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PRIME ENDS IN THE SOBOLEV MAPPING THEORY ON RIEMANN SURFACES

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We prove criteria for continuous and homeomorphic extension to the boundary of mappings with finite distortion between domains on the Riemann surfaces by prime ends of Caratheodory.

1. Introduction. The theory of the boundary behavior in the prime ends for the mappings with finite distortion has been developed in [11] for the plane domains and in [14] for the spatial domains. The pointwise boundary behavior of the mappings with finite distortion in regular domains on Riemann surfaces was recently studied by us in [26]. Moreover, the problem was investigated in regular domains on the Riemann manifolds for $n \geq 3$ as well as in metric spaces, see e.g. [1] and [28]. It is necessary to mention also that the theory of the boundary behavior of Sobolev's mappings has significant applications to the boundary value problems for the Beltrami equations and for analogs of the Laplace equation in anisotropic and inhomogeneous media, see e.g. [2], [7]–[10], [12], [13], [19], [22], [24] and relevant references therein. For basic definitions and notations, discussions and historic comments in the mapping theory on the Riemann surfaces, see our previous papers [25]–[27].

2. Definition of the prime ends and preliminary remarks. First recall the necessary definitions of some general notions. Given a topological space T, a *path in T* is a continuous map $\gamma\colon [a,b] \to T$. Given $A, B, C \subseteq T$, $\Delta(A,B,C)$ denotes the collection of all paths γ joining A and B in C, i.e., $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$ for all $t \in (a,b)$. In what follows, $|\gamma|$ denotes the *locus* of γ, i.e. the image $\gamma([a,b])$.

We act similarly to Caratheodory ([4]) under the definition of the prime ends of domains on a Riemann surface $\mathbb{S}$, see Chapter 9 in [5]. First of all, recall that a continuous mapping $\sigma\colon \mathbb{I} \to \mathbb{S}$, $\mathbb{I} = (0,1)$, is called a *Jordan arc* in $\mathbb{S}$ if $\sigma(t_1) \neq \sigma(t_2)$ for $t_1 \neq t_2$. We also use the notations σ, $\overline{\sigma}$ and $\partial\sigma$ for $\sigma(\mathbb{I})$, $\overline{\sigma(\mathbb{I})}$ and $\overline{\sigma(\mathbb{I})} \setminus \sigma(\mathbb{I})$, correspondingly. A *cross–cut* of a domain $D \subset \mathbb{S}$ is either a closed Jordan curve or a Jordan arc σ in the domain D with both ends on ∂D splitting D.

A sequence $\sigma_1, \ldots, \sigma_m, \ldots$ of cross-cuts of D is called a *chain* in D if:

(i) $\overline{\sigma_i} \cap \overline{\sigma_j} = \varnothing$ for every $i \neq j$, $i, j = 1, 2, \ldots$;

(ii) σ_m splits D into 2 domains one of which contains σ_{m+1} and another one σ_{m-1} for every $m > 1$;

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(iii) $\delta(\sigma_m) \to 0$ as $m \to \infty$.

Here

$$\delta(E) = \sup_{p_1, p_2 \in \mathbb{S}} \delta(p_1, p_2)$$

denotes the diameter of a set E in $\mathbb{S}$ with respect to a metric δ in $\mathbb{S}$ agreed with its topology, see [25]–[26].

Correspondingly to the definition, a chain of cross-cuts σ_m generates a sequence of domains $d_m \subset D$ such that $d_1 \supset d_2 \supset \ldots \supset d_m \supset \ldots$ and $D \cap \partial\, d_m = \sigma_m$. Two chains of cross-cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are called *equivalent* if, for every $m = 1, 2, \ldots$, the domain d_m contains all domains d'_k except a finite number and, for every $k = 1, 2, \ldots$, the domain d'_k contains all domains d_m except a finite number, too. A *prime end* P of the domain D is an equivalence class of chains of cross-cuts of D that are not contracted to a point in D. Later on, E_D denote the collection of all prime ends of a domain D and $\overline{D}_P = D \cup E_D$ is its completion by its prime ends.

Next, we say that a sequence of points $p_l \in D$ is *convergent to a prime end* P of D if, for a chain of cross–cuts $\{\sigma_m\}$ in P, for every $m = 1, 2, \ldots$, the domain d_m contains all points p_l except their finite collection. Further, we say that a sequence of prime ends P_l converge to a prime end P if, for a chain of cross–cuts $\{\sigma_m\}$ in P, for every $m = 1, 2, \ldots$, the domain d_m contains chains of cross–cuts $\{\sigma'_k\}$ in all prime ends P_l except their finite collection.

Now, let D be a domain in the compactification $\overline{\mathbb{S}}$ of a Riemann surface $\mathbb{S}$ by Kerekjarto-Stoilow, see a discussion in [25]–[26]. Open neighborhoods of points in D is induced by the topology of $\overline{\mathbb{S}}$. A basis of neighborhoods of a prime end P of D can be defined in the following way. Let d be an arbitrary domain from a chain in P. Denote by d^* the union of d and all prime ends of D having some chains in d. Just all such d^* form a basis of open neighborhoods of the prime end P. The corresponding topology on $\overline{D}_P$ is called the *topology of prime ends.*

Let P be a prime end of D on a Riemann surface $\mathbb{S}$, $\{\sigma_m\}$ and $\{\sigma'_m\}$ be two chains in P, d_m and d'_m be domains corresponding to σ_m and σ'_m. Then

$$\bigcap_{m=1}^{\infty} \overline{d_m} \subseteq \bigcap_{m=1}^{\infty} \overline{d'_m} \subset \bigcap_{m=1}^{\infty} \overline{d_m},$$

and, thus,

$$\bigcap_{m=1}^{\infty} \overline{d_m} = \bigcap_{m=1}^{\infty} \overline{d'_m},$$

i.e. the set named by a *body of the prime end* P

$$I(P) := \bigcap_{m=1}^{\infty} \overline{d_m} \tag{1}$$

depends only on P but not on a choice of a chain of cross–cuts $\{\sigma_m\}$ in P.

It is necessary to note also that, for any chain $\{\sigma_m\}$ in the prime end P,

$$\Omega := \bigcap_{m=1}^{\infty} d_m = \varnothing. \tag{2}$$

Indeed, every point p in Ω belongs to D. Moreover, some open neighborhood of p in D should belong to Ω. In the contrary case each neighborhood of p should have a point in some σ_m.

