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PFLUGER-TYPE THEOREM FOR FUNCTIONS OF REFINED REGULAR GROWTH

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Without a priori assumptions on zero distribution we prove that if an entire function f of noninteger order ρ has an asymptotic of the form $\log|f(re^{i\theta})| = r^\rho h_f(\theta) + O(\frac{r^\rho}{\delta(r)})$, $E \not\ni re^{i\theta} \to \infty$, where h is the indicator of f, δ is an unbounded regularly growing function, and E is an appropriate exceptional set, then the counting function of zeros and the integrated counting function of zeros in the angle $\{z : \alpha < \arg z < \beta\}$ have similar asymptotic for almost all $\alpha < \beta$. It complements results on functions of completely regular growth due to P. Agranovich and V. Logvinenko, B. Vynnyts'kyi and R. Khats'.

1. Introduction and the main result. A function $V\colon (0,\infty) \to (0,\infty)$ is called *regularly varying* with the exponent $\rho \ge 0$ ([8]) if $V(cr) \sim c^\rho V(r)$ $(r \to \infty)$, $c \in (0,\infty)$. If $\rho(r)$ is a proximate order ([11]), $\rho(r) \to \rho$, $r \to \infty$, the function $V(r) = r^{\rho(r)}$ is increasing and regularly varying with the exponent ρ. Conversely, given a regularly varying function V satisfying $V(r) \to \infty$ as $r \to \infty$, there exists a proximate order $\rho(r)$ such that $V(r) \sim r^{\rho(r)}$ as $r \to \infty$. We denote $D(z,r) = \{\zeta : |\zeta - z| < r\}$.

Let f be an entire function of proximate order ρ. The function

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r^{\rho(r)}}$$

is called the *indicator* of f. The indicator is a ρ-trigonometrically convex function (being a constant for $\rho = 0$), see [11].

An entire function f is called an *entire function of completely regular growth (CRG)* ([11]) if

$$\log|f(re^{i\theta})| = h_f(\theta) r^{\rho(r)} + \varepsilon(re^{i\theta}) r^{\rho(r)},$$

where $\varepsilon(re^{i\theta})$ tends to 0 uniformly outside E as $re^{i\theta} \to \infty$, and E is a C_0-*set*, i.e.

$$E \subset \bigcup_k D(z_k, r_k), \quad \text{where} \sum_{|z_k|\le r} r_k = o(r), \quad r \to \infty.$$

A set $E \subset \mathbb{C}$ is called a C_0^α-*set*, $0 < \alpha \le 2$ if

$$E \subset \bigcup_k D(z_k, r_k), \quad \sum_{|z_k|\le r} r_k^\alpha = o(r^\alpha), \quad r \to \infty.$$

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Let $Z_f = \{z : f(z) = 0\}$ be the zero set of f,

$$n(r, \alpha, \beta) = \#\{c_k \in Z_f : \alpha < \arg c_k \le \beta, |c_k| \le r\},\ 0 \le \alpha < \beta \le 2\pi,$$

be the counting function of zeros Z_f in the angle $\{z : \alpha < \arg z \le \beta, |z| \le r\}$.

The set Z_f is said to have the *angular density* for the exponent $\rho(r)$ if for all $\alpha < \beta$, except, perhaps, at most countable set there exists

$$\Delta(\alpha, \beta) = \lim_{r\to\infty} \frac{n(r, \alpha, \beta)}{r^{\rho(r)}}.$$

Theorem A (Main theorem of the theory of CRG functions [11]). *An entire function f of proximate order $\rho(r)$, $\rho(r) \to \rho \in (0, \infty) \setminus \mathbb{N}$ has completely regular growth if and only if Z_f has the angular density for the exponent $\rho(r)$.*

There is also a description of CRG functions of an integer order, those functions should satisfy an additional Lindelöf's condition ([11]). Here we restrict ourselves to the case of noninteger ρ for simplicity.

Remark 1. V. Azarin obtained a counterpart of the theory of CRG functions for subharmonic functions in $\mathbb{C}$ ([5]).

Remark 2. A. Kondratyuk generalized the theory of CRG functions for meromorphic functions in $\mathbb{C}$ ([10]).

P. Agranovich and V. Logvinenko considered the relation between the asymptotics for zeros of the form

$$n(r, \alpha, \beta) = \Delta_1(\alpha, \beta) r^{\rho_1} + \Delta_2(\alpha, \beta) r^{\rho_2} + \varphi(r, \alpha, \beta) \tag{1}$$

and the asymptotics for $\log |f|$ of the form

$$\log |f(re^{i\theta})| = H_1(\theta) r^{\rho_1} + H_2(\theta) r^{\rho_2} + \psi(r, \theta), \tag{2}$$

where H_j is uniquely defined by Δ_j.

