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# COMPOSITION OF ENTIRE FUNCTIONS AND BOUNDED $L$-INDEX IN DIRECTION

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In the present paper we give an answer to the following question: Let $f\colon \mathbb{C}\to\mathbb{C}$ be an entire function of bounded $l$-index, $\Phi\colon \mathbb{C}^n\to\mathbb{C}$ be an entire function, $n\geq 2$, $l\colon \mathbb{C}\to\mathbb{R}_+$ be a continuous function. What are a positive continuous function $L\colon \mathbb{C}^n\to\mathbb{R}_+$ and a direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ such that the composite function $f(\Phi(z))$ has bounded $L$-index in the direction $\mathbf{b}$?

**1. Introduction.** The present paper is devoted to the theory of entire functions of bounded $L$-index in direction. We need some notations and definitions. Let $L\colon \mathbb{C}^n\to\mathbb{R}_+$ be any fixed continuous function. An entire function $F(z)$, $z\in\mathbb{C}^n$, is called a *function of bounded $L$-index in a direction* $\mathbf{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ ( [1,3,4,7]), if there exists $m_0\in\mathbb{Z}_+$ such that for every $m\in\mathbb{Z}_+$ and every $z\in\mathbb{C}^n$

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial\mathbf{b}^m}\right|\leq\max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right|:0\leq k\leq m_0\right\},\tag{1}$$

where $\frac{\partial^0 F(z)}{\partial\mathbf{b}^0}:=F(z)$, $\frac{\partial F(z)}{\partial\mathbf{b}}:=\sum_{j=1}^n\frac{\partial F(z)}{\partial z_j}b_j=\langle\mathbf{grad}F,\overline{\mathbf{b}}\rangle$, $\frac{\partial^k F(z)}{\partial\mathbf{b}^k}:=\frac{\partial}{\partial\mathbf{b}}\big(\frac{\partial^{k-1}F(z)}{\partial\mathbf{b}^{k-1}}\big)$, $k\geq 2$. In the case $n=1$ and $\mathbf{b}=1$ we obtain the definition of entire function of one variable of bounded $l$-index (see [14,17,18]); in the case $n=1$, $\mathbf{b}=1$ and $L(z)\equiv 1$ it is reduced to the definition of function of bounded index, supposed by B. Lepson ( [15]).

Despite numerous papers about functions of bounded index (see bibliography in [4,18]), the index boundedness of composition of entire and analytic functions is only considered in four of them ( [12,13,16,18]). In paper [16], there was investigated boundedness of $l$-index of the composition $f(P(z))$, where $f$ is an entire function and $P$ is a polynomial. In paper [12], there are found conditions, which provide boundedness of $l$-index of the function $f(w(z))$, where $f$ is an analytic function in the unit disc $\mathbb{D}=\{z\in\mathbb{C}\colon\ |z|<1\}$, $w(z)=\frac{z-z_0}{1-z\overline{z}_0}e^{i\alpha}$, $z_0\in\mathbb{D}$, $\alpha\in\mathbb{R}$. The most general result of such type is obtained in [13] for the composition of analytic functions in arbitrary domains from $\mathbb{C}$. Also M.M. Sheremeta ( [18, p. 99]) proved that an entire function $f(z)$ has bounded index if and only if the analytic function $f(\frac{1}{z})$ in $\mathbb{C}\setminus\{0\}$ has bounded $l$-index with $l(z)=\frac{1}{|z|^2}$.

However, the multidimensional case ( [2]) is investigated only for the composite function $f\big(\sum_{j=1}^n z_jm_j\big)$, where $m=(m_1,\ldots,m_n)\in\mathbb{C}^n$ is fixed. Thus, the following question arises:

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*Let $f\colon \mathbb{C}\to\mathbb{C}$ be an entire function of bounded $l$-index, $\Phi\colon \mathbb{C}^n\to\mathbb{C}$ be an entire function, $l\colon \mathbb{C}\to\mathbb{R}_+$ be a continuous function. What are a positive continuous function $L$ and a direction $\mathbf{b}\in\mathbb{C}^n$ such that the composite function $f(\Phi(z))$ has bounded $L$-index in the direction $\mathbf{b}$?*

In the present paper, we give an answer to this question.

Note that the positivity and continuity of the function $L$ are weak restrictions. Therefore, we impose additional restriction on the function $L$.

