tag{53}$$

Denote $x_j=\frac{\|S\|}{s_j}\in(1,+\infty)$, $x=(x_1,\dots,x_n)$. Obviously, $\Theta(S)\to 1$ as $s_j\to\infty$, $j\in\{1,\dots,n\}$. Then (53) implies a constrained optimization problem

$$H(x):=\prod_{j=1}^{n}x_j^{1/(2x_j)}\to\max$$

$$\text{subject to } \sum_{j=1}^{n}\frac{1}{x_j}=1,\ x_j\in(1,+\infty). \tag{54}$$

If this problem has a solution, then $H(x)$ is not greater than some H^* and we choose $r>H^*$ in (53).

Let us introduce a Lagrange multiplier λ and study the Lagrange function $\mathcal{L}(x,\lambda)$ defined by

$$\mathcal{L}(x,\lambda)=\prod_{j=1}^{n}x_j^{1/(2x_j)}+\lambda\Big(\sum_{j=1}^{n}\frac{1}{x_j}-1\Big).$$

A necessary condition for optimality in constrained problems yields that

$$\frac{\partial\mathcal{L}}{\partial x_j}=\frac{1-\ln x_j}{2x_j^2}\prod_{k=1}^{n}x_k^{1/(2x_k)}+\lambda(-\frac{1}{x_j^2})=0$$

or

$$\frac{1-\ln x_j}{2}=\lambda/\prod_{k=1}^{n}x_k^{1/(2x_k)}.$$

Hence, $x_j=\exp(1-2\lambda/\prod_{k=1}^{n}x_k^{1/(2x_k)})$, i.e. $x_1=x_2=\ldots=x_n$. Constraint (54) implies that $\sum_{j=1}^{n}\frac{1}{x_j}=\frac{n}{x_1}=1$ or $x_j=n$ for every $j\in\{1,\dots,n\}$. Then $H(x)\le\prod_{j=1}^{n}n^{1/(2n)}=\sqrt{n}$.

We choose $r\ge\sqrt{n}$. For this r we have $\frac{1}{r}\prod_{j=1}^{n}\left(\frac{\|S\|}{s_j}\right)^{\frac{s_j}{2\|S\|}}\le 1$. In view of (53) it means that there exist n_2 such that

$$\frac{(n+\|S\|-1)!\prod_{j=1}^{n}\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}}\le 1 \tag{55}$$

for all $\|S\|\ge n_2$.

The asymptotic behavior of right part (51) in other cases S can be investigated similarly. Taking into account (51), (52) and (55) we have that for all $\|S\|\ge n_1+n_2$

$$\frac{|F^{(K+S)}(z^0)|}{(K+S)!\mathbf{L}^{S+K}(z^0)}\le\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}.$$

This means that for every $J \in \mathbb{Z}^n_+$

$$\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(z^0)} \le \max\left\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^K(z^0)} \colon \|K\| \le n_0+n_1+n_2\right\}$$

where n_0, n_1, n_2 are independent of z_0. Therefore, the function F has bounded $\mathbf{L}$-index in joint variables with $N(F, \mathbf{L}, \mathbb{B}^n) \le n_0 + n_1 + n_2$. □

If we impose additional constraint by the function $\mathbf{L}$ then Theorem 12 implies the following criterion

Theorem 13. *Let $\mathbf{L} \in Q'(\mathbb{B}^n)$ be such that $\sup_{z\in\mathbb{B}^n} \frac{\mathcal{L}(z)}{\ell(z)} = C < \infty$. An analytic function F in $\mathbb{B}^n$ has bounded $\mathbf{L}$-index in joint variables if and only if for each $r \in (0, \beta]$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ such that for every $z^0 \in \mathbb{B}^n$ there exists $K^0 \in \mathbb{Z}^n_+$, $\|K^0\| \le n_0$, such that inequality (42) holds.*

Proof. Sufficiency is proved in Theorem 12. As for necessity we choose $q = q(R) = [2(N + 1)Cr\prod_{j=1}^n(\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^{N+1}]+1$ and replace $\mathcal{L}(z^0)$ by $\ell(z^0)$ in the proof of Theorem 12. No other changes. □

Theorem 14. *Let $\mathbf{L} \in Q'(\mathbb{B}^n)$. In order that an analytic function F in $\mathbb{B}^n$ be of bounded $\mathbf{L}$-index in joint variables it is necessary that for every $r \in (0, \beta]$ $\exists n_0 \in \mathbb{Z}_+$ $\exists p \ge 1$ $\forall z^0 \in \mathbb{B}^n$ $\exists K^0 \in \mathbb{Z}^n_+$, $\|K^0\| \le n_0$, and*

$$\max\left\{|F^{(K^0)}(z)| \colon z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\} \le p|F^{(K^0)}(z^0)| \tag{56}$$

and it is sufficient that for every $r \in (0, \beta]$

$$\exists n_0 \in \mathbb{Z}_+\ \exists p \ge 1\ \forall z^0 \in \mathbb{B}^n \quad \forall j \in \{1, \ldots, n\}\ \exists K^0_j = (0, \ldots, 0, \underbrace{k^0_j}_{j\text{-th place}}, 0, \ldots, 0)$$

such that $k^0_j \le n_0$ and

$$\max\left\{|F^{(K^0_j)}(z)| \colon z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\right\} \le p|F^{(K^0_j)}(z^0)|\ \forall j \in \{1, \ldots, n\}, \tag{57}$$

Proof. Proof of Theorem 12 implies that the inequality (41) is true for some K^0. Therefore, we have

$$\frac{p_0}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} \ge \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z)} \colon z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\} =$$
$$= \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!}\frac{\mathbf{L}^{K^0}(z^0)}{\mathbf{L}^{K^0}(z^0)\mathbf{L}^{K^0}(z)} \colon z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\} \ge$$
$$\ge \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!}\frac{\prod_{j=1}^n(\lambda_{2,j}(r))^{-n_0}}{\mathbf{L}^{K^0}(z^0)} \colon z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\}.$$

This inequality implies

$$\frac{p_0\prod_{j=1}^n(\lambda_{2,j}(r))^{n_0}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} \ge \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z^0)} \colon z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\}. \tag{58}$$

From (58) we obtain inequality (56) with $p = p_0 \prod_{j=1}^{n} (\lambda_{2,j}(r))^{n_0}$. The necessity of condition (56) is proved.

Now we prove the sufficiency of (57). Suppose that for every $r \in (0, \beta]$ $\exists n_0 \in \mathbb{Z}_+$, $p > 1$ such that $\forall z_0 \in \mathbb{B}^n$ and some $K_j^0 \in \mathbb{Z}_+^n$ with $k_j^0 \le n_0$ the inequality (57) holds.

$$\frac{F^{(K_j^0+S)}(z^0)}{S!} = \frac{1}{(2\pi i)^2} \int\limits_{T^n(z^0, R/\mathbf{L}(z^0))} \frac{F^{(K_j^0)}(z)}{(z-z^0)^{S+\mathbf{e}}} dz.$$

In view of (49) we write Cauchy's formula as following $\forall z^0 \in \mathbb{B}^n$ $\forall S \in \mathbb{Z}_+^n$

$$F^{(K_j^0+S)}(z^0) = \frac{(n+\|S\|-1)!}{(n-1)!} \int\limits_{\mathbb{S}^n(z^0, r/\ell(z^0))} \frac{(\overline{\xi - z^0})^S F^{(K_j^0)}(\xi)}{|\xi - z^0|^{2(n+\|S\|)-1}} d\sigma(\xi)$$