Theorem B (Agranovich, Logvinenko, 1987). *Let $[\rho_1] < \rho_2 < \rho_1$ and (1) hold, where for some $q \ge 1$ and any $T > 0$*

$$\int_\alpha^{\alpha+T} d\beta \int_r^{2r} |\varphi(t, \alpha, \beta)|^q\, dt = o(r^{\rho_2 q+1}), \quad r \to \infty.$$

Then (2) holds with $\psi(r, \theta) = o(r^{\rho_2})$ as $re^{i\theta} \to \infty$ outside some C_0^2-set. And C_0^2-set cannot be replaced by a $C_0^{2-\varepsilon}$-set, $\varepsilon > 0$.

Theorem C (Agranovich, Logvinenko, 1985). *Let either*

$$[\rho_1] < \rho_2 < \rho_1 < [\rho_1] + \frac{1}{2} \quad \text{or} \quad [\rho_1] + \frac{1}{2} < \rho_2 < \rho_1.$$

If all zeros are positive and (2) holds for $\theta = 0$, and for some $q \ge 1$

$$\int_r^{2r} |\psi(t, 0)|^q\, dt = o(r^{\rho_2 q+1}), \quad r \to \infty, \tag{3}$$

then (1) *holds and the reminder term* $\varphi(\cdot, \alpha, \beta)$ *satisfies*

$$\int_r^{2r} |\varphi(t, \alpha, \beta)|^q \, dt = o(r^{\rho_2 q+1}), \quad r \to \infty,$$

uniformly in α *and* β *for every* $q > 1$.

Remark 3. $\psi(t, \theta) = o(t^{\rho_2})$ as $te^{i\theta} \to \infty$ outside some exceptional set does not imply $\varphi(t, \alpha, \beta) = o(t^{\rho_2})$ as $t \to +\infty$.

Remark 4. Agranovich and Logvinenko relaxed slightly the restriction that zeros are located on a ray in [4] (see also [1], [6].)

B. Vynnytskyi and R. Khats' introduced the following concept ([13]).

Definition 1. An entire function f is said to be a *function of refined regular growth* ([13]) if for some $\rho \in (0, \infty)$ there exist $\rho_2 \in (0, \rho)$ and a set $E \subset \bigcup_k D(z_k, r_k)$ with $\sum_k r_k < \infty$ such that

$$\log |f(re^{i\theta})| = r^\rho h_f(\theta) + o(r^{\rho_2}), \quad E \not\ni re^{i\theta} \to \infty. \tag{4}$$

Theorem D (Vynnyts'kyi, Khats', 2005). *Let* f *be an entire function of noninteger order* ρ *with zeros on a ray (a finite system of rays). The function* f *is of refined regular growth if and only if there exist* $\rho_1 \in (0, \rho)$ *and* $\Delta \geq 0$ *such that*

$$n(t) := n(t, 0, 2\pi) = \Delta t^\rho + o(t^{\rho_1}), \quad t \to \infty. \tag{5}$$

This result has the following disadvantages. Firstly, the restriction on zero location is very strong. Secondly, it is not clear how ρ_1 and ρ_2 are connected, though one can try to find this connection following the proof from [13].

One can consider equalities (4) and (5) as closeness of the subharmonic functions $\log |f(re^{i\theta})|$ and $r^\rho h_f(\theta)$ and the counting functions of their Riesz measures, respectively. Necessary and sufficient conditions for the relation

$$u_1(z) - u_2(z) = O(|z|^\sigma), \quad \sigma \geq 0, z \notin E,$$

for subharmonic functions of order ρ and an exceptional set E were established by R. Yulmukhametov [14]. B. Khabibullin indicated ([9]) that for integer σ sufficiency of Yulmukhametov's theorem fails to hold and gave another sufficient conditions providing $u_1(z) - u_2(z) = O(|z|^\sigma \log |z|)$, $\sigma \geq 0$, $z \notin E$.

We note that exceptional sets in the results of Yulmukhametov, Khabibullin, Vynnyts'kyi and Khats' are essentially smaller than C_0^α-sets, in general. To the best of our knowledge, the first results stating that a function of completely regular growth has regular distribution of zeros are due to A. Pfluger ([12]). We are interested in assertions of such type, that we shall call Pfluger-type theorems, in the case where an entire function has an asymptotic stronger than that of a function of completely regular growth. More precisely, the aim of this paper is to relax the assumption that zeros are located on a finite system of rays. Also we try to control the rate of the error term. Exceptional sets that appear in out result are similar to C_0^α-sets and related to magnitude of the error term.