However, in view of condition (iii) then $p \in \partial D$ that should contradict the inclusion $p \in D$. Thus, Ω is an open set and if Ω would be not empty, then the connectedness of D would be broken because $D = \Omega \cup \Omega^*$ with the open set $\Omega^* := D \setminus I(P)$.

In view of conditions (i) and (ii), we have by (2) that

$$I(P) = \bigcap_{m=1}^{\infty} (\partial d_m \cap \partial D) = \partial D \cap \bigcap_{m=1}^{\infty} \partial d_m .$$

Thus, we obtain the following statement.

Proposition 1. *For each prime end P of a domain D on a Riemann surface,*

$$I(P) \subseteq \partial D. \tag{3}$$

Remark 1. If D is a domain in $\overline{\mathbb{S}}$ with $\partial D \subset \mathbb{S}$, then $I(P)$ is a continuum, i.e. it is a connected compact set, see e.g. I(9.12) in [31], see also I.9.3 in [3], and $I(P)$ belongs to only one (connected) component Γ of ∂D. Hence we say that the component Γ is *associated with the prime end P*.

Moreover, every prime end of D in the case contains a *convergent chain* $\{\sigma_m\}$, i.e., that is contracted to a point $p_0 \in \partial D$. Furthermore, each prime end P contains a *spherical chain* $\{\sigma_m\}$ lying on circles $S(p_0, r_m) = \{p \in \mathbb{S} \colon \delta(p, p_0) = r_m\}$ with $p_0 \in \partial D$ and $r_m \to 0$ as $m \to \infty$. The proof is perfectly similar to Lemma 1 in [14] after the replacement of metrics, see also Theorem VI.7.1 in [21], and hence we omit it. Note by the way that the condition (iii) does not depend in the case on the choice of the metric δ agreed with the topology of $\mathbb{S}$ because ∂D has a compact neighborhood.

It is known that the conformal modulus M of the family of all paths joining a pair of the opposite sides of a rectangle is equal to the ratio of lengths of other pair of opposite sides and their own, see e.g. I.4.3 in [18]. This simple fact gives a series of useful consequences.

Corollary 1. *Let S be the open sector of the ring $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, $z_0 \in \mathbb{C}$, between the rays $R_k = \{z \in \mathbb{C} : z = z_0 + re^{i\alpha_k},\ r \in (0, \infty)\}$, $k = 1, 2$, $0 \le \alpha_1 < \alpha_2 \le 2\pi$. Then*

$$M(\Delta(R_1, R_2, S)) \ = \ \frac{\log \frac{r_2}{r_1}}{\alpha_2 - \alpha_1}, \qquad M(\Delta(C_1, C_2, S)) \ = \ \frac{\alpha_2 - \alpha_1}{\log \frac{r_2}{r_1}} \tag{4}$$

where C_k are the boundary circles $\{z \in \mathbb{C} : |z - z_0| = r_k\}$, $k = 1, 2$, of the ring A.

Indeed, the conclusion follows from the invariance of the modulus M under conformal mappings because the sector S is mapped with the mapping $\log(z - z_0)$ onto the rectangle $R = \{\zeta = \xi + i\eta \in \mathbb{C} \colon \ \log r_1 < \xi < \log r_2, \alpha_1 < \eta < \alpha_2\}$.

Corollary 2. *Under notations of Corollary 1 and $\alpha_2 - \alpha_1 = \Delta$, the modulus of all Jordan arcs joining the rays R_1 and R_2 in the sector S is greater or equal to the number $\frac{1}{\Delta} \log \frac{r_2}{r_1}$.*

Indeed, every path $\gamma \colon [a, b] \to \mathbb{C}$ in $\Delta(R_1, R_2, S)$ has a countable collection of loops because its preimage (without the the corresponding point of cusp in $\mathbb{C}$) is open in (a, b). Thus, numbering its loops and removing them by induction, we come to a Jordan arc γ_* in $\Delta(R_1, R_2, S)$ with its locus $|\gamma_*| \subseteq |\gamma|$.

3. Some general topological lemmas. The following statement is an analog of Proposition 2.3 in [23], see also Proposition 13.3 in [19].

Proposition 2. *Let T be a topological space. Suppose that E_1 and E_2 are sets in T with $\overline{E_1} \cap \overline{E_2} = \varnothing$. Then*

$$\Delta\,(\,E_1\,,\,E_2\,,\,T\,) > \Delta\,(\,\partial E_1\,,\,\partial E_2\,,\,T \setminus (\overline{E}_1 \cup \overline{E}_2)\,). \tag{5}$$

Proof. Indeed, let $\gamma \in \Delta(E_1, E_2, T)$, i.e. the path $\gamma\colon [a,b] \to T$ is such that $\gamma(a) \in E_1$ and $\gamma(b) \in E_2$. Note that the set $\alpha := \gamma^{-1}(\overline{E_1})$ is a closed subset of the segment $[a,b]$ because γ is continuous, see e.g. Theorem 1 in Section I.2.1 of [3]. Consequently, α is compact because $[a,b]$ is a compact space, see e.g. I.9.3 in [3]. Then there is

$$a_* := \max_{t\in\alpha} t < b$$

because $\gamma(b) \in E_2$ and by the hypothesis of the proposition $\overline{E_1} \cap \overline{E_2} = \varnothing$. Thus, $\gamma' := \gamma|_{[a_*,b]}$ belongs to $\Delta(\partial E_1, E_2, T \setminus \overline{E}_1)$ because γ is continuous and hence $\gamma'(a_*)$ cannot be an inner point of E_1.

Arguing similarly in the space $T' = T \setminus E_1$ with $E_1' := E_2$ and $E_2' := \partial E_1$, we obtain that there is

$$b_* := \min_{\gamma'(t)\in\overline{E_2}} t > a_*.$$

Thus, by the given construction $\gamma_* := \gamma|_{[a_*,b_*]}$ just belongs to $\Delta(\partial E_1, \partial E_2, T \setminus (\overline{E}_1 \cup \overline{E}_2))$. □

Lemma 1. *In addition to the hypothesis of Proposition 2, let T be a subspace of a metric space (M,ρ). Suppose that*

$$\partial E_1 \subseteq C_1 := \{p \in M\colon \rho(p,p_0) = R_1\},\ \partial E_2 \subseteq C_2 := \{p \in M\colon \rho(p,p_0) = R_2\}$$

with $p_0 \in M \setminus T$ and $R_1 < R_2$. Then

$$\Delta\,(\,E_1\,,\,E_2\,,\,T\,) > \Delta\,(\,C_1\,,\,C_2\,,\,A\,) \tag{6}$$

where $A = A(p_0,R_1,R_2) := \{p \in M\colon R_1 < \rho(p,p_0) < R_2\}$.