For $\eta>0$, $z\in\mathbb{C}^n$, $\mathbf{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ and a positive continuous function $L\colon\mathbb{C}^n\to\mathbb{R}_+$ we define

$$\lambda(\eta)=\sup_{z\in\mathbb{C}^n}\sup_{t_1,t_2\in\mathbb{C}}\left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})}\colon |t_1-t_2|\le\frac{\eta}{\min\{L(z+t_1\mathbf{b}),L(z+t_2\mathbf{b})\}}\right\}.$$

By $Q^n_{\mathbf{b}}$ we denote the class of functions $L$ such that $\lambda(\eta)$ is finite for any $\eta>0$. We also use notation $Q=Q^1_1$ for a class of positive continuous function $l(z)$, when $z\in\mathbb{C}$, $\mathbf{b}=1$, $n=1$, $L\equiv l$.

Our main result is following

**Theorem 1.** *Let $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, $f$ be an entire function in $\mathbb{C}$, $\Phi$ be an entire function in $\mathbb{C}^n$ such that $\frac{\partial\Phi(z)}{\partial\mathbf{b}}\neq 0$ and*

$$\left|\frac{\partial^j\Phi(z)}{\partial\mathbf{b}^j}\right|\le K\left|\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right|^j,\quad K\equiv\text{const}>0,\tag{2}$$

*for all $z\in\mathbb{C}^n$ and every $j\le p$, where $p$ is defined in* (3).

*Let $l\in Q$, $l(w)\ge 1$, $w\in\mathbb{C}$ and $L\in Q^n_{\mathbf{b}}$, where $L(z)=\left|\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right|l(\Phi(z))$. The entire function $f$ has bounded $l$-index if and only if $F(z)=f(\Phi(z))$ has bounded $L$-index in the direction $\mathbf{b}$.*

In paper [13], there was obtained a similar proposition for analytic functions of one variable in an arbitrary domain in the complex plane.

**2. Proof of Main Theorem.** To prove the main theorem we need auxiliary propositions.

**Theorem 2** ([1, 4])**.** *Let $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ and $L\in Q^n_{\mathbf{b}}$. An entire function $F(z)$ has bounded $L$-index in direction $\mathbf{b}$ if and only if there exist numbers $p\in\mathbb{Z}_+$, $R>0$ and $C>0$ such that for each $z\in\mathbb{C}^n$, $|z|\ge R$,*

$$\left|\frac{1}{L^{p+1}(z)}\frac{\partial^{p+1}F(z)}{\partial\mathbf{b}^{p+1}}\right|\le C\max\left\{\left|\frac{1}{L^k(z)}\frac{\partial^kF(z)}{\partial\mathbf{b}^k}\right|\colon\ 0\le k\le p\right\}.\tag{3}$$

In previous investigations of bounded $L$-index in direction ([1, 3, 4, 7]) there was used another equivalent definition of class $Q^n_{\mathbf{b}}$. The presented definition appeared for the first time in [6]. For $n=1$ Theorem 2 is Sheremeta's result ([16]). W. K. Hayman [10] proved Theorem 2 for entire functions of bounded index. Analogs of the Hayman Theorem are also known for other classes of holomorphic functions of bounded index ([5, 8, 9, 11]).

*Proof of Theorem 1.* At first, we prove that

$$\frac{\partial^kF(z)}{\partial\mathbf{b}^k}=f^{(k)}(\Phi(z))\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^k+\sum_{j=1}^{k-1}f^{(j)}(\Phi(z))Q_{j,k}(z),\tag{4}$$

where

$$Q_{j,k}(z)=\sum_{\substack{n_1+2n_2+\ldots+kn_k=k\\ 0\le n_1\le j-1}} c_{j,k,n_1,\ldots,n_k}\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{n_1}\left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{n_2}\ldots\left(\frac{\partial^k\Phi(z)}{\partial\mathbf{b}^k}\right)^{n_k},$$

and $c_{j,k,n_1,\ldots,n_k}$ are some non-negative integer coefficients. We also deduce that

$$f^{(k)}(\Phi(z))=\frac{\frac{\partial^k F(z)}{\partial\mathbf{b}^k}}{\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^k}+\frac{1}{\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{2k}}\sum_{j=1}^{k-1}\frac{\partial^j F(z)}{\partial\mathbf{b}^j}\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^j Q^*_{j,k}(z), \tag{5}$$

where

$$Q^*_{j,k}(z)=\sum_{m_1+2m_2+\ldots+km_k=2(k-j)} b_{j,k,m_1,\ldots,m_k}\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{m_1}\left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{m_2}\ldots\left(\frac{\partial^k\Phi(z)}{\partial\mathbf{b}^k}\right)^{m_k},$$

and $b_{j,k,m_1,\ldots,m_k}$ are some integer coefficients.

