УДК 517.53

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ON THE l-INDEX BOUNDEDNESS OF SOME COMPOSITION OF FUNCTIONS

M. M. Sheremeta. *On the l-index boundedness of some composition of functions*, Mat. Stud. **47** (2017), 207–210.

It is suggested that for an entire function f the function $F(z) = f(\frac{q}{(1-z)^n})$, $n \in \mathbb{N}$, is of bounded l-index with $l(|z|) = \frac{\beta}{(1-|z|)^{n+1}}$, $\beta > 1$, if and only if f is of bounded index.

1. Introduction. Let $0 < R \le +\infty$, $\mathbb{D}_R = \{z\colon |z| < R\}$ and l be a positive continuous function on $[0, R)$, which satisfies

$$l(r) > \frac{\beta}{R-r}, \quad \beta = \mathrm{const} > 1. \tag{1}$$

An analytic in $\mathbb{D}_R$ function f is said ([1, p. 67]) to be of bounded l-index if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{D}_R$

$$\frac{|f^{(n)}(z)|}{n! l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k! l^k(|z|)}\colon 0 \le k \le N\right\}. \tag{2}$$

The least such integer is called the l-index of f and is denoted by $N(l; f)$. If $R = +\infty$ (i. e. f is an entire function) then the condition (1) is unnecessary. We remark also that if f is an entire function and $l(|z|) \equiv 1$ then f is said to be of bounded index.

A series of works is dedicated to the research of the l-index boundedness for different classes of analytic functions. For example, the l-index boundedness of entire functions represented by canonical products and Laguerre-Pólya functions is investigated in the papers [2-7]. The same problem is studied for analytic in the unit disc functions represented by Blaschke and Naftalevich-Tsuji products in [8–10].

In [11] it is proved that if f is an entire function and $F(z) = f(qz^n)$ $n \ge 2$, then the function F is of bounded l-index with $l(|z|) = |z|^{n-1}$ for $|z| \ge 1$ if and only if f is of bounded index. The following question arises: whether it is possible in this statement replace qz^n by $\frac{q}{(1-z)^n}$ and $l(|z|) = |z|^{n-1}$ by $l(|z|) = \frac{\beta}{(1-|z|)^{n+1}}$, $\beta > 1$. Here we give some elementary functions for which such a replacement is possible.

We need some notations. Suppose that f is an analytic in $\mathbb{D} = \mathbb{D}_1$ function and $l(|z|) = L(\frac{1}{1-|z|})$, $L(x)/x > \beta > 1$ for $x \ge 1$. Then (2) is equivalent to

$$\frac{|f^{(n)}(z)|}{n! L^n(1/(1-|z|))} \le \max\left\{\frac{|f^{(k)}(z)|}{k! L^k(1/(1-|z|))}\colon 0 \le k \le N\right\}. \tag{3}$$

2010 *Mathematics Subject Classification:* 30B50.
Keywords: analytic function; index boundedness; composition of functions.
doi:10.15330/ms.47.2.207-210

For $r \in [0,\beta)$ we define

$$\lambda_1(r) = \inf\left\{\frac{1}{L(x)}L\left(\frac{x}{1+tx/L(x)}\right):\ -r\le t\le r, x\ge 1\right\},$$
$$\lambda_2(r) = \sup\left\{\frac{1}{L(x)}L\left(\frac{x}{1+tx/L(x)}\right):\ -r\le t\le r, x\ge 1\right\}.$$

By Q_β we denote the class of the continuous in $[0,\beta)$ functions L such that $L(x)/x > \beta > 1$ for $x \ge 1$ and $0 < \lambda_1(r) \le \lambda_2(r) < +\infty$ for all $r \in [0,\beta)$. Then [12] (see also [1, p. 21]) the following statement is true.

Lemma 1. *If $\beta > 1$ and $L \in Q_\beta$ then (3) holds if and only if there exist numbers $p \in \mathbb{Z}_+$ and $C > 0$ such that for each $z \in \mathbb{D}$*

$$\frac{|f^{(p+1)}(z)|}{L^{p+1}(1/(1-|z|))} \le C\max\left\{\frac{|f^{(k)}(z)|}{L^k(1/(1-|z|))}: 0\le k\le p\right\}.$$

The function $L(x) = \beta x^{n+1}$ belongs to Q_β. Therefore, Lemma 1 implies the following lemma.