Note that here, generally speaking, $C_1 \cap T \neq E_1$ and $C_2 \cap T \neq E_2$ as well as γ_* in the proof of Proposition 2 is not in R.

Proof. First of all, note that by the continuity of γ_* the set $\omega := \gamma_*^{-1}(R)$ is open in $[a_*,b_*]$ and ω is the union of a countable collection of disjoint intervals (a_1,b_1), (a_2,b_2), … with ends in $\Gamma := \gamma_*^{-1}(\partial R)$. If there is a pair a_k and b_k in the different sets $\Gamma_i := \gamma_*^{-1}(C_i)$, $i = 1,2$, $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \varnothing$, then the proof is complete.

Let us assume that such a pair is absent. Then the given collection is split into 2 collections of disjoint intervals (a_l', b_l') and (a_l'', b_l'') with ends $a_l', b_l' \in \Gamma_1$ and $a_l'', b_l'' \in \Gamma_2$, $l = 1,2,\ldots$. Set $\alpha_1 = \bigcup_l (a_l', b_l')$ and $\alpha_2 = \bigcup_l (a_l'', b_l'')$.

Arguing by contradiction, it is easy to show that $\gamma_*\colon [a_*,b_*] \to (M,\rho)$ is uniformly continuous because $[a_*,b_*]$ is a compact space. Indeed, let us assume that there is $\varepsilon > 0$ and a sequence of pairs a_n^* and $b_n^* \in [a_*,b_*]$, $n = 1,2,\ldots$, such that $|b_n^* - a_n^*| \to 0$ as $n \to \infty$ and simultaneously $\rho(\gamma_*(a_n^*), \gamma_*(b_n^*)) \geq \varepsilon$. However, by compactness of $[a_*,b_*]$ there is a subsequence $a_{n_k}^* \to a_0 \in [a_*,b_*]$ and then also $b_{n_k}^* \to a_0$ as $k \to \infty$. Hence by the continuity of γ_* it should be $\rho(\gamma_*(a_{n_k}^*), \gamma_*(a_0)) \to 0$ as well as $\rho(\gamma_*(b_{n_k}^*), \gamma_*(a_0)) \to 0$ and then by the triangle inequality also $\rho(\gamma_*(a_{n_k}^*), \gamma_*(b_{n_k}^*)) \to 0$ as $k \to \infty$. The contradiction disproves the assumption.

Note that $b'_l - a'_l \to 0$ as $l \to \infty$ and by the uniform continuity of γ_* on $[a_*, b_*]$ we have that $|\gamma'_l| \to C_1$ in the sense that

$$\sup_{p\in|\gamma'_l|} \inf_{q\in C_1} \rho(p,q) \to 0 \quad \text{as } l \to \infty$$

where $\gamma'_l := \gamma_*|_{[a'_l,b'_l]}$, $l = 1, 2, \ldots$. Thus, there is $R'_2 \in (R_1, R_2)$ such that the set $L_1 := \bigcup_l |\gamma'_l|$ lies outside of $B_2 := \{p \in M : \rho(p, p_0) > R'_2\}$.

Arguing similarly, we obtain that there is $R'_1 \in (R_1, R'_2)$ such that the set $L_2 := \bigcup_l |\gamma''_l|$ lies outside of $B_1 := \{p \in M : \rho(p, p_0) < R'_1\}$. Remark that the sets $\beta_1 := \gamma_*^{-1}(B_1)$ and $\beta_2 := \gamma_*^{-1}(B_2)$ are open in $[a_*, b_*]$ because γ_* is continuous and by the construction $\delta_1 := \alpha_1 \cup \beta_1$ and $\delta_2 := \alpha_2 \cup \beta_2$ are open, mutually disjoint and together cover the segment $[a_*, b_*]$. The latter contradicts to connectedness of the segment and, thus, disproves the above assumption. □

4. The main lemma.

Lemma 2. *Let $\mathbb{S}$ be a Riemann surface, D be a domain in $\overline{\mathbb{S}}$ with $\partial D \subset \mathbb{S}$ and let Γ be an isolated component of ∂D. Then Γ has a neighborhood U with a conformal mapping H of $U^* := U \cap D$ onto a ring $R = \{z \in \mathbb{C} : 0 \le r < |z| < 1\}$ where $\gamma := \partial U^* \cap D$ is a closed Jordan curve,*

$$C(\gamma, H) = \{z \in \mathbb{C} : |z| = 1\}, \quad C(\Gamma, H) = \{z \in \mathbb{C} : |z| = r\}$$

and $r = 0$ if and only if Γ is degenerated to a point. Furthermore, the mapping H can be extended to a homeomorphism $\tilde{H}$ of $\overline{U^*}_P$ onto $\overline{R}$.

Here we use the notation of the *cluster set* of the mapping H for $B \subseteq \partial D$,

$$C(B, H) := \left\{ z \in \mathbb{C} : z = \lim_{k\to\infty} H(p_k), p_k \to p \in B, p_k \in D \right\}.$$

Proof. By the Kerekjarto–Stoilow representation of $\mathbb{S}$, Γ has an open neighborhood V in $\mathbb{S}$ of a finite genus and we may assume that $\overline{V}$ is a compact subset of $\mathbb{S}$, V is connected and does not intersect $\partial D \setminus \Gamma$ because Γ is an isolated component of ∂D. Thus, $V \cap D$ is a Riemann surface of finite genus with an isolated boundary element g corresponding to Γ. However, a Riemann surface of finite genus has boundary elements only of the first kind, see, e.g., IV.II.6 in [29]. Consequently, Γ has a neighborhood U^* from the side of D of genus zero with a closed Jordan curve $\gamma = \partial U^* \cap D$. The latter means that U^* is homeomorphic to a plane domain and, consequently, by the general principle of Koebe, see e.g. Section II.3 in [15], U^* is conformally equivalent to a planar domain D^*. Note that by the construction U^* has two nondegenerate boundary components. Hence there is a conformal mapping H of U^* onto a ring $D^* = R = \{z \in \mathbb{C} : 0 \le r < |z| < 1\}$ with $C(\gamma, H) = C_1 := \{z \in \mathbb{C} : |z| = 1\}$ and $C(\Gamma, H) = C_r := \{z \in \mathbb{C} : |z| = r\}$, see e.g. Proposition 2.5 in [23] or Proposition 13.5 in [19]. Set $U = U^* \cup (V \setminus D)$.