As in (50), this yields

$$|F^{(K_j^0+S)}(z^0)| \le$$
$$\le \frac{(n+\|S\|-1)!}{(n-1)!} \left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1} \max\{|F^{(K_j^0)}(z)| \colon z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\} \times$$
$$\times \int\limits_{\mathbb{S}_n(z^0, r/\ell(z^0))} |(\overline{\xi - z^0})^S| d\sigma(\xi) \le \frac{(n+\|S\|-1)!}{(n-1)!} \left(\frac{\ell(z^0)}{r}\right)^{\|S\|} \times$$
$$\times \max\{|F^{(K_j^0)}(z)| \colon z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\} \int\limits_{\mathbb{S}^n(z^0, r/\ell(z^0))} \frac{|(\xi - z^0)^S|}{(r/\ell(z^0))^{\|S\|}} d\sigma\left(\frac{\xi - z^0}{r/\ell(z^0)}\right) \le$$
$$\le \frac{(n+\|S\|-1)!}{(n-1)!} \left(\frac{\ell(z^0)}{r}\right)^{\|S\|} \max\{|F^{(K_j^0)}(z)| \colon z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\} \times$$
$$\times \int\limits_{\mathbb{S}^n(\mathbf{0}, 1)} |\xi^S| d\sigma(\xi) = \frac{(n+\|S\|-1)!}{(n-1)!} \left(\frac{\ell(z^0)}{r}\right)^{\|S\|} \times$$
$$\times \max\{|F^{(K_j^0)}(z)| \colon z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\} \frac{\Gamma(n) \prod_{j=1}^{n} \Gamma(s_j/2+1)}{\Gamma(n+\|S\|/2)}.$$

Now we put $r = \beta$ and use (57)

$$|F^{(K_j^0+S)}(z^0)| \le \left(\frac{\ell(z^0)}{\beta}\right)^{\|S\|} \frac{(n+\|S\|-1)! \prod_{j=1}^{n} \Gamma(s_j/2+1)}{\Gamma(n+\|S\|/2)} \times$$
$$\times \max\{|F^{(K_j^0)}(z)| \colon z \in \mathbb{B}^n\left[z^0, \beta/\ell(z^0)\right]\} \le$$
$$\le p \left(\frac{\ell(z^0)}{\beta}\right)^{\|S\|} \frac{(n+\|S\|-1)! \prod_{j=1}^{n} \Gamma(s_j/2+1)}{\Gamma(n+\|S\|/2)} |F^{(K_j^0)}(z^0)|. \tag{59}$$

Therefore (59) implies for all $j \in \{1, \ldots, n\}$ and $k_j^0 \le n_0$

$$\frac{|F^{(K_j^0+S)}(z^0)|}{\mathbf{L}^{K_j^0+S}(z^0)(K_j^0+S)!} \le p \frac{K_j^0!(n+\|S\|-1)! \prod_{j=1}^{n} \Gamma(s_j/2+1)}{\beta^{\|S\|}(K_j^0+S)!\Gamma(n+\|S\|/2)} \frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!} \le$$

$$\le pn_0!\frac{(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{\beta^{\|S\|}S!\Gamma(n+\|S\|/2)}\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}.$$

In view of (55) there exists n_1 such that for all $\|S\|\ge n_1$

$$\frac{(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{\beta^{\|S\|}S!\Gamma(n+\|S\|/2)}\le 1.$$

Obviously, there exists n_2 such that for all $\|S\|\ge n_2$ $\frac{pn_0!}{\beta^{\|S\|}}\le 1$. Consequently, we have

$$\frac{|F^{(K_j^0+S)}(z^0)|}{\mathbf{L}^{K_j^0+S}(z^0)(K_j^0+S)!}\le\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}\text{ for all }\|S\|\ge n_1+n_2$$

i. e. $N(F,\mathbf{L},\mathbb{B}^n)\le n_0+n_1+n_2$. □

Lemma 2. *Let $\mathbf{L}_1$, $\mathbf{L}_2$ be positive continuous functions in $\mathbb{B}^n$ and for every $z\in\mathbb{B}^n$ $\mathbf{L}_1(z)\le\mathbf{L}_2(z)$. If an analytic function F in $\mathbb{B}^n$ has bounded $\mathbf{L}_1$-index in joint variables then F is of bounded $\mathbf{L}_2$-index in joint variables. If, in addition, for every $z\in\mathbb{B}^n$ $\mathcal{L}_1(z)\le\ell_2(z)$ then $N(F,\mathbf{L}_2,\mathbb{B}^n)\le N(F,\mathbf{L}_1,\mathbb{B}^n)$.*

Proof. Let $N(F,\mathbf{L}_1,\mathbb{B}^n)=n_0$. Using (2) we deduce

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}_2^J(z)}=\frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\frac{|F^{(J)}(z)|}{J!\mathbf{L}_1^J(z)}\le\frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_1^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\le n_0\right\}\le$$
$$\le\frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\max\left\{\frac{\mathbf{L}_2^K(z)}{\mathbf{L}_1^K(z)}\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\le n_0\right\}\le$$
$$\le\max_{\|K\|\le n_0}\left(\frac{\mathbf{L}_1(z)}{\mathbf{L}_2(z)}\right)^{J-K}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\le n_0\right\}. \tag{60}$$

Since $\mathbf{L}_1(z)\le\mathbf{L}_2(z)$ it means that for all $\|J\|\ge nn_0$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}_2^J(z)}\le\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\le n_0\right\}.$$

Thus, F has bounded $\mathbf{L}_2$-index in joint variables.

If, in addition, for every $z\in\mathbb{B}^n$ $\mathcal{L}_1(z)\le\ell_2(z)$ then for all $\|J\|\ge n_0$ (60) yields

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}_2^J(z)}\le\max_{\|K\|\le n_0}\left(\frac{\mathcal{L}_1(z)}{\ell_2(z)}\right)^{\|J-K\|}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\le n_0\right\}\le$$
$$\le\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\le n_0\right\}$$

and $N(F,\mathbf{L}_2,\mathbb{B}^n)\le N(F,\mathbf{L}_1,\mathbb{B}^n)$. □

Denote $\widetilde{\mathbf{L}}(z)=(\widetilde{l}_1(z),\dots,\widetilde{l}_n(z))$. The notation $\mathbf{L}\asymp\widetilde{\mathbf{L}}$ means that there exist $\Theta_1=(\theta_{1,j},\dots,\theta_{1,n})\in\mathbb{R}_+^n$, $\Theta_2=(\theta_{2,j},\dots,\theta_{2,n})\in\mathbb{R}_+^n$ such that $\forall z\in\mathbb{B}^n$ $\theta_{1,j}\widetilde{l}_j(z)\le l_j(z)\le\theta_{2,j}\widetilde{l}_j(z)$ for each $j\in\{1,\dots,n\}$.

Theorem 15. *Let* $\mathbf{L}\in Q'(\mathbb{B}^n)$, $\mathbf{L}\asymp\widetilde{\mathbf{L}}$, $\sup_{z\in\mathbb{B}^n}\frac{\mathcal{L}(z)}{\ell(z)}=C<\infty$, $\min_{1\le j\le n}\theta_{1,j}>\frac{\sqrt{n}}{\beta}$. *An analytic function* F *in* $\mathbb{B}^n$ *has bounded* $\widetilde{\mathbf{L}}$*-index in joint variables if and only if* F *has bounded* $\mathbf{L}$*-index.*

Proof. It is easy to prove that if $\mathbf{L}\in Q'(\mathbb{B}^n)$ and $\mathbf{L}\asymp\widetilde{\mathbf{L}}$ then $\widetilde{\mathbf{L}}\in Q'(\mathbb{B}^n)$ with $\beta'=\beta\min_{1\le j\le n}\theta_{1,j}>\sqrt{n}$ instead of β in (1).