The validity of (4) and (5) will be checked by the method of mathematical induction. Obviously, for $k=1$ equalities (4) and (5) hold. Assume that they are true for $k=s$. Let us to prove them for $k=s+1$. Evaluate directional derivative in (4)

$$\frac{\partial^{s+1}F(z)}{\partial\mathbf{b}^{s+1}}=f^{(s+1)}(\Phi(z))\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{s+1}+sf^{(s)}(\Phi(z))\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{s-1}\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}+$$
$$+\sum_{j=1}^{s-1}\left(f^{(j+1)}(\Phi(z))\frac{\partial\Phi(z)}{\partial\mathbf{b}}Q_{j,s}(z)+f^{(j)}(\Phi(z))\frac{\partial Q_{j,s}(z)}{\partial\mathbf{b}}\right)=$$
$$=f^{(s+1)}(\Phi(z))\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{s+1}+f^{(s)}(\Phi(z))\left(s\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{s-1}\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}+\frac{\partial\Phi(z)}{\partial\mathbf{b}}Q_{s-1,s}(z)\right)+$$
$$+\sum_{j=2}^{s-1}f^{(j)}(\Phi(z))\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}Q_{j-1,s}(z)+\frac{\partial Q_{j,s}(z)}{\partial\mathbf{b}}\right)+f'(\Phi(z))\frac{\partial Q_{1,s}(z)}{\partial\mathbf{b}}.$$

Since

$$s\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{s-1}\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}+$$
$$+\sum_{\substack{n_1+2n_2+\ldots+sn_s=s\\ 0\le n_1\le s-2}} c_{s-1,s,n_1,\ldots,n_s}\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{n_1+1}\left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)\ldots\left(\frac{\partial^s\Phi(z)}{\partial\mathbf{b}^s}\right)^{n_s}=$$
$$=\sum_{\substack{m_1+2m_2+\ldots+sm_s=s+1\\ 0\le m_1\le s-1}}\tilde{c}_{s,s+1,m_1,\ldots,m_s}\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{m_1}\left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{m_2}\ldots\left(\frac{\partial^s\Phi(z)}{\partial\mathbf{b}^s}\right)^{m_s}=Q_{s,s+1}(z),$$

$$\frac{\partial Q_{1,s}(z)}{\partial\mathbf{b}}=\sum_{2n_2+\ldots+sn_s=s}c_{1,s,0,n_2,\ldots,n_s}\left(n_2\left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{n_2-1}\left(\frac{\partial^3\Phi(z)}{\partial\mathbf{b}^3}\right)^{n_3+1}\ldots\left(\frac{\partial^s\Phi(z)}{\partial\mathbf{b}^s}\right)^{n_s}+\right.$$
$$\left.+\ldots+n_s\left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{n_2}\left(\frac{\partial^3\Phi(z)}{\partial\mathbf{b}^3}\right)^{n_3}\ldots\left(\frac{\partial^s\Phi(z)}{\partial\mathbf{b}^s}\right)^{n_s-1}\frac{\partial^{s+1}\Phi(z)}{\partial\mathbf{b}^{s+1}}\right)=$$

$$= \sum_{2m_2+\ldots+(s+1)m_{s+1}=s+1} \tilde{c}_{1,s+1,0,m_2,\ldots,m_{s+1}} \left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{m_2} \ldots \left(\frac{\partial^s\Phi(z)}{\partial\mathbf{b}^s}\right)^{m_s} \left(\frac{\partial^{s+1}\Phi(z)}{\partial\mathbf{b}^{s+1}}\right)^{m_{s+1}} =$$

$$= Q_{1,s+1}(z),$$

$$\frac{\partial\Phi(z)}{\partial\mathbf{b}} Q_{j-1,s}(z) + \frac{\partial Q_{j,s}(z)}{\partial\mathbf{b}} =$$

$$= \sum_{\substack{n_1+2n_2+\ldots+sn_s=s\\ 0\le n_1\le j-2}} c_{j-1,s,n_1,\ldots,n_s} \left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{n_1+1} \left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{n_2} \ldots \left(\frac{\partial^s\Phi(z)}{\partial\mathbf{b}^s}\right)^{n_s} +$$

$$+ \sum_{\substack{n_1+2n_2+\ldots+kn_s=s\\ 0\le n_1\le j-1}} c_{j,s,n_1,n_2,\ldots,n_s} \left( n_1 \left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{n_1-1} \left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{n_2+1} \ldots \left(\frac{\partial^s\Phi(z)}{\partial\mathbf{b}^s}\right)^{n_s} + \right.$$