If Γ is not degenerated into a point, then $r \ne 0$. Indeed, in the contrary case the images of the closed Jordan curves around the origin in the punctured disk $\mathbb{D}_\varepsilon = \{z \in \mathbb{C} : 0 < |z| < \varepsilon\}$ under the mapping H^{-1} should be contracted to Γ as $\varepsilon \to 0$ and hence their lengths are not less than $\delta := \operatorname{diam} \Gamma > 0$ for small enough ε. However, the latter contradicts to the conformal invariance of the modulus because by Corollary 2 the modulus of all such closed Jordan curves is equal to ∞. Inversely, if Γ is degenerated into a point $p_0 \in \mathbb{S}$, then it is obvious that $r = 0$ because p_0 has arbitrarily small neighborhoods that are conformally mapped onto the unit disk in $\mathbb{C}$. Hence we omit the consideration of this trivial case and restrict ourselves by the case $r > 0$.

Now, by the condition (i) in the definition of prime ends and the invariance of M we have, for every chain $\{\sigma_m\}$ in a prime end P associated with Γ and localized in U^*, that

$$M(\Delta(\sigma_m, \sigma_{m+1}, U^*)) < \infty \quad \forall m = 1, 2, \dots \tag{7}$$

Moreover, by Remark 1 P contains a chain $\{\sigma_m\}$ lying on circles $S_m = S(p_0, r_m) = \{p \in \mathbb{S}\colon \delta(p, p_0) = r_m\}$ with $p_0 \in \partial D$ and $r_m \to 0$ as $m \to \infty$ for which and any continuum C in U^*

$$\lim_{m\to\infty} M(\Delta(\sigma_m, C, U^*)) \leq \lim_{m\to\infty} M(\Delta(\sigma_m, \sigma_{m_0}, U^*)) = 0. \tag{8}$$

Indeed, for every continuum C in U^*, there is m_0 such that $C \subset D \setminus d_{m_0}$ and the closed ball $B_0 = B(p_0, r_{m_0}) = \{p \in \mathbb{S}\colon \delta(p, p_0) \leq r_{m_0}\}$ is compact and lies in a chart U_0 of p_0. Then $\Delta(\sigma_m, C, U^*) \subseteq \Delta(\sigma_m, D \setminus d_{m_0}, U^*)$, by Proposition 2 $\Delta(\sigma_m, D \setminus d_{m_0}, U^*) > \Delta(\sigma_m, \sigma_{m_0}, U^*)$ and by Lemma 1 $\Delta(\sigma_m, \sigma_{m_0}, U^*) > \Delta(S_m, S_{m_0}, A)$ where $A := \{p \in \mathbb{S}\colon r_m < \delta(p, p_0) < r_{m_0}\}$ belongs to the chart U_0 of the point p_0. Note, $M(\Delta(S_m, S_{m_0}, A)) \leq M(\Delta(S_m, S_{m_0}, U_0)) \to 0$ as $m \to \infty$ because S_{m_0} is a compact set in $B_0 \setminus \{p_0\}$ and S_m is contracted to p_0 as $m \to \infty$, see also 7.5 in [30]. Finally, we obtain (8) by the minorization principle, see e.g. [6], p. 178. Similarly, it is proved that prime ends associated with γ also satisfy conditions (7) and (8).

Thus, the prime ends of U^* in the sense (i)–(iii) and their images in R are the prime ends in the sense of Section 4 in [20]. By Lemma 3.5 in [20] the prime ends of Näkki in R coincide with prime ends of Caratheodory. Moreover, the Näkki prime ends in R has a one-to-one correspondence with the points of ∂R whose extension to the mapping between $\overline{R}$ and $\overline{R}_P$ by the identity in R is a homeomorphism with respect to the topologies of $\overline{R}$ and $\overline{R}_P$ or with respect to convergence of points and prime ends, respectively, see Theorems 4.1 and 4.2 in [20]. Consequently, if p_k is a sequence of points in U^* which is convergent to a prime end P of U^*, then $H(p_k)$ is convergent to a unique point $z_0 \in \partial R$ that depends only on P.

Denote by $\tilde{H}$ the extension of H to $\overline{U^*}_P$. It is clear by definitions of prime ends of Näkki and Caratheodory as classes of equivalence that $\tilde{H}(P_1) \neq \tilde{H}(P_2)$ for every prime ends $P_1 \neq P_2$ of the domain U^*. Let us consider the metric $\rho(P, P^*) := |\tilde{H}(P) - \tilde{H}(P^*)|$ on the space $\overline{U^*}_P$. It is obvious by definitions that $\rho(P_k, P_0) \to 0$ implies that $P_k \to P_0$ as $k \to \infty$. The inverse conclusion follows because of the mapping $\tilde{H}\colon \overline{U^*}_P \to \overline{R}$ is continuous. Indeed, let $P_k \to P_0$, $k = 1, 2, \dots$, be a sequence in $\overline{U^*}_P$. It is obvious, $\tilde{H}(P_k) \to \tilde{H}(P_0)$ for $P_0 \in U^*$. If $P_0 \in E_{U^*}$, then we are able to choose $p_k \in U^*$ such that $|\tilde{H}(P_k) - \tilde{H}(p_k)| < 2^{-k}$, $k = 1, 2, \dots$, and $p_k \to P_0$ as $k \to \infty$. The latter implies that $\tilde{H}(p_k) \to \tilde{H}(P_0)$ and then the former implies that $\tilde{H}(P_k) \to \tilde{H}(P_0)$. Thus, the space $\overline{U^*}_P$ is metrizable with the given metric ρ and $\tilde{H}$ is an isometric embedding of $\overline{U^*}_P$ in $\overline{R}$. By construction $\tilde{H}(U^*) = R$ and, by Proposition 2.5 in [23] or Proposition 13.5 in [19], $\tilde{H}(E_{U^*}) \subseteq \partial R$. Let us show that $\tilde{H}(E_{U^*}) = \partial R$.