Let $N(F,\widetilde{\mathbf{L}},\mathbb{B}^n)=\widetilde{n}_0<+\infty$. Then by Theorem 12 for every $\widetilde{r}\in(0,\beta)$ there exists $\widetilde{p}\ge 1$ such that for each $z^0\in\mathbb{B}^n$ and some K^0 with $\|K^0\|\le\widetilde{n}_0$, the inequality (41) holds with $\widetilde{\mathbf{L}}$ and $\widetilde{r}$ instead of $\mathbf{L}$ and r. Hence,

$$\frac{\widetilde{p}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)}=\frac{\widetilde{p}}{K^0!}\frac{\Theta_2^{K^0}|F^{(K^0)}(z^0)|}{\Theta_2^{K^0}\mathbf{L}^{K^0}(z^0)}\ge\frac{\widetilde{p}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\Theta_2^{K^0}\widetilde{\mathbf{L}}^{K^0}(z^0)}\ge$$
$$\ge\frac{1}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!\widetilde{\mathbf{L}}^K(z)}\colon \|K\|\le\widetilde{n}_0, z\in\mathbb{B}^n\left[z^0,\widetilde{r}/\widetilde{\mathcal{L}}(z)\right]\right\}\ge$$
$$\ge\frac{1}{\Theta_2^{K^0}}\max\left\{\frac{\Theta_1^K|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\le\widetilde{n}_0, z\in\mathbb{B}^n\left[z^0,\min_{1\le j\le n}\Theta_{1,j}\widetilde{r}/\mathcal{L}(z)\right]\right\}\ge$$
$$\ge\frac{\min\limits_{0\le\|K\|\le n_0}\{\Theta_1^K\}}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\le\widetilde{n}_0, z\in\mathbb{B}^n\left[z^0,\min_{1\le j\le n}\Theta_{1,j}\widetilde{r}/\mathcal{L}(z)\right]\right\}\ge$$
$$\ge\frac{\min\limits_{0\le\|K\|\le n_0}\{\Theta_1^K\}}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\le\widetilde{n}_0, z\in\mathbb{B}^n\left[z^0,\frac{\widetilde{r}\min_{1\le j\le n}\Theta_{1,j}}{C\ell(z)}\right]\right\}.$$

In view of Theorem 12, we obtain that function F has bounded $\mathbf{L}$-index. □

Theorem 16. *Let* $\mathbf{L}\in Q'(\mathbb{B}^n)$, *a function* F *be analytic in* $\mathbb{B}^n$. *If there exist* $r\in(0,\beta]$, $n_0\in\mathbb{Z}_+$, $p_0>1$ *such that for each* $z^0\in\mathbb{B}^n$ *and for some* $K^0\in\mathbb{Z}_+^n$ *with* $\|K^0\|\le n_0$ *the inequality* (42) *holds then* F *has bounded* $\mathbf{L}$*-index in joint variables.*

Proof. The proof of sufficiency in Theorem 12 for $r=\beta$ implies that $N(F,\mathbf{L},\mathbb{B}^n)<+\infty$.

Let $\mathbf{L}^*(z)=\frac{r_0\mathbf{L}(z)}{r}$, $\ell^*(z)=\frac{r_0\ell(z)}{r}$, $r^0=\beta$ and r is radius for which (42) is true. In a general case from validity of (42) for F and $\mathbf{L}$ for $r<\beta$ we obtain

$$\max\left\{\frac{|F^{(K)}(z)|}{K!(\mathbf{L}^*(z))^K}\colon \|K\|\le n_0, z\in\mathbb{B}^n\left[z^0,r_0/\ell^*(z^0)\right]\right\}\le$$
$$\le\max\left\{\frac{|F^{(K)}(z)|}{K!(r_0\mathbf{L}(z)/r)^K}\colon \|K\|\le n_0, z\in\mathbb{B}^n\left[z^0,r_0/(r_0\ell(z^0)/r)\right]\right\}\le$$
$$\le\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\le n_0, z\in\mathbb{B}^n\left[z^0,r/\ell(z^0)\right]\right\}\le$$
$$\le\frac{p_0}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)}=\frac{\beta^{\|K^0\|}p_0}{r^{\|K^0\|}K^0!}\frac{|F^{(K^0)}(z)|}{(r_0\mathbf{L}(z)/r)^{K^0}}=\frac{p_0\beta^{n_0}}{r^{n_0}}\frac{|F^{(K^0)}(z)|}{K^0!(\mathbf{L}^*(z))^{K^0}}.$$

i. e. (41) holds for F, $\mathbf{L}^*$ and $r_0=\beta$. As above now we apply Theorem 12 to the function $F(z)$ and $\mathbf{L}^*(z)=r_0\mathbf{L}(z)/r$. This implies that F is of bounded $\mathbf{L}^*$-index in joint variables. Therefore, by Lemma 2 the function F has bounded $\mathbf{L}$-index in joint variables. □

8. Boundedness of L-index in joint variables of analytic solutions of systems of partial differential equations. Using Theorems 11 and 1 we obtain this corollary.

Corollary 5. *Let $\mathbf{L} \in Q^n$, $F(z)$ be an analytic function in $\mathbb{B}^n$, G be a bounded domain in $\mathbb{B}^n$ such that $d = \inf_{z\in\overline{G}}(1-|z|) > 0$. The function $F(z)$ is of bounded $\mathbf{L}$-index in joint variables and only if there exist $p \in \mathbb{Z}_+$ and $C > 0$ such that for all $z \in \mathbb{B}^n \setminus G$ the inequality (8) holds.*

In one-dimensional case, this corollary was obtained in [33]. It was proposed to apply the corollary in investigation of index boundedness for entire solutions of linear higher order differential equations ([20]).

Let us denote $a^+ = \max\{a, 0\}$, $u_j(t) = u_j(t, R, \Theta) = l_j(\frac{tR}{r^*}e^{i\Theta})$, where $a \in \mathbb{R}$, $t \in [0, r^*]$, $j \in \{1, \dots, n\}$, $r^* = \max_{1\le j\le n} r_j \neq 0$ that is $\frac{t}{r^*}|R| < 1$.

Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable r_k, $k \in \{1, \dots, n\}$, $|R| < 1$, $\Theta \in [0, 2\pi]^n$. By $W(\mathbb{B}^n)$ we denote the class of the functions $\mathbf{L}$ such that

$$r^*(-(u_j(t, R, \Theta))'_{t=r^*})^+/(r_j l_j^2(Re^{i\Theta})) \to 0 \tag{61}$$

uniformly in $\Theta \in [0, 2\pi]^n$, $j \in \{1, \dots, n\}$, as $|R| \to 1-0$, $W := W^1$.

It is easy to check that condition (61) can be replaced by the stronger restriction

$$\frac{\langle z, \nabla l_j(z)\rangle}{|z_j| l_j^2(z)} \to 0$$

as $|z| \to 1-0$.

Lemma 3. *If $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where every $l_j(z)\colon \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function satisfying (1) then*

$$\max_{\Theta\in[0,2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau Re^{i\Theta})\rangle d\tau \to +\infty \text{ as } |R| \to 1-0.$$

Proof. Using (1) we obtain

$$\max_{\Theta\in[0,2\pi]^n} \int_0^{r^*} \sum_{j=1}^n \frac{r_j}{r^*} l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right) d\tau \ge \int_0^{r^*} \sum_{j=1}^n \frac{r_j}{r^*} \frac{\beta}{1-\frac{\tau}{r^*}|R|} d\tau =$$
$$= -\sum_{j=1}^n \frac{r_j\beta}{R} \ln(1-|R|) \to +\infty \text{ as } |R| \to 1-0.$$

□

Lemma 4. *Let $\mathbf{L} \in W(\mathbb{B}^n)$, F be an analytic function in $\mathbb{B}^n$. If there exists $R' \in \mathbb{R}^n_+$, $|R'| < 1$, and $p \in \mathbb{Z}_+$, $c > 0$ such that for all $z \in \mathbb{B}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$ inequality (8) holds then*

$$\overline{\lim_{|R|\to 1-0}} \frac{\ln \max\{|F(z)|\colon z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max\limits_{\Theta\in[0,2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau Re^{i\Theta})\rangle d\tau} \le c. \tag{62}$$

Proof. Let $R \in \mathbb{R}^n_+$ be such that $1 > |R| > |R'|$, $\Theta \in [0, 2\pi]^n$. Denote $\alpha_j = \frac{r_j}{r^*}$, $j \in \{1, \dots, n\}$ and $A = (\alpha_1, \dots, \alpha_n)$. We consider the function

$$g(t) = \max\left\{\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})}\colon \ \|S\| \le p\right\},$$

where $Ate^{i\Theta}=(\alpha_1te^{i\theta_1},\ldots,\alpha_nte^{i\theta_n})$ and $|At|>|R'|$.