Theorem 1. *Let f be an entire function of noninteger order ρ, $f(0)\neq 0$, δ be an increasing unbounded regularly varying function with the exponent $\tau\in[0,\min\{1,\rho\})$ and such that*

$$\sum_k \frac{1}{\delta(2^k)}<\infty. \tag{6}$$

Suppose that there exists a set $E\subset\bigcup_k D(a_k,s_k)$ such that

$$\sum_{|a_k|\le r} s_k = O\Big(\frac{r}{\delta(r)}\Big),$$

and

$$\log|f(re^{i\theta})| = r^\rho h_f(\theta)+O\Big(\frac{r^\rho}{\delta(r)}\Big), \quad E\not\ni re^{i\theta}\to\infty.$$

Then for almost all $\{\alpha,\beta\}\subset[0,2\pi]$, $\alpha<\beta$

$$N(r,\alpha,\beta):=\int_0^r \frac{n(t,\alpha,\beta)}{t}\,dt=\frac{s_f(\alpha,\beta)}{2\pi\rho^2}r^\rho+O\Big(\frac{r^\rho}{\sqrt{\delta(r)}}\Big), \tag{7}$$

$$n(r,\alpha,\beta)=\frac{s_f(\alpha,\beta)}{2\pi\rho}r^\rho+O\Big(\frac{r^\rho}{\sqrt[4]{\delta(r)}}\Big), \quad r\to\infty, \tag{8}$$

where

$$s_f(\alpha,\beta):=\tilde s(\beta)-\tilde s(\alpha),\ \ \tilde s(\theta)=h'_+(\theta)+\rho^2\int_0^\theta h(\varphi)d\varphi.$$

Remark 5. The condition (6) could be probably relaxed using the arguments similar to that from [11]. It allows us to choose $\delta(r)=r^\sigma$, $\sigma\in(0,\min\{1,\rho\})$, $\delta(r)=\log^s r$, $s>1$, but not $s\le 1$.

Remark 6. Assumption (6) implies that the function $\delta(r)$ is unbounded.

2. Proof of the theorem.

Proof. Multiplying the function $\delta(r)$ by a constant if necessary, we may assume that

$$\sum_{|a_k|\le r} s_k\le \frac14\frac{r}{\delta(r)}, \quad r\to+\infty. \tag{9}$$

Let F be the radial projection of E, i.e. $F=\{|z|:z\in E\}$. It follows from (9) and properties of the function δ that for all r there exists $r^*\in[r,r+\frac{r}{\delta(r)}]\setminus F$ for r sufficiently large. In fact, suppose on the contrary, that

$$\Big[r,r+\frac{r}{\delta(r)}\Big]\subset F=\bigcup_k[|a_k|-s_k,|a_k|+s_k].$$

Note that $[|a_k|-s_k,|a_k|+s_k]\cap[r,r+\frac{r}{\delta(r)}]\neq\varnothing$ implies $a_k-s_k\le r$, and, in view of Remark 6, $|a_k|\le r+s_k=(1+o(1))r$ $(r\to\infty)$. Therefore,

$$\frac{r}{\delta(r)}\le\sum_{k:|a_k|-s_k\le r}2s_k\le\frac{r(1+o(1))}{2\delta(r(1+o(1)))}<\frac{r}{\delta(r)}, \quad r\to\infty.$$

that is a contradiction.

Hence we can choose a sequence (r_k) with the properties: $r_k \uparrow \infty$ $(k \to \infty)$,

$$r_k \notin F, \quad r_{k+1} \le r_k\Big(1 + \frac{1}{\delta(r_k)}\Big). \tag{10}$$

Lemma 1. *For almost all $\varphi \in [0, 2\pi)$ one has*

$$\log|f(re^{i\varphi})| = r^\rho h(\varphi) + O\Big(\frac{r^\rho}{\delta(r)}\Big), \quad r \to \infty \tag{11}$$

uniformly in φ provided that (6) *holds.*

Proof of Lemma 1. It follows from (9) and (6) that

$$\sum_{k=1}^{\infty} \sum_{2^{k-1} \le |a_j| < 2^k} \frac{s_j}{|a_j|} \le \sum_{k=1}^{\infty} \sum_{2^{k-1} \le |a_j| < 2^k} \frac{2 s_j}{2^k} < \sum_{k=1}^{\infty} \frac{1}{2} \frac{1}{\delta(2^k)} < \infty.$$

Then, for arbitrary $\varepsilon > 0$ there exists $R_0 > 0$ such that the angular measure circular projection of $E \cap \{z : |z| \ge R_0\}$ is smaller than ε, i.e. for all θ there exists φ such that $|\varphi - \theta| < \varepsilon/2$ and $E \cap \{re^{i\varphi} : r \ge R_0\} = \varnothing$. Thus (11) holds on the ray $\{re^{i\varphi} : r \ge R_0\}$. □

Lemma 2. *For all $\varphi \in [0, 2\pi)$ one has*

$$\log|f(re^{i\varphi})| \le r^\rho h(\varphi) + O\Big(\frac{r^\rho}{\delta(r)}\Big), \quad r \to \infty,$$

uniformly in φ.