For this goal, fixing $z_0 \in \partial C_r$ and $\varepsilon \in (0, 1)$, consider the family $\mathfrak{F}$ of all Jordan arcs in the open disk $B_\varepsilon = B(z_0, \varepsilon) := \{z \in \mathbb{C}\colon |z - z_0| < \varepsilon\}$ joining in R the two open arcs A_1 and A_2 of $C_r \cap B_\varepsilon \setminus \{z_0\}$. By the minorization principle, see e.g. [6], and the invariance of M (with respect to the conformal mapping consisting of the composition of the inversion with respect to the unit circle and the reflection with respect to the straight line L_0 passing through the origin and the point z_0) we obtain from Corollary 2 that the conformal modulus of the family $\mathfrak{F}$ is equal to ∞. By the invariance of the modulus under conformal mappings we have that the modulus of the family $\mathfrak{F}_* = H^{-1}(\mathfrak{F})$ is also equal to ∞. Consequently, the length of elements of $\mathfrak{F}_*$ cannot be restricted from below and, by arbitrariness of ε, there is a sequence of mutually disjoint cross-cuts $\sigma_m \in \mathfrak{F}$ of R with $\sigma_m(0) \in A_1$ and $\sigma_m(1) \in A_2$ that is contracted to the point z_0 such that $\delta(\sigma_m^*) \to 0$ as $m \to \infty$ where $\sigma_m^* = H^{-1}(\sigma_m)$ and, moreover, $\sigma_{m+1}^* \subset d_m^*$ where d_m^* is the corresponding component of D generated by σ_m^*, $\partial d_m^* \cap U^* = \sigma_m^*$ for all $m = 1, 2, \ldots$. Note that such rectifiable $\sigma_m^* \colon (0,1) \to D$ have limits $p_m^{(1)} = \lim\limits_{t\to+0} \sigma_m^*(t)$ and $p_m^{(2)} = \lim\limits_{t\to1-0} \sigma_m^*(t)$ because $\overline{U^*}$ is a compact subset of $\mathbb{S}$, see e.g. Proposition I.9.3 in [3], cf. also Theorem 1.3.2 in [30], moreover, the points $p_m^{(1)}$ and $p_m^{(2)}$ belongs to Γ, see e.g. Proposition 2.5 in [23] or Proposition 13.5 in [19].

Finally, it remains to show that $\overline{\sigma_m^*} \cap \overline{\sigma_{m+1}^*} = \varnothing$, passing in case of need to a suitable subchain of cross–cuts σ_m in R. First of all, by the above construction we may assume that

$$\delta_m := \inf_{z\in\sigma_m} |z - z_0| > \delta_m^* := \sup_{z\in\sigma_{m+1}} |z - z_0| > 0 \qquad \forall m = 1, 2, \ldots$$

and also that σ_m^* is contracted to a point $p_0 \in \Gamma$ because Γ is compact and $\delta(\sigma_m^*) \to 0$. It is clear that the desired subchain exists if $\sigma_m^*(0) \neq p_0 \neq \sigma_m^*(1)$ for all large enough m.

In the contrary case, it would exist a subchain $\tilde{\sigma}_k := \sigma_{m_k}$, $k = 1, 2, \ldots$, such that either $\tilde{\sigma}_k^*(0) = p_0 = \tilde{\sigma}_{k+1}^*(0)$ or $\tilde{\sigma}_k^*(1) = p_0 = \tilde{\sigma}_{k+1}^*(1)$ for all $k = 1, 2, \ldots$, where $\tilde{\sigma}_k^* := H^{-1}(\tilde{\sigma}_k)$, $k = 1, 2, \ldots$. In the first case, consider the ring $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ with $0 < \delta_{m_k}^* < r_1 < r_2 < \delta_{m_k}$. As above, by the minorization principle, the invariance of M and Corollary 1 the conformal modulus of the family $\tilde{\mathfrak{F}}$ of all paths in $A \cap R$ joining the open arc $A_0 := A \cap A_1$ of the circle C_r and the interval $\mathrm{I}_0 := A \cap L_0$ of the straight line L_0 is not less than $\frac{2}{\pi} \log \frac{r_2}{r_1} > 0$. The modulus of the family $\tilde{\mathfrak{F}}_* = H^{-1}(\tilde{\mathfrak{F}})$ should be the same. However, the modulus of $\tilde{\mathfrak{F}}_*$ is equal to zero because all paths in $\tilde{\mathfrak{F}}_*$ are ended at the point p_0.

Indeed, denote by I the maximal open interval of L_0 containing I_0 and not intersecting $\tilde{\sigma}_k$ and $\tilde{\sigma}_{k+1}$, and by t_0 and t_* the parameter numbers in $(0,1)$ corresponding to its ends on $\tilde{\sigma}_k$ and $\tilde{\sigma}_{k+1}$. Then $H^{-1}(\mathrm{I})$, $\tilde{\sigma}_k^*((0,t_0])$, $\tilde{\sigma}_{k+1}^*((0,t_*])$ and the point p_0 form a closed Jordan curve in $\overline{U^*}$ with the only point on ∂U^*. Note that the corresponding Jordan domain contains the family $\tilde{\mathfrak{F}}_*$ of paths γ that should be ended on Γ and, consequently, at the point p_0. The second possibility is similarly disproved.

Thus, $\tilde{H}$ is isometry between $\overline{U^*}_P$ with the given metric ρ and $\overline{R}$. □

Remark 2. By the proof we have that $\overline{U^*}_P$ is a compact space with the metric ρ. Moreover, it follows from the proof that the spaces of prime ends by Caratheodory and Näkki coincide not only in the ring R but also in U^* because the Näkki prime ends are invariant under conformal mappings.

Furthermore, if D be a domain in the Kerekjarto-Stoilow compactification $\overline{\mathbb{S}}$ of a Riemann surface $\mathbb{S}$ and ∂D is a set in $\mathbb{S}$ with a finite collection of components, then their prime ends by Caratheodory and Näkki also coincide, the whole space $\overline{D}_P$ can be metrized through the theory of pseudometric spaces, see e.g. Section 2.21.XV in [16], and $\overline{D}_P$ is compact.

Namely, let ρ_0 be one of the metrics on $\overline{\mathbb{S}}$ and let $\rho_1, \ldots, \rho_n$ be the above metrics on $\overline{U_1^*}_P, \ldots, \overline{U_n^*}_P$ for the corresponding components $\Gamma_1, \ldots, \Gamma_n$ of ∂D. Here we may assume that the sets $\overline{U_j^*}$ are mutually disjoint. Then $\rho_j^* := \rho_j/(1+\rho_j) \le 1$, $j = 0, 1, \ldots, n$, are also metrics generating the same topologies on $D_0 := D \setminus \cup U_j^*$, $\overline{U_1^*}_P, \ldots, \overline{U_n^*}_P$, correspondingly, see e.g. Section 2.21.V in [16], and the topology of prime ends on $\overline{D}_P$ is generated by the metric

$$\rho = \sum_{j=0}^{n} 2^{-(j+1)} \tilde{\rho}_j < 1$$

where the pseudometrics $\tilde{\rho}_j$ are extensions of ρ_j^* onto $\overline{D}_P$ by 1, see e.g. Remark 2 in point 2.21.XV of [16]. Note that the space $\overline{D}_P$ is compact because $\overline{D}_P = \bigcup \overline{U_j^*}_P \cup D_0$ where D_0 is a compact space as a closed subset of the compact space $\overline{\mathbb{S}}$, see e.g. Proposition I.9.3 in [3].