Lemma 2. *If $l(|z|) = \frac{\beta}{(1-|z|)^{n+1}}$, $\beta > 1$, then an analytic function f in $\mathbb{D} = \mathbb{D}_1$ is of bounded l-index if and only if there exist numbers $p \in \mathbb{Z}_+$ and $C > 0$ such that for each $z \in \mathbb{D}$*

$$\frac{|f^{(p+1)}(z)|}{l^{p+1}(|z|)} \le C\max\left\{\frac{|f^{(k)}(z)|}{l^k(|z|)}: 0\le k\le p\right\}. \tag{4}$$

If $f(\xi) = e^\xi$ then $F(z) = \exp\{\frac{q}{(1-z)^n}\}$, $F'(z) = \exp\{\frac{q}{(1-z)^n}\}\frac{qn}{(1-z)^{n+1}}$ and

$$F''(z) = \exp\left\{\frac{q}{(1-z)^n}\right\}\frac{q^2n^2}{(1-z)^{2n+2}} + \exp\left\{\frac{q}{(1-z)^n}\right\}\frac{qn(n+1)}{(1-z)^{n+2}} =$$
$$= \frac{n+1}{1-z}F'(z) + \frac{q^2n^2}{(1-z)^{2n+2}}F(z),$$

whence

$$\frac{|F''(z)|}{l^2(|z|)} \le \frac{n+1}{(1-|z|)l(|z|)}\frac{|F'(z)|}{l(|z|)} + \frac{|q|^2n^2}{(1-|z|)^{2n+2}l^2(|z|)}|F(z)| \le$$
$$\le \frac{n+1}{\beta}\frac{|F'(z)|}{l(|z|)} + \frac{|q|^2n^2}{\beta^2}|F(z)| \le \left(\frac{n+1}{\beta} + \frac{|q|^2n^2}{\beta^2}\right)\max\left\{\frac{|F'(z)|}{l(|z|)}, |F(z)|\right\}$$

that is (4) holds with $p = 2$ and $C = \frac{n+1}{\beta} + \frac{|q|^2n^2}{\beta^2}$ and by Lemma 2 the function $F(z) = \exp\{\frac{q}{(1-z)^n}\}$ is of bounded l-index with $l(|z|) = \frac{\beta}{(1-|z|)^{n+1}}$, $\beta > 1$.

It is easy to show that for the functions $f(\xi) = \operatorname{ch}\xi$ and $f(\xi) = \operatorname{sh}\xi$ the equality

$$F''(z) = \frac{n+1}{1-z}F'(z) + \frac{q^2n^2}{(1-z)^{2n+2}}F(z),$$

is correct and for the functions $f(\xi) = \cos\xi$ and $f(\xi) = \sin\xi$ we have

$$F''(z) = \frac{n+1}{1-z}F'(z) - \frac{q^2n^2}{(1-z)^{2n+2}}F(z).$$

Therefore, as above we get that the functions $F(z)=\operatorname{ch}\{\frac{q}{(1-z)^n}\}$, $F(z)=\operatorname{sh}\{\frac{q}{(1-z)^n}\}$ $F(z)=\cos\{\frac{q}{(1-z)^n}\}$ and $F(z)=\sin\{\frac{q}{(1-z)^n}\}$ are of bounded l-index with $l(|z|)=\frac{\beta}{(1-|z|)^{n+1}}$, $\beta>1$.

We remark that the entire functions e^z, $\operatorname{ch} z$, $\operatorname{ch} z$, $\cos z$ and $\sin z$ are of bounded index and satisfy a differential equation of the form $w''+aw=0$. S.M. Shah ([13]) considered a more general differential equation

$$z^2w''+(\beta_0z^2+\beta_1z)w'+(\gamma_0z^2+\gamma_1z+\gamma_2)w=0, \tag{5}$$

where β_0, β_1, γ_0, γ_1, γ_2 are constant parameters, and investigated the close-to-convexity of its solutions.