Since the function $\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})}$ is continuously differentiable by real $t\in[0,r^*]$, outside the zero set of function $|F^{(S)}(Ate^{i\Theta})|$, the function $g(t)$ is a continuously differentiable function on $[0,r^*]$, except, perhaps, for a countable set of points.

Therefore, using the inequality $\frac{d}{dr}|g(r)|\le|g'(r)|$ which holds except for the points $r=t$ such that $g(t)=0$, we deduce

$$\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})}\right)=\frac{1}{\mathbf{L}^S(Ate^{i\Theta})}\frac{d}{dt}|F^{(S)}(Ate^{i\Theta})|+$$
$$+|F^{(S)}(Ate^{i\Theta})|\frac{d}{dt}\frac{1}{\mathbf{L}^S(Ate^{i\Theta})}\le\frac{1}{\mathbf{L}^S(Ate^{i\Theta})}\left|\sum_{j=1}^nF^{(S+\mathbf{e}_j)}(Ate^{i\Theta})\alpha_je^{i\theta_j}\right|-$$
$$-\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n\frac{s_ju_j'(t)}{l_j(Ate^{i\Theta})}\le\sum_{j=1}^n\frac{|F^{(S+\mathbf{e}_j)}(Ate^{i\Theta})|}{\mathbf{L}^{S+\mathbf{e}_j}(Ate^{i\Theta})}\alpha_jl_j(Ate^{i\Theta})+$$
$$+\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\tag{63}$$

For absolutely continuous functions h_1, h_2, $\ldots$, h_k and $h(x):=\max\{h_j(z)\colon 1\le j\le k\}$, $h'(x)\le\max\{h_j'(x)\colon 1\le j\le k\}$, $x\in[a,b]$ (see [35, Lemma 4.1, p. 81]). The function g is absolutely continuous, therefore, from (8) and (63) it follows that

$$g'(t)\le\max\left\{\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})}\right)\colon\|S\|\le p\right\}\le$$
$$\le\max_{\|S\|\le p}\left\{\sum_{j=1}^n\frac{\alpha_jl_j(Ate^{i\Theta})|F^{(S+\mathbf{e}_j)}(Ate^{i\Theta})|}{\mathbf{L}^{S+\mathbf{e}_j}(Ate^{i\Theta})}+\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}\le$$
$$\le g(t)\left(\max\{1,c\}\sum_{j=1}^n\alpha_jl_j(Ate^{i\Theta})+\max_{\|S\|\le p}\left\{\sum_{j=1}^n\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}\right)=g(t)(\beta(t)+\gamma(t)),$$

where

$$\beta(t)=\max\{1,c\}\sum_{j=1}^n\alpha_jl_j(Ate^{i\Theta}),\ \gamma(t)=\max_{\|S\|\le p}\left\{\sum_{j=1}^n\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}.$$

Thus, $\frac{d}{dt}\ln g(t)\le\beta(t)+\gamma(t)$ and

$$g(t)\le g(t_0)\exp\int_{t_0}^t(\beta(\tau)+\gamma(\tau))d\tau,$$

where t_0 is chosen such that $g(t_0)\ne0$. The condition $\mathbf{L}\in W(\mathbb{B}^n)$ gives

$$\frac{\gamma(t)}{\beta(t)}=\frac{\sum_{j=1}^n\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}}{c\sum_{j=1}^n\alpha_jl_j(Ate^{i\Theta})}\le p\sum_{j=1}^n\frac{(-u_j'(t))^+}{\alpha_jl_j^2(Ate^{i\Theta})}\le p\varepsilon,$$

where $\varepsilon=\varepsilon(R)\to0$ uniformly in $\Theta\in[0,2\pi]^n$, $t=r^*$ as $|R|\to1-0$.

But $|F(Ate^{i\Theta})| \le g(t) \le g(t_0)\exp\int\limits_{t_0}^{t}(\beta(\tau)+\gamma(\tau))d\tau$ and $r^*A=R$. Then we put $t=r^*$ and obtain

$$\ln\max\{|F(z)|\colon\ z\in\mathbb{T}^n(\mathbf{0},R)\} = \ln\max_{\Theta\in[0,2\pi]^n}|F(Re^{i\Theta})| \le \ln\max_{\Theta\in[0,2\pi]^n} g(r^*) \le$$

$$\le \ln g(t_0) + \max_{\Theta\in[0,2\pi]^n}\int\limits_{t_0}^{r^*}(\beta(\tau)+\gamma(\tau))d\tau \le$$

$$\le \ln g(t_0) + \max_{\Theta\in[0,2\pi]^n}\int\limits_{t_0}^{r^*} c\sum_{j=1}^{n}\alpha_j l_j(A\tau e^{i\Theta})\,(1+p\varepsilon)\,d\tau =$$

$$= \ln g(t_0) + \max\{1,c\}\max_{\Theta\in[0,2\pi]^n}\int\limits_{t_0}^{r^*}\sum_{j=1}^{n}\frac{r_j}{r^*}l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)(1+p\varepsilon)\,d\tau.$$

This implies (62). □

Lemma 5. *Let* $\mathbf{L}\in W(\mathbb{B}^n)$, *$F$ be an analytic function in* $\mathbb{B}^n$. *If there exists* $R'\in\mathbb{R}^n_+$, $|R'|<1$ *and* $p\in\mathbb{Z}_+$, $c>0$ *such that for all* $z\in\mathbb{B}^n\setminus\mathbb{D}^n(\mathbf{0},R')$ *inequality*

$$\max\left\{\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)}\colon \|J\|=p+1\right\} \le c\cdot\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\le p\right\} \tag{64}$$

holds then

$$\varlimsup_{|R|\to 1-0}\frac{\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^n}\int\limits_0^1\langle R,\mathbf{L}(\tau Re^{i\Theta})\rangle d\tau} \le (p+1)\max\{1,c\}. \tag{65}$$

Proof. The proof of Lemma 5 is similar to that of Lemma 4.

Let $R\in\mathbb{R}^n_+$ be such that $1>|R|>|R'|$, $\Theta\in[0,2\pi]^n$. Denote $\alpha_j=\frac{r_j}{r^*}$, $j\in\{1,\ldots,n\}$ and $A=(\alpha_1,\ldots,\alpha_n)$. We consider the function

$$g(t)=\max\left\{\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\colon\ \|S\|\le p\right\},$$

where $At=(\alpha_1 t,\ldots,\alpha_n t)$, $Ate^{i\Theta}=(\alpha_1 te^{i\theta_1},\ldots,\alpha_n te^{i\theta_n})$ and $|At|>|R'|$.

As above the function $\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}$ is continuously differentiable by real $t\in[0,r^*]$, outside the zero set of the function $|F^{(S)}(Ate^{i\Theta})|$, the function $g(t)$ is a continuously differentiable function on $[0,r^*]$, except, perhaps, for a countable set of points.