$$\left. + \ldots + n_s \left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{n_1} \left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{n_2} \ldots \left(\frac{\partial^s\Phi(z)}{\partial\mathbf{b}^s}\right)^{n_s-1} \frac{\partial^{s+1}\Phi(z)}{\partial\mathbf{b}^{s+1}} \right) =$$

$$= \sum_{\substack{m_1+2m_2+\ldots+(s+1)m_{s+1}=s+1\\ 0\le m_1\le j-1}} \tilde{c}_{j,s+1,m_1,\ldots,m_{s+1}} \left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{n_1} \ldots \left(\frac{\partial^s\Phi(z)}{\partial\mathbf{b}^s}\right)^{n_s} \left(\frac{\partial^{s+1}\Phi(z)}{\partial\mathbf{b}^{s+1}}\right)^{n_{s+1}} = Q_{j,s+1}(z),$$

we obtain (4) with $s+1$ instead $k$.

By the mathematical induction as (4) it can be proved equality (5).

Let $f$ be entire function of bounded $l$-index. By Theorem 2 inequality (3) is valid for $n=1$, $F=f$, $L=l$, $\mathbf{b}=1$. Taking into account (2) and (4), for $k=p+1$ we have

$$\frac{1}{L^{p+1}(z)} \left|\frac{\partial^{p+1}F(z)}{\partial\mathbf{b}^{p+1}}\right| \le \frac{|f^{(p+1)}(\Phi(z))|}{L^{p+1}(z)} \left|\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right|^{p+1} + \sum_{j=1}^{p} \frac{|f^{(j)}(\Phi(z))||Q_{j,p+1}(z)|}{L^{p+1}(z)} \le$$

$$\le \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:\ 0\le k\le p\right\} \left(C + \sum_{j=1}^{p} \frac{|Q_{j,p+1}(z)|}{l^{p+1-j}(\Phi(z))|\frac{\partial\Phi(z)}{\partial\mathbf{b}}|^{p+1}}\right) \le$$

$$\le \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:\ 0\le k\le p\right\} \times$$

$$\times \left(C + \sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} c_{j,p+1,n_1,\ldots,n_{p+1}} \frac{\left|\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{n_1} \left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{n_2} \ldots \left(\frac{\partial^{p+1}\Phi(z)}{\partial\mathbf{b}^{p+1}}\right)^{n_{p+1}}\right|}{l^{p+1-j}(\Phi(z)) \left|\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right|^{p+1}}\right) \le$$

$$\le \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:\ 0\le k\le p\right\} \left(C + \sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} \frac{c_{j,p+1,n_1,\ldots,n_{p+1}} K^{p+1}}{l^{p+1-j}(\Phi(z))}\right) \le$$

$$\le C_1 \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))} : 0\le k\le p\right\}.$$

Using equality (5), we can estimate the fraction $\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}$ :

$$\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))} \le \frac{|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}|}{l^k(\Phi(z))|\frac{\partial\Phi(z)}{\partial\mathbf{b}}|^k} + \sum_{j=1}^{k-1} \frac{|\frac{\partial^j F(z)}{\partial\mathbf{b}^j}||Q^*_{j,k}(z)|}{l^k(\Phi(z))|\frac{\partial\Phi(z)}{\partial\mathbf{b}}|^{2k-j}} \le$$

$$\le \max\left\{\frac{1}{L^j(z)}\left|\frac{\partial^j F(z)}{\partial \mathbf{b}^j}\right| : 1\le j\le k\right\}\left(1+\sum_{j=1}^{k-1}\frac{|Q^*_{j,k}(z)|}{l^{k-j}(\Phi(z))|\frac{\partial\Phi(z)}{\partial\mathbf{b}}|^{2(k-j)}}\right)\le$$

$$\le \max\left\{\frac{1}{L^j(z)}\left|\frac{\partial^j F(z)}{\partial \mathbf{b}^j}\right| : 1\le j\le k\right\}\times$$

$$\times\left(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\ldots,m_k}|\frac{|\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{m_1}\left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{m_2}\ldots\left(\frac{\partial^k\Phi(z)}{\partial\mathbf{b}^k}\right)^{m_k}|}{l^{k-j}(\Phi(z))|\frac{\partial\Phi(z)}{\partial\mathbf{b}}|^{2(k-j)}}\right)\le$$

$$\le \max\left\{\frac{1}{L^j(z)}\left|\frac{\partial^j F(z)}{\partial \mathbf{b}^j}\right| : 1\le j\le k\right\}\left(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}\frac{|b_{j,k,m_1,\ldots,m_k}|K^k}{l^{k-j}(\Phi(z))}\right)\le$$

$$\le C_2\max\left\{\frac{1}{L^j(z)}\left|\frac{\partial^j F(z)}{\partial \mathbf{b}^j}\right| : 1\le j\le k\right\}.$$

Hence, it follows that

$$\frac{1}{L^{p+1}(z)}\left|\frac{\partial^{p+1}F(z)}{\partial\mathbf{b}^{p+1}}\right|\le C_1C_2\max\left\{\frac{1}{L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right| :\ 0\le k\le p\right\}. \tag{6}$$

Therefore, by Theorem 2 inequality (6) means that the function $F$ has bounded $L$-index in the direction $\mathbf{b}$.