Proof of Lemma 2. By (9) for every $z = re^{i\theta}$ there exist $\tau \in (0, \frac{r}{\delta(r)})$ such that

$$\partial D(z, \tau) = \{w : |w - z| = \tau\} \cap E = \varnothing.$$

Choose $\tilde{w}$ satisfying $|\tilde{w} - z| = \tau$ and $|f(\tilde{w})| = \max\{|f(w)| : w \in \partial D(z, \tau)\}$. By the maximum modulus principle

$$\begin{aligned} \ln|f(re^{i\theta})| &\le \ln|f(\tilde{w})| = |\tilde{w}|^\rho h(\tilde{\theta}) + O\Big(\frac{|\tilde{w}|^\rho}{\delta(|\tilde{w}|)}\Big) \le \\ \le \Big(r + \frac{r}{\delta(r)}\Big)^\rho h(\tilde{\theta}) &+ O\Big(\frac{r^\rho}{\delta(r)}\Big) \le r^\rho\Big(1 + \frac{1}{\delta(r)}\Big)^\rho h(\tilde{\theta}) + O\Big(\frac{r^\rho}{\delta(r)}\Big) = \\ &= r^\rho h(\theta) + r^\rho(h(\tilde{\theta}) - h(\theta)) + O\Big(\frac{r^\rho}{\delta(r)}\Big), \quad r \to \infty. \end{aligned}$$

By [10, Lemma 8.1], h satisfies the Lipschitz condition, thus $|h(\tilde{\theta}) - h(\theta)| \le \frac{K}{\delta(r)}$. Finally,

$$\ln|f(re^{i\theta})| \le r^\rho h(\theta) + O\Big(\frac{r^\rho}{\delta(r)}\Big), \quad r \to \infty.$$

□

Lemma 3. *If $f(0)\neq 0$, then for almost all α, β, $\alpha<\beta\le\alpha+2\pi$,*

$$N(r,\alpha,\beta)=\frac{s_f(\alpha,\beta)}{2\pi\rho^2}r^\rho+O\Big(\frac{r^\rho}{\sqrt{\delta(r)}}\Big),\quad r\to\infty.$$

Proof of Lemma 3. Let $\mathcal{R}$ be the set of $\varphi\in[0,2\pi)$ such that (11) holds. It is known that $h'(\theta)$ exists outside at most countable set, moreover the function

$$s(\theta)=h'_+(\theta)+\rho^2\int_{\theta_0}^{\theta}h(\varphi)\,d\varphi$$

is nondecreasing for arbitrary fixed θ_0. Therefore, there exists

$$s'(\theta)=h''(\theta)+\rho^2h(\theta),\quad \theta\in[0,2\pi)\setminus\mathcal{R}_0,$$

where $\mathrm{mes}(\mathcal{R}_0)=0$.

Let $\varphi\in\mathcal{R}\setminus\mathcal{R}_0$, i.e. there exists $h''(\varphi)$ and the assertion of Lemma 1 holds. Note that $\mathrm{mes}(\mathcal{R}\setminus\mathcal{R}_0)=2\pi$. We then follow the arguments from [11, Chapter III] (cf. [13]).

Let $J_f^t(\varphi)=\int_0^t\frac{\ln|f(ue^{i\varphi})|}{u}\,du$. By the generalized Jensen formula ([11, Chap. III])

$$N(r,\alpha,\beta)=\frac{1}{2\pi}\left[\frac{d}{d\varphi}\int_0^r J_f^t(\varphi)\frac{dt}{t}\right]_{\varphi=\beta}-\frac{1}{2\pi}\left[\frac{d}{d\varphi}\int_0^r J_f^t(\varphi)\frac{dt}{t}\right]_{\varphi=\alpha}+ \\ +\frac{1}{2\pi}\int_\alpha^\beta\ln|f(re^{i\varphi})|\,d\varphi. \tag{12}$$