Therefore, using the inequality $\frac{d}{dr}|g(r)|\le|g'(r)|$ which holds except for the points $r=t$ such that $g(t)=0$, we deduce

$$\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\right) = \frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\frac{d}{dt}|F^{(S)}(Ate^{i\Theta})|+$$

$$+|F^{(S)}(Ate^{i\Theta})|\frac{d}{dt}\frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})} \le \frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\left|\sum_{j=1}^{n}F^{(S+\mathbf{e}_j)}(Ate^{i\Theta})\alpha_j e^{i\theta_j}\right| -$$

$$-\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n \frac{s_j u_j'(t)}{l_j(Ate^{i\Theta})} \leq \sum_{j=1}^n \frac{|F^{(S+\mathbf{e}_j)}(Ate^{i\Theta})|}{(S+\mathbf{e}_j)!\mathbf{L}^{S+\mathbf{e}_j}(Ate^{i\Theta})}\alpha_j(s_j+1)l_j(Ate^{i\Theta})+ \\ +\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n \frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}. \tag{66}$$

For absolutely continuous functions h_1, h_2, $\ldots$, h_k and $h(x) := \max\{h_j(z)\colon 1 \leq j \leq k\}$, $h'(x) \leq \max\{h_j'(x)\colon 1 \leq j \leq k\}$, $x \in [a,b]$ (see [35, Lemma 4.1, p. 81]). The function g is absolutely continuous. Therefore, (8) and (66) yield

$$g'(t) \leq \max\left\{\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\right) : \|S\| \leq N\right\} \leq \\ \leq \max_{\|S\|\leq p}\left\{\sum_{j=1}^n \frac{\alpha_j(s_j+1)l_j(Ate^{i\Theta})|F^{(S+\mathbf{e}_j)}(Ate^{i\Theta})|}{(S+\mathbf{e}_j)!\mathbf{L}^{S+\mathbf{e}_j}(Ate^{i\Theta})} + \frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n \frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\} \leq \\ \leq g(t)\left(\max\{1,c\}\max_{\|S\|\leq p}\left\{\sum_{j=1}^n \alpha_j(s_j+1)l_j(Ate^{i\Theta})\right\}+ \right. \\ \left. + \max_{\|S\|\leq p}\left\{\sum_{j=1}^n \frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}\right) = g(t)(\beta(t)+\gamma(t)),$$

where

$$\beta(t) = \max\{1,c\}\max_{\|S\|\leq p}\left\{\sum_{j=1}^n \alpha_j(s_j+1)l_j(Ate^{i\Theta})\right\}, \quad \gamma(t) = \max_{\|S\|\leq p}\left\{\sum_{j=1}^n \frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}.$$

Thus, $\frac{d}{dt}\ln g(t) \leq \beta(t)+\gamma(t)$ and

$$g(t) \leq g(t_0)\exp\int_{t_0}^{t}(\beta(\tau)+\gamma(\tau))d\tau,$$

where t_0 is chosen such that $g(t_0) \neq 0$. Denote $\widetilde{\beta}(t) = \sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})$. Since $\mathbf{L} \in W(\mathbb{B}^n)$, for some S^*, $\|S^*\| \leq p$ and $\widetilde{S}$, $\|\widetilde{S}\| \leq p$, we obtain

$$\frac{\gamma(t)}{\widetilde{\beta}(t)} = \frac{\sum_{j=1}^n \frac{s_j^*(-u_j'(t))^+}{l_j(Ate^{i\Theta})}}{\sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})} \leq \sum_{j=1}^n s_j^* \frac{(-u_j'(t))^+}{\alpha_j l_j^2(Ate^{i\Theta})} \leq p\sum_{j=1}^n \frac{(-u_j'(t))^+}{\alpha_j l_j^2(Ate^{i\Theta})} \leq p\varepsilon,$$

$$\frac{\beta(t)}{\widetilde{\beta}(t)} = \frac{\max\{1,c\}\sum_{j=1}^n \alpha_j(\tilde{s}_j+1)l_j(Ate^{i\Theta})}{\sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})} = \\ = \max\{1,c\} + \max\{1,c\}\frac{\sum_{j=1}^n \alpha_j\tilde{s}_j l_j(Ate^{i\Theta})}{\sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})} \leq \\ \leq \max\{1,c\} + \max\{1,c\}\sum_{j=1}^n \tilde{s}_j \leq \max\{1,c\}(1+p),$$

where $\varepsilon = \varepsilon(R) \to 0$ uniformly in $\Theta \in [0,2\pi]^n$, $t = r^*$ as $|R| \to 1-0$

But $|F(Ate^{i\Theta})| \leq g(t) \leq g(t_0)\exp\int\limits_{t_0}^{t}(\beta(\tau)+\gamma(\tau))d\tau$ and $r^*A = R$. Then we put $t = r^*$ and obtain

$$\ln\max\{|F(z)\colon\ z\in\mathbb{T}^n(\mathbf{0},R)\} = \ln\max_{\Theta\in[0,2\pi]^n}|F(Re^{i\Theta})| \leq \ln\max_{\Theta\in[0,2\pi]^n} g(r^*) \leq$$
$$\leq \ln g(t_0) + \max_{\Theta\in[0,2\pi]^n}\int\limits_{t_0}^{r^*}(\beta(\tau)+\gamma(\tau))d\tau \leq$$
$$\leq \ln g(t_0) + \max_{\Theta\in[0,2\pi]^n}\int\limits_{t_0}^{r^*}\sum_{j=1}^{n}\alpha_j l_j(A\tau e^{i\Theta})\left(\max\{1,c\}(1+p)+p\varepsilon\right)d\tau =$$
$$= \ln g(t_0) + \max_{\Theta\in[0,2\pi]^n}\int\limits_{t_0}^{r^*}\sum_{j=1}^{n}\frac{r_j}{r^*}l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)\left(\max\{1,c\}(1+p)+p\varepsilon\right)d\tau.$$

This implies (65). □

Using proved lemmas we will formulate and prove propositions that provide growth estimates for analytic solutions of the following system of partial differential equations:

$$G_{p_j\mathbf{e}_j}(z)F^{(p_j\mathbf{e}_j)}(z) + \sum_{\|S_j\|\leq p_j-1} G_{S_j}(z)F^{(S_j)}(z) = H_j(z),\ j\in\{1,\ldots,n\} \tag{67}$$

$p_j\in\mathbb{N}$, $S_j\in\mathbb{Z}^n_+$, H_j and G_{S_j} are analytic functions in $\mathbb{B}^n$. Denote $QW(\mathbb{B}^n) = Q(\mathbb{B}^n)\cap W(\mathbb{B}^n)$. Accordingly, $QW(\mathbb{D}) = Q(\mathbb{D})\cap W(\mathbb{D})$.

We will say that non-homogeneous system of PDE's (67) belongs to class $\mathcal{A}(\mathbf{G},\mathbf{H},\mathbf{L})$, if $\mathbf{L}\in QW(\mathbb{B}^n)$, for all $z\in\mathbb{B}^n$ and for every $j\in\{1,\ldots,n\}$ the analytic functions H_j and G_{S_j} in $\mathbb{B}^n$ satisfy the following conditions:

1) for every $\|S_j\|\leq p_j-1$ and for each $M\in\mathbb{Z}^n_+$,

$$\|M\|\leq 1+\sum_{\substack{k=1\\k\neq j}}^{n}p_k,\ |G^{(M)}_{S_j}(z)|\mathbf{L}^{S_j-M}(z)\leq B_{S_j,M}l_j^{p_j}(z)|G_{p_j\mathbf{1}_j}(z)|,$$
$$|G^{(M)}_{p_j\mathbf{1}_j}(z)|\leq B_{p_j\mathbf{1}_j,M}\mathbf{L}^M(z)|G_{p_j\mathbf{1}_j}(z)|,$$

2) for every $I\in\mathbb{Z}^n_+$,

$$\|I\| = 1+\sum_{\substack{k=1\\k\neq j}}^{n}p_k\quad |H^{(I)}_j(z)|\leq D_{I,j}\mathbf{L}^I(z)|H_j(z)|,$$

3) $G_{p_j\mathbf{1}_j}(z)\neq 0$, where $B_{S_j,M}$, $D_{I,j}$, $B_{p_j\mathbf{1}_j,M}$ are positive constants, $\mathbf{H}(z) = (H_1(z),\ldots, H_n(z))$, $\mathbf{G}(z)$ is a matrix consisting of coefficients $G_{S_J}(z)$ of system (67).