Conversely, suppose that $F$ is a function of bounded $L$-index in the direction $\mathbf{b}$. Then it satisfies (3). In view of (2) and (5), we obtain

$$\frac{|f^{(p+1)}(\Phi(z))|}{l^{p+1}(\Phi(z))}\le\frac{|\frac{\partial^{p+1}F(z)}{\partial\mathbf{b}^{p+1}}|}{l^{p+1}(\Phi(z))|\frac{\partial\Phi(z)}{\partial\mathbf{b}}|^{p+1}}+\sum_{j=1}^{p}\frac{|\frac{\partial^j F(z)}{\partial\mathbf{b}^j}||Q^*_{j,p+1}(z)|}{l^{p+1}(\Phi(z))|\frac{\partial\Phi(z)}{\partial\mathbf{b}}|^{2p+2-j}}\le$$

$$\le \max\left\{\frac{1}{L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right| : 0\le k\le p\right\}\left(C+\sum_{j=1}^{p}\frac{|Q^*_{j,p+1}(z)|}{l^{p+1-j}(\Phi(z))|\frac{\partial\Phi(z)}{\partial\mathbf{b}}|^{2(p+1-j)}}\right)\le$$

$$\le \max\left\{\frac{1}{L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right| : 0\le k\le p\right\}\times$$

$$\times\left(C+\sum_{j=1}^{p}\sum_{\substack{m_1+\ldots+(p+1)m_{p+1}=\\=2(p+1-j)}}|b_{j,p+1,m_1,\ldots,m_{p+1}}|\frac{\left|\left(\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right)^{m_1}\left(\frac{\partial^2\Phi(z)}{\partial\mathbf{b}^2}\right)^{m_2}\ldots\left(\frac{\partial^{p+1}\Phi(z)}{\partial\mathbf{b}^{p+1}}\right)^{m_{p+1}}\right|}{l^{p+1-j}(\Phi(z))\left|\frac{\partial\Phi(z)}{\partial\mathbf{b}}\right|^{2(p+1-j)}}\right)\le$$

$$\le \max\left\{\frac{1}{L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right| : 0\le k\le p\right\}\left(C+\sum_{j=1}^{p}\sum_{\substack{m_1+\ldots+(p+1)m_{p+1}=\\=2(p+1-j)}}\frac{|b_{j,p+1,m_1,\ldots,m_{p+1}}|K^{2p+2-2j}}{l^{p+1-j}(\Phi(z))}\right)\le$$

$$\le C_3\max\left\{\frac{1}{L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right| : 0\le k\le p\right\}.$$

According to equality (4) we estimate

$$\frac{1}{L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right|\le\frac{|f^{(k)}(\Phi(z))||\varphi'(z)|^k}{L^k(z)}+\sum_{j=1}^{k-1}\frac{|f^{(j)}(\Phi(z))||Q_{j,k}(z)|}{L^k(z)}\le$$

$$\leq \max\left\{\frac{|f^{(j)}(\Phi(z))|}{l^j(\Phi(z))} : 1 \leq j \leq k\right\}\left(1+\sum_{j=1}^{k-1}\frac{|Q_{j,k}(z)|}{l^{k-j}(\Phi(z))|\frac{\partial\Phi(z)}{\partial\mathbf{b}}|^k}\right)\leq$$

$$\leq C_4\max\left\{\frac{|f^{(j)}(\Phi(z))|}{l^j(\Phi(z))} : 1 \leq j \leq k\right\}.$$

It implies that $\frac{|f^{(p+1)}(\Phi(z))|}{l^{p+1}(\Phi(z))} \leq C_3C_4\max\left\{\frac{|f^{(j)}(\Phi(z))|}{l^j(\Phi(z))} : 0 \leq j \leq p\right\}$.

Thus, by Theorem 2 ($n=1$, $F=f$, $L=l$, $\mathbf{b}=1$) the function $f$ has bounded $l$-index. □

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