Suppose that an entire function $f(\xi)$ satisfies (5), that is

$$\xi^2f''(\xi)+(\beta_0\xi^2+\beta_1\xi)f'(\xi)+(\gamma_0\xi^2+\gamma_1\xi+\gamma_2)f(\xi)\equiv 0. \tag{6}$$

Let $F(z)=f\{\frac{q}{(1-z)^n}\}$. Since

$$f'\Big\{\frac{q}{(1-z)^n}\Big\}=\frac{(1-z)^{n+1}}{qn}F'(z),\ f''\Big\{\frac{q}{(1-z)^n}\Big\}=\frac{(1-z)^{2n+2}}{q^2n^2}F''(z)-\frac{(n+1)(1-z)^{n+1}}{q^2n^2}F'(z),$$

from (6) we have

$$\frac{q^2}{(1-z)^{2n}}\left(\frac{(1-z)^{2n+2}}{q^2n^2}F''(z)-\frac{(n+1)(1-z)^{n+1}}{q^2n^2}F'(z)\right)+$$
$$+\left(\beta_0\frac{q^2}{(1-z)^{2n}}+\beta_1\frac{q}{(1-z)^n}\right)\frac{(1-z)^{n+1}}{qn}F'(z)+$$
$$+\left(\gamma_0\frac{q^2}{(1-z)^{2n}}+\gamma_1\frac{q}{(1-z)^n}+\gamma_2\right)F(z)\equiv 0,$$

that is

$$F''(z)+\left(\frac{\beta_0nq-(n+1)}{(1-z)^{n+1}}+\frac{\beta_1n}{(1-z)}\right)F'(z)+$$
$$+n^2\left(\gamma_0\frac{q^2}{(1-z)^{2n+2}}+\gamma_1\frac{q}{(1-z)^{n+2}}+\frac{\gamma_2}{(1-z)^2}\right)F(z)\equiv 0.$$

If $l(|z|)=\frac{\beta}{(1-|z|)^{n+1}}$, $\beta>1$, hence we obtain

$$\frac{|F''(z)|}{l^2(|z|)}\le\frac{|nq\beta_0-n-1|+n|\beta_1|}{(1-|z|)^{n+1}l(|z|)}\frac{|F'(z)|}{l(|z|)}+n^2\frac{|\gamma_0q^2|+|\gamma_1q|+|\gamma_2|}{(1-|z|)^{2n+2}l^2(|z|)}|F(z)|\le$$
$$\le\left(\frac{|nq\beta_0-n-1|+n|\beta_1|}{\beta}+\frac{n^2(|\gamma_0q^2|+|\gamma_1q|+|\gamma_2|)}{\beta^2}\right)\max\left\{\frac{|F'(z)|}{l(|z|)},|F(z)|\right\}$$

that is by Lemma 2 F is of bounded l-index. We remark also that from (6) it follows that f is of bounded index. Indeed, for $|\xi|\ge 1$

$$|f''(\xi)|\le(|\beta_0|+|\beta_1|)|f'(\xi)|+(|\gamma_0|+|\gamma_1|+|\gamma_2|)|f(\xi)|\le$$
$$\le(|\beta_0|+|\beta_1|+|\gamma_0|+|\gamma_1|+|\gamma_2|)\max\{|f'(\xi)|,|f(\xi)|\},$$

that is by Hayman's theorem [14] (see also Theorem 1.5 from [1] with $l(|z|)\equiv 1$) f is of bounded index in $\mathbb{C}\setminus\mathbb{D}$ and, thus [1, p. 32], f is of bounded index.

In view of the results given above we can propound following conjecture.

Conjecture 1. *For an entire function f the function $F(z) = f(\frac{q}{(1-z)^n})$, $n \in \mathbb{N}$, is of bounded l-index with $l(|z|) = \frac{\beta}{(1-|z|)^{n+1}}$, $\beta > 1$, if and only if f is of bounded index.*

Finally, we remark that there is a number of works [15-17] devoted to entire solutions of the differential equation (5). Their main results are estimates of the index with additional conditions on the parameters.

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Received 09.09.2016