Homogeneous system of PDE's (67) belongs to class $\mathcal{A}(\mathbf{G},\mathbf{0},\mathbf{L})$, if condition 1) holds for $M\in\mathbb{Z}^n_+$, such that $\|M\|\leq\sum_{\substack{k=1\\k\neq j}}^{n}p_k$ and $G_{p_j\mathbf{1}_j}(z)\neq 0$. Conditon 2) is not required in this case.

Instead of $G_{p_j\mathbf{1}_j}(z)\neq 0$ we can require validity of conditions 1) and 2) for all $z\in\mathbb{B}^n\setminus\mathbb{D}^n(\mathbf{0},R')$. It is possible in view of Theorem 11. If for some $M\in\mathbb{Z}^n_+$ $G^{(M)}_{S_j}(z)\equiv 0$ or $H^{(M)}_j(z)\equiv 0$ then we suppose that $B_{S_j,M}=0$ or $D_{M,j}=0$, respectively.

Theorem 17. *If non-homogeneous system of PDE's* (67) *belongs to class* $\mathcal{A}(\mathbf{G},\mathbf{H},\mathbf{L})$ *and an analytic function* $F(z)$ *in* $\mathbb{B}^n$ *satisfies* (67) *then* F *has bounded* **L**-*index in joint variables and*

$$\varlimsup_{|R|\to 1-0}\frac{\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^n}\int\limits_0^1\langle R,\mathbf{L}(\tau Re^{i\Theta})\rangle d\tau}\leq\max\{1,c\},\tag{68}$$

where c *is defined in* (73).

Proof. Taking into account that the function $F(z)$ satisfies system (67), we calculate the partial derivative $I\in\mathbb{Z}^n_+$ in each equation of the system

$$\sum_{\mathbf{0}\leq M\leq I}C_I^M G^{(M)}_{p_j\mathbf{e}_j}(z)F^{(p_j\mathbf{e}_j+I-M)}(z)+$$
$$+\sum_{\mathbf{0}\leq M\leq I}C_I^M\sum_{\|S_j\|\leq p_j-1}G^{(M)}_{S_j}(z)F^{(S_j+I-M)}(z)=H^{(I)}_j(z),\tag{69}$$

where $C_I^M=\frac{i_1!\ldots i_n!}{m_1!(i_1-m_1)!\ldots m_n!(i_n-m_n)!}$ and

$$\|I\|=1-p_j+\sum_{k=1}^n p_k=1+\sum_{\substack{k=1\\k\neq j}}^n p_k.$$

Using the second condition of the class definition $\mathcal{A}(\mathbf{G},\mathbf{H},\mathbf{L})$, we obtain

$$|H^{(I)}_j(z)|\leq D_{I,j}\mathbf{L}^I(z)|H_j(z)|\leq$$
$$\leq D_{I,j}\mathbf{L}^I(z)\Big(|G_{p_j\mathbf{e}_j}(z)||F^{(p_j\mathbf{e}_j)}(z)|+\sum_{\|S_j\|\leq p_j-1}|G_{S_j}(z)||F^{(S_j)}(z)|\Big).\tag{70}$$

Equation (69) yields

$$F^{(p_j\mathbf{e}_j+I)}(z)=\frac{1}{G_{p_j\mathbf{e}_j}(z)}\Big(H^{(I)}_j(z)-\sum_{\substack{\mathbf{0}\leq M\leq I\\M\neq\mathbf{0}}}C_I^M G^{(M)}_{p_j\mathbf{e}_j}(z)F^{(p_j\mathbf{e}_j+I-M)}(z)-$$
$$-\sum_{\mathbf{0}\leq M\leq I}C_I^M\sum_{\|S_j\|\leq p_j-1}G^{(M)}_{S_j}(z)F^{(S_j+I-M)}(z)\Big).\tag{71}$$

From (71) and the first condition it follows

$$\left|F^{(p_j\mathbf{e}_j+I)}(z)\right|=\frac{1}{|G_{p_j\mathbf{e}_j}(z)|}\left(D_{I,j}\mathbf{L}^I(z)\left(|G_{p_j\mathbf{e}_j}(z)||F^{(p_j\mathbf{e}_j)}(z)|+\sum_{\|S_j\|\leq p_j-1}|G_{S_j}(z)||F^{(S_j)}(z)|\right)+\right.$$
$$\left.+\sum_{\substack{\mathbf{0}\leq M\leq I\\M\neq\mathbf{0}}}C_I^M|G^{(M)}_{p_j\mathbf{e}_j}(z)||F^{(p_j\mathbf{e}_j+I-M)}(z)|+\sum_{\mathbf{0}\leq M\leq I}C_I^M\sum_{\|S_j\|\leq p_j-1}|G^{(M)}_{S_j}(z)||F^{(S_j+I-M)}(z)|\right)\leq$$

$$\leq D_{I,j}\mathbf{L}^I(z)\left(|F^{(p_j\mathbf{e}_j)}(z)| + \sum_{\|S_j\|\leq p_j-1} B_{S_j,\mathbf{0}} L^{p_j\mathbf{e}_j-S_j}(z)|F^{(S_j)}(z)|\right) +$$
$$+\sum_{\substack{\mathbf{0}\leq M\leq I\\ M\neq\mathbf{0}}} C_I^M B_{p_j\mathbf{e}_j,M}\mathbf{L}^M(z)|F^{(p_j\mathbf{e}_j+I-M)}(z)| +$$
$$+\sum_{\mathbf{0}\leq M\leq I} C_I^M \sum_{\|S_j\|\leq p_j-1} B_{S_j,M}\mathbf{L}^{p_j\mathbf{e}_j-S_j+M}(z)|F^{(S_j+I-M)}(z)|. \tag{72}$$

Dividing this inequality by $L^{p_j\mathbf{e}_j+I}(z)$, we obtain that for every

$$\|I\| = 1 + \sum_{\substack{k=1\\ k\neq j}}^n p_k$$

and $j \in \{1,\ldots,n\}$

$$\frac{\left|F^{(p_j\mathbf{e}_j+I)}(z)\right|}{\mathbf{L}^{p_j\mathbf{e}_j+I}(z)} \leq D_{I,j}\left(\frac{|F^{(p_j\mathbf{e}_j)}(z)|}{\mathbf{L}^{p_j\mathbf{e}_j}(z)} + \sum_{\|S_j\|\leq p_j-1} B_{S_j,\mathbf{0}}\frac{|F^{(S_j)}(z)|}{\mathbf{L}^{S_j}(z)}\right) +$$
$$+\sum_{\substack{\mathbf{0}\leq M\leq I\\ M\neq\mathbf{0}}} C_I^M B_{p_j\mathbf{e}_j,M}\frac{|F^{(p_j\mathbf{e}_j+I-M)}(z)|}{\mathbf{L}^{p_j\mathbf{e}_j+I-M}(z)} + \sum_{\mathbf{0}\leq M\leq I} C_I^M \sum_{\|S_j\|\leq p_j-1} B_{S_j,M}\frac{|F^{(S_j+I-M)}(z)|}{\mathbf{L}^{S_j+I-M}(z)} \leq$$
$$\leq \left(D_{I,j}(1+\sum_{\|S_j\|\leq p_j-1} B_{S_j,\mathbf{0}}) + \sum_{\substack{\mathbf{0}\leq M\leq I\\ M\neq\mathbf{0}}} C_I^M B_{p_j\mathbf{e}_j,M} + \sum_{\mathbf{0}\leq M\leq I} C_I^M \sum_{\|S_j\|\leq p_j-1} B_{S_j,M}\right)\times$$
$$\times \max\left\{\frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)} \colon \|S\| \leq \sum_{j=1}^n p_j\right\}.$$

Obviously, $\|p_j\mathbf{e}_j + I\| = 1 + \sum_{j=1}^n p_j$. This implies

$$\max\left\{\frac{\left|F^{(K)}(z)\right|}{\mathbf{L}^K(z)} \colon \|K\| = 1 + \sum_{j=1}^n p_j\right\} \leq \max\{1,c\}\cdot\max\left\{\frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)} \colon \|S\| \leq \sum_{j=1}^n p_j\right\},$$

where

$$c = \max_{\substack{\|I\|=1-p_j+\sum_{k=1}^n p_k,\\ j\in\{1,\ldots,n\}}} \left(D_{I,j}(1+\sum_{\|S_j\|\leq p_j-1} B_{S_j,\mathbf{0}}) + \sum_{\substack{\mathbf{0}\leq M\leq I\\ M\neq\mathbf{0}}} C_I^M B_{p_j\mathbf{e}_j,M} + \right.$$
$$\left. + \sum_{\mathbf{0}\leq M\leq I} C_I^M \sum_{\|S_j\|\leq p_j-1} B_{S_j,M}\right) \tag{73}$$

for all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$.