Corollary 3. *Under hypothesis of Lemma 2, the space of all prime ends associated with a nondegenerate isolated component of ∂D is homeomorphic to a circle.*

5. On boundary behavior in prime ends of inverse maps. The main base for extending inverse mappings is the following fact.

Lemma 3. *Let $\mathbb{S}$ and $\mathbb{S}'$ be Riemann surfaces, D and D' be domains in $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}'}$, $\partial D \subset \mathbb{S}$ and $\partial D' \subset \mathbb{S}'$ have finite collections of components, and let $f : D \to D^*$ be a homeomorphism of finite distortion with $K_f \in L^1_{\text{loc}}$. Then*

$$C(P_1, f) \cap C(P_2, f) = \varnothing \tag{9}$$

for all prime ends $P_1 \neq P_2$ in the domain D.

Here we use the notation of the *cluster set* of the mapping f at $P \in E_D$,

$$C(P, f) := \left\{ P' \in E_{D'} : P' = \lim_{k\to\infty} f(p_k), p_k \to P,\ p_k \in D \right\}$$

As usual, we also assume here that the dilatation K_f of the mapping f is extended by zero outside of the domain D.

Proof. First of all note that $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}'}$ are metrizable spaces. Hence their compactness is equivalent to their sequential compactness, see e.g. Remark 41.I.3 in [17], and $\overline{D}$, $\overline{D}'$, ∂D and $\partial D'$ are compact subsets of $\mathbb{S}$ and $\mathbb{S}'$, correspondingly, see e.g. Proposition I.9.3 in [3]. Thus, by Lemma 2, Remarks 1 and 2, we may assume that P_1 and P_2 are associated with the same nondegenerate component Γ of ∂D, $K_f \in L^1(D)$, D' is a ring $R = \{z \in \mathbb{C} : 0 \leq r < |z| < 1\}$ and $A_k := C(P_k, f)$, $k = 1, 2$ are sets of points in the circle $C_r := \{z \in \mathbb{C} : |z| = r\}$, ∂D consists of 2 components: Γ and a closed Jordan curve γ, f is extended to a homeomorphism of $D \cup \gamma$ onto $D' \cup C_1$, $C(C_r, f^{-1}) = \Gamma$, see Proposition 2.5 in [23] or Proposition 13.5 in [19]. Note that the sets A_k are continua, i.e. closed arcs of the circle C_r, because

$$A_k = \bigcap_{m=1}^{\infty} \overline{f\left(d_m^{(k)}\right)}, \quad k = 1, 2,$$

where $d_m^{(k)}$ are domains corresponding to chains of cross–cuts $\{\sigma_m^{(k)}\}$ in the prime ends P_k, $k = 1, 2$, see e.g. I(9.12) in [31] and also I.9.3 in [3]. In addition, by Remark 1 we may assume also that $\sigma_m^{(k)}$ are open arcs of the circles $C_m^{(k)} := \{p \in \mathbb{S} : h(p, p_k) = r_m^{(k)}\}$ on $\mathbb{S}$ with $p_k \in \partial D$ and $r_m^{(k)} \to 0$ as $m \to \infty$, $k = 1, 2$.

Set $p_0 = p_1$. By the definition of the topology of the prime ends in the space $\overline{D}_P$, we have that $d_m^{(1)} \cap d_m^{(2)} = \varnothing$ for all large enough m because $P_1 \neq P_2$. For such m, set $R_1 = r_{m+1}^{(1)} < R_2 = r_m^{(1)}$ and

$$U_k = d_m^{(k)},\ \Sigma_k = \sigma_m^{(k)},\ C_k = \{p \in \mathbb{S} : h(p, p_0) = R_k\},\ k = 1, 2.$$

Let K_1 and K_2 be arbitrary continua in U_1 and U_2, correspondingly. Applying Proposition 2 and Lemma 1 with $T = D$, $E_1 = d^{(1)}_{m+1}$ and $E_2 = D \setminus d^{(1)}_m$, and taking into account the inclusion $\Delta(K_1, K_2, D) \subset \Delta(E_1, E_2, D)$, we obtain that

$$\Delta(K_1, K_2, D) > \Delta(C_1, C_2, A),\ A := \{p \in \mathbb{S} : R_1 < h(p, p_0) < R_2\}, \tag{10}$$

which means that any path $\alpha\colon [a, b] \to \mathbb{S}$ joining K_1 and K_2 in D, $\alpha(a) \in K_1$, $\alpha(b) \in K_2$ and $\alpha(t) \in D$, $t \in (a, b)$, has a subpath joining C_1 and C_2 in A. Thus, since f is a homeomorphism, we have also that

$$\Delta(fK_1, fK_2, fD) > \Delta(fC_1, fC_2, fA) \tag{11}$$

and by the minorization principle, see e.g. [6], p. 178, we obtain that

$$M(\Delta(fK_1, fK_2, fD)) \le M(\Delta(fC_1, fC_2, fA)). \tag{12}$$

So, by Lemma 3.1 in [26] we conclude that

$$M(\Delta(fK_1, fK_2, fD)) \leqslant \int\limits_A K_f(p) \cdot \xi^2(h(p, p_0))dh(p) \tag{13}$$

for all measurable functions $\xi\colon (R_1, R_2) \to [0, \infty]$ such that

$$\int\limits_{R_1}^{R_2} \xi(R)dR \geqslant 1. \tag{14}$$

In particular, for $\xi(R) \equiv 1/\delta$, $\delta = R_2 - R_1 > 0$, we get from here that

$$M(\Delta(fK_1, fK_2, fD)) \leqslant M_0 := \frac{1}{\delta}\int\limits_D K_f(p)dh(p) < \infty. \tag{15}$$

Since f is a homeomorphism, (15) means that

$$M(\Delta(\mathcal{K}_1, \mathcal{K}_2, D')) \leqslant M_0 < \infty \tag{16}$$

for all continua $\mathcal{K}_1$ and $\mathcal{K}_2$ in the domains $V_1 = fU_1$ and $V_2 = fU_2$, correspondingly.