Thus, by Lemma 4 estimate (68) holds, and by Corollary 5 the analytic function F in $\mathbb{B}^n$ has bounded $\mathbf{L}$-index in joint variables. □

If system (67) is homogeneous ($H_j(z)\equiv 0$), the previous theorem can be simplified.

Theorem 18. *If homogeneous system of PDE's* (67) *belongs to class* $\mathcal{A}(\mathbf{G},\mathbf{0},\mathbf{L})$ *and an analytic function* F *in* $\mathbb{B}^n$ *is a solution of the system then* F *has bounded* **L**-*index in joint variables and*

$$\varlimsup_{|R|\to 1-0}\frac{\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^n}\int\limits_0^1\langle R,\mathbf{L}(\tau Re^{i\Theta})\rangle d\tau}\le\max\{1,c\}, \tag{74}$$

where c *is defined in* (73) *with* $D_{I,j}=0$ *and* $\|I\|=-p_j+\sum_{k=1}^n p_k$ *instead of* $\|I\|=1-p_j+\sum_{k=1}^n p_k$.

Proof. If $H_j(z)\equiv 0$ then (71) implies

$$F^{(p_j\mathbf{e}_j+I)}(z)=\frac{1}{G_{p_j\mathbf{e}_j}(z)}\Bigg(-\sum_{\substack{\mathbf{0}\le M\le I\\ M\neq\mathbf{0}}}C_I^M G_{p_j\mathbf{e}_j}^{(M)}(z)F^{(p_j\mathbf{e}_j+I-M)}(z)-$$
$$-\sum_{\mathbf{0}\le M\le I}C_I^M\sum_{\|S_j\|\le p_j-1}G_{S_j}^{(M)}(z)F^{(S_j+I-M)}(z)\Bigg). \tag{75}$$

Hence, we obtain

$$|F^{(p_j\mathbf{e}_j+I)}(z)|\le\frac{1}{|G_{p_j\mathbf{e}_j}(z)|}\Bigg(\sum_{\substack{\mathbf{0}\le M\le I\\ M\neq\mathbf{0}}}C_I^M|G_{p_j\mathbf{e}_j}^{(M)}(z)||F^{(p_j\mathbf{e}_j+I-M)}(z)|+$$
$$+\sum_{\mathbf{0}\le M\le I}C_I^M\sum_{\|S_j\|\le p_j-1}|G_{S_j}^{(M)}(z)||F^{(S_j+I-M)}(z)|\Bigg).$$

Dividing the obtained inequality by $\mathbf{L}^{p_j\mathbf{e}_j+I}(z)$ and using assumptions of the theorem on the functions G_{S_j}, we deduce

$$\frac{|F^{(p_j\mathbf{e}_j+I)}(z)|}{\mathbf{L}^{p_j\mathbf{e}_j+I}(z)}\le\frac{1}{|G_{p_j\mathbf{e}_j}(z)|\mathbf{L}^{p_j\mathbf{e}_j+I}(z)}\Bigg(\sum_{\substack{\mathbf{0}\le M\le I\\ M\neq\mathbf{0}}}C_I^M B_{p_j\mathbf{e}_j,M}\mathbf{L}^M(z)|G_{p_j\mathbf{e}_j}(z)||F^{(p_j\mathbf{e}_j+I-M)}(z)|+$$
$$+\sum_{\mathbf{0}\le M\le I}C_I^M\sum_{\|S_j\|\le p_j-1}B_{S_j,M}\mathbf{L}^{p_j\mathbf{e}_j-S_j+M}(z)|G_{p_j\mathbf{e}_j}(z)||F^{(S_j+I-M)}(z)|\Bigg)=$$
$$=\sum_{\substack{\mathbf{0}\le M\le I\\ M\neq\mathbf{0}}}C_I^M B_{p_j\mathbf{e}_j,M}\frac{|F^{(p_j\mathbf{e}_j+I-M)}(z)|}{\mathbf{L}^{p_j\mathbf{e}_j+I-M}(z)}+\sum_{\mathbf{0}\le M\le I}C_I^M\sum_{\|S_j\|\le p_j-1}B_{S_j,M}\frac{|F^{(S_j+I-M)}(z)|}{\mathbf{L}^{S_j+I-M}(z)}\le$$
$$\le\Bigg(\sum_{\substack{\mathbf{0}\le M\le I\\ M\neq\mathbf{0}}}C_I^M B_{p_j\mathbf{e}_j,M}+\sum_{\mathbf{0}\le M\le I}C_I^M\sum_{\|S_j\|\le p_j-1}B_{S_j,M}\Bigg)\times$$

$$\times\max\left\{\frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)}\colon \|S\|\le -1+\sum_{j=1}^n p_j\right\}.$$

Obviously, $\|p_j\mathbf{e}_j+I\|=\sum_{j=1}^n p_j$. Therefore,

$$\max\left\{\frac{\left|F^{(K)}(z)\right|}{\mathbf{L}^K(z)}\colon \|K\|=\sum_{j=1}^n p_j\right\}\le \max\{1,c\}\cdot\max\left\{\frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)}\colon \|S\|\le -1+\sum_{j=1}^n p_j\right\},$$

where

$$c=\max_{\substack{\|I\|=-p_j+\sum_{k=1}^n p_k,\\ j\in\{1,\ldots,n\}}}\left(\sum_{\substack{\mathbf{0}\le M\le I\\ M\ne\mathbf{0}}} C_I^M B_{p_j\mathbf{e}_j,M}+\sum_{\mathbf{0}\le M\le I} C_I^M\sum_{\|S_j\|\le p_j-1} B_{S_j,M}\right)$$

for all $z\in\mathbb{B}^n\setminus\mathbb{D}^n(\mathbf{0},R')$.

Thus, all conditions of Corollary 5 are satisfying. Hence, the function F has bounded $\mathbf{L}$-index in joint variables and by Lemma 4 estimate (74) holds. □

Note that estimate (68) and (74) cannot be improved (see examples for $n=1$ in [20]).

Moreover, using Corollary 5 and Lemma 5 we can supplement two previous Theorems 17 and 18 with propositions that contain estimates of $\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}$, which can sometimes be better than (74) and (68). Two following theorems have proofs that of to Theorems 17 and 18.

Theorem 19. *If non-homogeneous system of PDE's* (67) *belongs to class* $\mathcal{A}(\mathbf{G},\mathbf{H},\mathbf{L})$ *and an analytic function* $F(z)$ *in* $\mathbb{B}^n$ *satisfies* (67) *then* F *has bounded* $\mathbf{L}$*-index in joint variables and*

$$\overline{\lim_{|R|\to 1-0}}\frac{\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^n}\int_0^1\langle R,\mathbf{L}(\tau Re^{i\Theta})\rangle d\tau}\le \max\{1,c\},\tag{76}$$

where c' *is defined in* (77).

Proof. As in proof of Theorem 17, dividing (72) by $(p_j\mathbf{e}_j+I)!L^{p_j\mathbf{e}_j+I}(z)$, we obtain that for every $\|I\|=1+\sum_{\substack{k=1\\k\ne j}}^n p_k$ and $j\in\{1,\ldots,n\}$

$$\frac{|F^{(p_j\mathbf{e}_j+I)}(z)|}{(p_j\mathbf{e}_j+I)!\mathbf{L}^{p_j\mathbf{e}_j+I}(z)}\le D_{I,j}\left(\frac{|F^{(p_j\mathbf{e}_j)}(z)|}{(p_j\mathbf{e}_j+I)!\mathbf{L}^{p_j\mathbf{e}_j}(z)}+\sum_{\|S_j\|\le p_j-1}B_{S_j,\mathbf{0}}\frac{|F^{(S_j)}(z)|}{(p_j\mathbf{e}_j+I)!\mathbf{L}^{p_j\mathbf{e}_j-S_j}(z)}\right)+$$

$$+\sum_{\substack{\mathbf{0}\le M\le I\\ M\ne\mathbf{0}}} C_I^M B_{p_j\mathbf{e}_j,M}\frac{|F^{(p_j\mathbf{e}_j+I-M)}(z)|}{(p_j\mathbf{e}_j+I)!\mathbf{L}^{p_j\mathbf{e}_j+I-M}(z)}+$$

$$+\sum_{\mathbf{0}\le M\le I} C_I^M\sum_{\|S_j\|\le p_j-1} B_{S_j,M}\frac{|F^{(S_j+I-M)}(z)|}{(p_j\mathbf{e}_j+I)!\mathbf{L}^{S_j+I-M}(z)}\le$$

$$\le D_{I,j}\left(\frac{|F^{(p_j\mathbf{e}_j)}(z)|}{(p_j\mathbf{e}_j+I)!\mathbf{L}^{p_j\mathbf{e}_j}(z)}+B\sum_{\|S_j\|\le p_j-1}\frac{|F^{(S_j)}(z)|}{(p_j\mathbf{e}_j+I)!\mathbf{L}^{p_j\mathbf{e}_j-S_j}(z)}\right)+$$

$$+B\sum_{\substack{\mathbf{0}\le M\le I\\ M\ne\mathbf{0}}} C_I^M \frac{|F^{(p_j\mathbf{e}_j+I-M)}(z)|}{(p_j\mathbf{e}_j+I)!\mathbf{L}^{p_j\mathbf{e}_j+I-M}(z)}+$$

$$+B\sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1}\frac{|F^{(S_j+I-M)}(z)|}{(p_j\mathbf{e}_j+I)!\mathbf{L}^{S_j+I-M}(z)}\le$$

$$\le\Bigg(D_{I,j}\Bigg(\frac{p_j!}{(p_j\mathbf{e}_j+I)!}+B\sum_{\|S_j\|\le p_j-1}\frac{(p_j\mathbf{e}_j-S_j)!}{(p_j\mathbf{e}_j+I)!}\Bigg)+$$

$$+B\sum_{\substack{\mathbf{0}\le M\le I\\ M\ne\mathbf{0}}} C_I^M \frac{(p_j\mathbf{e}_j+I-M)!|}{(p_j\mathbf{e}_j+I)!}+B\sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1}\frac{(S_j+I-M)!}{(p_j\mathbf{e}_j+I)!}\Bigg)\times$$

$$\times\max\left\{\frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)}\colon \|S\|\le\sum_{j=1}^n p_j\right\}.$$

where $B=\max\{B_{S_j,M},B_{p_j\mathbf{e}_j,M}\colon j\in\{1,\dots,n\},\mathbf{0}\le M\le I,\|I\|=1+\sum_{\substack{k=1\\k\ne j}}^n p_k.\}$

Obviously, $\|p_j\mathbf{e}_j+I\|=1+\sum_{j=1}^n p_j$. For all $z\in\mathbb{C}^n\setminus\mathbb{D}^n(\mathbf{0},R')$ it implies

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|=1+\sum_{j=1}^n p_j\right\}\le\max\{1,c'\}\cdot\max\left\{\frac{|F^{(S)}(z)|}{S!\mathbf{L}^S(z)}\colon \|S\|\le\sum_{j=1}^n p_j\right\},$$

where

$$c'=\max_{\substack{\|I\|=1-p_j+\sum_{k=1}^n p_k,\\ j\in\{1,\dots,n\}}}\Bigg(D_{I,j}\Bigg(\frac{p_j!}{(p_j\mathbf{e}_j+I)!}+B\sum_{\|S_j\|\le p_j-1}\frac{(p_j\mathbf{e}_j-S_j)!}{(p_j\mathbf{e}_j+I)!}\Bigg)+$$

$$+B\sum_{\substack{\mathbf{0}\le M\le I\\ M\ne\mathbf{0}}} C_I^M \frac{(p_j\mathbf{e}_j+I-M)!|}{(p_j\mathbf{e}_j+I)!}+B\sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1}\frac{(S_j+I-M)!}{(p_j\mathbf{e}_j+I)!}\Bigg). \qquad (77)$$

In view of Corollary 5 the analytic function F in $\mathbb{B}^n$ has bounded **L**-index in joint variables. And by Lemma 5 estimate (76) holds. □

By analogy to the proofs of Theorems 18 and 19 it can be proved the following assertion.

Theorem 20. *If homogeneous system of PDE's* (67) *belongs to class* $\mathcal{A}(\mathbf{G},\mathbf{0},\mathbf{L})$ *and* F *is an analytic solution of the system in* $\mathbb{B}^n$ *then* F *has bounded* **L***-index in joint variables and*

$$\varlimsup_{|R|\to 1-0}\frac{\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^n}\int_0^1\langle R,\mathbf{L}(\tau Re^{i\Theta})\rangle d\tau}\le\max\{1,c'\},$$

where c' *is defined in* (77) *with* $D_{I,j}=0$ *and* $\|I\|=-p_j+\sum_{k=1}^n p_k$ *instead* $\|I\|=1-p_j+\sum_{k=1}^n p_k$.

Remark 2. The obtained propositions in this section are new even for functions analytic in a disc. Analytic functions in the unit disc of bounded l-index are considered in [35,36].

For example, if $n = 1$ then system (67) reduces to the following differential equation

$$g_p(z)f^{(p)}(z) + \sum_{j-0}^{p-1} g_j(z)f^{(j)}(z) = h(z), \tag{78}$$

where h and g_j are analytic functions in $\mathbb{D}$. Then Theorem 17 implies the corollary for $n = 1$.

Corollary 6. *Let $l \in QW(\mathbb{D})$ and for all $z \in \mathbb{C}$ such that $|z| > r'$ analytic functions h and g_j in $\mathbb{D}$ satisfy the following conditions*

1) $\left|g_j^{(m)}(z)\right| \leq B_{j,m} l^{p-j+m}(z)|g_p(z)|$ *and* $\left|g_p'(z)\right| < B_{p,1} l^m(z)|g_p(z)|$ *for every* $j \in \{1, \ldots, p-1\}$, $m \in \{0, 1\}$,

2) $|h'(z)| \leq Dl(z)|h(z)|$,

where $B_{j,m}$ and D are nonnegative constants, and $B_{p,1}$ is positive constant. If an analytic function f in $\mathbb{D}$ satisfies (78) *then f has bounded l-index and*

$$\varlimsup_{r\to 1-0} \frac{\ln \max\{|f(z)|\colon |z| = r\}}{\max\limits_{\theta\in[0,2\pi]} \int_0^r l\left(\tau e^{i\theta}\right) d\tau} \leq c,$$

where $c = D(1 + \sum_{j=0}^{p-1} B_{j,0}) + B_{p,1} + \sum_{m=0}^{1} \sum_{j=0}^{p-1} B_{j,m}$.

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