Let us assume that $A_1 \cap A_2 \neq \varnothing$. Then by the construction there is $p_0 \in \partial R \cap \partial V_1 \cap \partial V_2$. However, the latter contradicts (16) because the ring $D' = R$ is a QED (quasiextremal distance) domains, see e.g. Theorem 3.2 in [19], see also Theorem 10.12 in [30]. □

Theorem 1. *Let $\mathbb{S}$ and $\mathbb{S}'$ be Riemann surfaces, D and D' be domains in $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}'}$, correspondingly, $\partial D \subset \mathbb{S}$ and $\partial D' \subset \mathbb{S}'$ have finite collections of components, and let $f\colon D \to D'$ be a homeomorphism of finite distortion with $K_f \in L^1_{\rm loc}$. Then the inverse mapping $g = f^{-1}\colon D' \to D$ can be extended to a continuous mapping $\tilde{g}$ of $\overline{D'}_P$ onto $\overline{D}_P$.*

Proof. Recall that by Remark 2 the spaces $\overline{D}_P$ and $\overline{D'}_P$ are compact and metrizable with metrics ρ and ρ'. Let a sequence $p_n \in D'$ converge as $n \to \infty$ to a prime end $P' \in E_{D'}$. Then any subsequence of $p^*_n := g(p_n)$ has a convergent subsequence by compactness of $\overline{D}_P$. By Lemma 3 any such convergent subsequence should have the same limit. Thus, the sequence $\{p^*_n\}$ is convergent in $\overline{D}_P$, see e.g. Theorem 2 of Section 2.20.II in [16]. Similarly, by Lemma 3 the sequence $\tilde{p}_n^* := g(\tilde{p}_n)$ has the same limit for any other sequence $\tilde{p}_n \in D'$ as $n \to \infty$. Consequently, g generates the natural mapping $\tilde{g}\colon \overline{D'}_P \to \overline{D}_P$.

Note that $\{p_n^*\}$ cannot converge to an inner point of D because $I(P') \subseteq \partial D'$ by Proposition 1 and, consequently, the cluster set of p_n^* belongs to ∂D, see e.g. Proposition 2.5 in [23] or Proposition 13.5 in [19]. Thus, $E_{D'}$ is mapped into E_D under this extension $\tilde{g}$ of g. In fact, $\tilde{g}$ maps $E_{D'}$ onto E_D because $p_n = f(p_n^*)$ has a convergent subsequence for every sequence $p_n^* \in D$ that is convergent to a prime end P of the domain D because $\overline{D'}_P$ is compact.

The map $\tilde{g}$ is continuous. Indeed, let a sequence $P_n' \in \overline{D'}_P$ be convergent to $P' \in \overline{D'}_P$. Then by the first item there is a sequence $p_n \in D'$ with $\rho'(P_n', p_n) < 2^{-n}$ and $\rho(p_n^*, P_n^*) < 2^{-n}$ where $p_n^* := g(p_n)$ and $P_n^* := \tilde{g}(P_n)$. Then $p_n \to P'$ and, by the first item, $p_n^* \to P^*$ as well as $P_n^* \to P^*$ as $n \to \infty$ where $P^* = \tilde{g}(P')$. □

Corollary 4. *Under the hypothesis of Lemma 3, if Γ is a nondegenerate component of ∂D, then $C(\Gamma, f)$ is a nondegenerate component of $\partial D'$.*

6. Lemma on extension to boundary of direct mappings. In contrast with the case of the inverse mappings, as it was already established in the plane, no degree of integrability of the dilatation leads to the extension to the boundary of direct mappings with finite distortion, see the example in the proof of Proposition 6.3 in [19]. The nature of the corresponding conditions has a much more refined character as the following lemma demonstrates.

Lemma 4. *Under the hypothesis of Theorem 1, let in addition*

$$\int\limits_{R(p_0,\varepsilon,\varepsilon_0)} K_f(p) \cdot \psi^2_{p_0,\varepsilon,\varepsilon_0}(h(p,p_0))dh(p) = o\left(I^2_{p_0,\varepsilon_0}(\varepsilon)\right) \quad \forall p_0 \in \partial D \tag{17}$$

as $\varepsilon \to 0$ for all $\varepsilon_0 < \delta(p_0)$ where $R(p_0,\varepsilon,\varepsilon_0) = \{p \in \mathbb{S}\colon \varepsilon < h(p,p_0) < \varepsilon_0\}$ and $\psi_{p_0,\varepsilon,\varepsilon_0}(t)\colon (0,\infty) \to [0,\infty]$, $\varepsilon \in (0,\varepsilon_0)$, is a family of measurable functions such that

$$0 < I_{p_0,\varepsilon_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{p_0,\varepsilon,\varepsilon_0}(t)dt < \infty \qquad \forall \varepsilon \in (0,\varepsilon_0).$$

Then f can be extended to a continuous mapping $\tilde{f}$ of $\overline{D}_P$ onto $\overline{D'}_P$.

We assume here as above that the function K_f is extended by zero outside of D.

Proof. By Lemma 2, Remarks 1 and 2, arguing as in the beginning of the proof of Lemma 3, we may assume with no loss of generality that $\overline{D}$ is a compact set in $\mathbb{S}$, ∂D consists of 2 components: a closed Jordan curve γ and one more nondegenerate component Γ, D' is a ring $R = \{z \in \mathbb{C}\colon 0 < r < |z| < 1\}$, $\overline{D'}_P = \overline{R}$,

$$C(\Gamma, f) = C_r := \{z \in \mathbb{C}\colon |z| = r\},\ C(\gamma, f) = C_1 := \{z \in \mathbb{C}\colon |z| = 1\}$$

and that f is extended to a homeomorphism of $D \cup \gamma$ onto $D' \cup C_1$.

Let us first prove that the set $L := C(P, f)$ consists of a single point of C_r for a prime end P of the domain D associated with Γ. Note that $L \neq \varnothing$ by compactness of the set $\overline{R}$ and, moreover, $L \subseteq C_r$ by Proposition 1.

Let us assume that there is at least two points ζ_0 and $\zeta_* \in L$. Set $U = \{\zeta \in \mathbb{C}\colon |\zeta - \zeta_0| < \rho_0\}$ where $0 < \rho_0 < |\zeta_* - \zeta_0|$.

Let σ_k, $k = 1, 2, \ldots$, be a chain in the prime end P from Remark 1 lying on the circles $S_k := \{p \in \mathbb{S}\colon h(p, p_0) = r_k\}$ where $p_0 \in \Gamma$ and $r_k \to 0$ as $k \to \infty$. Let d_k be the domains associated with σ_k. Then there exist points ζ_k and ζ_k^* in the domains $d_k' = f(d_k) \subset R$ such that $|\zeta_0 - \zeta_k| < \rho_0$ and $|\zeta_0 - \zeta_k^*| > \rho_0$ and, moreover, $\zeta_k \to \zeta_0$ and $\zeta_k^* \to \zeta_*$ as $k \to \infty$. Let γ_k be paths joining ζ_k and ζ_k^* in d_k'. Note that by the construction $\partial U \cap \gamma_k \neq \varnothing$, $k = 1, 2, \ldots$.

By the condition of strong accessibility of the point ζ_0 in the ring R, there is a continuum $E \subset R$ and a number $\delta > 0$ such that

$$M(\Delta(E, \gamma_k; R)) \geqslant \delta \tag{18}$$

for all large enough k. Note that $C = f^{-1}(E)$ is a compact subset of D and hence

$$h(p_0, C)) > 0.$$

Let $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 := \min\left(\delta(p_0), h(p_0, C)\right)$. Without loss of generality, we may assume that $r_k < \varepsilon_0$ and that (18) holds for all $k = 1, 2, \ldots$.

Let Γ_m be the family of paths joining the circle $S_0 := \{p \in \mathbb{S}\colon h(p, p_0) = \varepsilon_0\}$ and σ_m, $m = 1, 2, \ldots$, in the intersection of $D \setminus d_m$ and the ring $R_m := \{p \in \mathbb{S}\colon r_m < h(p, p_0) < \varepsilon_0\}$. Applying Proposition 2 and Lemma 1 with $T = D$, $E_1 = d_m$ and $E_2 = B_0 := \{p \in \mathbb{S}\colon h(p, p_0) > \varepsilon_0\}$, and taking into account the inclusion $\Delta(C, C_k, D) \subset \Delta(E_1, E_2, D) = \Delta(B_0, d_m, D)$ where $C_k = f^{-1}(\gamma_k)$, we have that $\Delta(C, C_k, D) > \Gamma_m$ for all $k \geqslant m$ because by the construction $C_k \subset d_k \subset d_m$. Thus, since f is a homeomorphism, we have also that $\Delta(E, \gamma_k, D) > f\Gamma_m$ for all $k \geqslant m$, and by the principle of minorization, see e.g. [6], p. 178, we obtain that $M(f(\Gamma_m)) \geqslant \delta$ for all $m = 1, 2, \ldots$.

On the other hand, every function $\xi(t) = \xi_m(t) := \psi_{p_0, r_m, \varepsilon_0}(t)/I_{p_0, \varepsilon_0}(r_m)$, $m = 1, 2, \ldots$, satisfies the condition (14) and by Lemma 3.1 in [26]

$$M(f\Gamma_m) \leqslant \int\limits_{R_m} K_f(p) \cdot \xi_m^2(h(p, p_0))dh(p),$$

i.e., $M(f\Gamma_m) \to 0$ as $m \to \infty$ in view of (17).

The obtained contradiction disproves the assumption that the cluster set $C(P, f)$ consists of more than one point.

Thus, we have the extension $\tilde{f}$ of f to $\overline{D}_P$ such that $\tilde{f}(E_D) \subseteq E_{D'}$. In fact, $\tilde{f}(E_D) = E_{D'}$. Indeed, if $\zeta_0 \in D'$, then there is a sequence ζ_n in D' that is convergent to ζ_0. We may assume with no loss of generality that $f^{-1}(\zeta_n) \to P_0 \in \overline{D}_P$ because $\overline{D}_P$ is compact, see Remark 2. Hence $\zeta_0 \in E_D$ because $\zeta_0 \notin D$, see e.g. Proposition 2.5 in [23] or Proposition 13.5 in [19].

Finally, let us show that the extended mapping $\tilde{f}\colon \overline{D}_P \to \overline{D'}_P$ is continuous. Indeed, let $P_n \to P_0$ in $\overline{D}_P$. The statement is obvious for $P_0 \in D$. If $P_0 \in E_D$, then by the last item we are able to choose $P_n^* \in D$ such that $\rho(P_n, P_n^*) < 2^{-n}$ and $\rho'(\tilde{f}(P_n), \tilde{f}(P_n^*)) < 2^{-n}$ where ρ and ρ' are some metrics on $\overline{D}_P$ and $\overline{D'}_P$, correspondingly, see Remark 2. Note that by the first part of the proof $f(P_n^*) \to f(P_0)$ because $P_n^* \to P_0$. Consequently, $\tilde{f}(P_n) \to \tilde{f}(P_0)$. □

7. On the homeomorphic extension to the boundary. Combining Lemma 4 and Theorem 1, we obtain the following significant conclusion.

Theorem 2. *Under the hypothesis of Lemma 4, the homeomorphism $f\colon D \to D'$ can be extended to a homeomorphism $\tilde{f}\colon \overline{D}_P \to \overline{D'}_P$.*

Proof. Indeed, by Lemma 3 the mapping $\tilde{f}\colon \overline{D}_P \to \overline{D'}_P$ from Lemma 4 is injective and hence it has the well defined inverse mapping $\tilde{f}^{-1}\colon \overline{D'}_P \to \overline{D}_P$ and the latter coincides with the mapping $\tilde{g}\colon \overline{D'}_P \to \overline{D}_P$ from Theorem 1 because a limit under a metric convergence is unique. The continuity of the mappings $\tilde{g}$ and $\tilde{f}$ follows from Theorem 1 and Lemma 4, respectively. □

Remark 3. Note that condition (17) holds, in particular, if

$$\int\limits_{D(p_0,\varepsilon_0)} K_f(p)\cdot\psi^2(h(p,p_0))dh(p) < \infty \quad \forall p_0 \in \partial D \tag{19}$$

where $D(p_0,\varepsilon_0) = \{p \in \mathbb{S}\colon h(p,p_0) < \varepsilon_0\}$ and where $\psi(t)\colon (0,\infty) \to [0,\infty]$ is a locally integrable function such that $I_{p_0,\varepsilon_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the extendability of f to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$, it suffices for the integrals in (19) to be convergent for an arbitrary nonnegative function $\psi(t)$ that is locally integrable on $(0,\infty)$ but that has a non-integrable singularity at zero.

Thus, Theorem 2 will have a great number of interesting corollaries for the theory of the boundary behavior of the Sobolev mappings on the Riemann surfaces that will be published elsewhere.

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