It follows from Lemmas 1, 2, and properties of the function δ that uniformly in $\varphi\in\mathcal{R}$

$$J_f^t(\varphi)=\int_0^t\frac{\ln|f(ue^{i\varphi})|}{u}\,du=\frac{t^\rho}{\rho}h(\varphi)+O\Big(\int_0^t\frac{u^\rho}{\delta(u)}\,du\Big)= \\ =\frac{t^\rho}{\rho}h(\varphi)+O\Big(\frac{t^\rho}{\delta(t)}\Big),\quad t\to\infty. \tag{13}$$

Following the proof of Theorem 3 ([11]) we integrate (12)

$$\frac{1}{q_1q_2}\int_\beta^{\beta+q_1}\int_\alpha^{\alpha+q_2}N(r,\alpha^*,\beta^*)\,d\alpha^*d\beta^*=\frac{1}{2\pi}\int_0^r\frac{J_f^t(\beta+q_1)-J_f^t(\beta)}{q_1}\frac{dt}{t}- \\ -\frac{1}{2\pi}\int_0^r\frac{J_f^t(\alpha+q_2)-J_f^t(\alpha)}{q_1}\frac{dt}{t}+\frac{1}{2\pi q_1q_2}\int_\beta^{\beta+q_1}\int_\alpha^{\alpha+q_2}\int_{\alpha^*}^{\beta^*}\ln|f(re^{i\varphi})|\,d\varphi\,d\alpha^*d\beta^*. \tag{14}$$

Assume that $\alpha,\beta\in\mathcal{R}\setminus\mathcal{R}_0$. Then

$$h(\beta+q)=h(\beta)+h'(\beta)q+\frac{h''(\beta)q^2}{2}+o(q^2),\quad q\to 0; \tag{15}$$

the same holds for α instead of β. Correlations (13) and (15) imply

$$\int_0^r\frac{J_f^t(\beta+q_1)-J_f^t(\beta)}{q_1}\frac{dt}{t}=\int_0^r\frac{t^\rho}{\rho}\frac{h(\beta+q_1)-h(\beta)}{q_1}\frac{dt}{t}+O\Big(\frac{r^\rho}{q_1\delta(r)}\Big)= \\ =\frac{r^\rho}{\rho^2}(h'(\beta)+O(q_1))+O\Big(\frac{r^\rho}{q_1\delta(r)}\Big),\quad r\to\infty, \tag{16}$$

and similarly

$$\int_0^r \frac{J_f^t(\alpha+q_2)-J_f^t(\alpha)}{q_2}\frac{dt}{t}=\frac{r^\rho}{\rho^2}(h'(\alpha)+O(q_2))+O\Big(\frac{r^\rho}{q_2\delta(r)}\Big),\quad r\to\infty, \tag{17}$$

For $r=r_k$ we have

$$\int_{\alpha^*}^{\beta^*}\ln|f(r_ke^{i\varphi})|\,d\varphi=r_k^\rho\int_{\alpha^*}^{\beta^*}h(\varphi)\,d\varphi+O\Big(\frac{r_k^\rho}{\delta(r_k)}\Big),\quad k\to\infty. \tag{18}$$

Then

$$\frac{1}{q_1q_2}\int_\beta^{\beta+q_1}\int_\alpha^{\alpha+q_2}\int_{\alpha^*}^{\beta^*}\ln|f(r_ke^{i\varphi})|\,d\varphi\,d\alpha^*d\beta^*=$$
$$=\frac{r_k^\rho}{q_1q_2}\int_\beta^{\beta+q_1}\int_\alpha^{\alpha+q_2}\int_{\alpha^*}^{\beta^*}h(\varphi)\,d\varphi\,d\alpha^*d\beta^*+O\Big(\frac{r_k^\rho}{\delta(r_k)}\Big)=$$
$$=\frac{r_k^\rho}{q_2}\int_\alpha^{\alpha+q_2}\int_{\alpha^*}^{\beta'}h(\varphi)\,d\varphi\,d\alpha^*+O\Big(\frac{r_k^\rho}{\delta(r_k)}\Big)=r_k^\rho\int_{\alpha'}^{\beta'}h(\varphi)\,d\varphi+O\Big(\frac{r_k^\rho}{\delta(r_k)}\Big)$$

for some α' between α and $\alpha+q_2$, β between β and $\beta+q_1$. Since

$$\Big|\int_\alpha^{\alpha'}h(\varphi)\,d\varphi\Big|\le C|q_2|,\quad \Big|\int_\beta^{\beta'}h(\varphi)\,d\varphi\Big|\le C|q_1|,$$

where C is defined by the indicator, we have

$$\frac{1}{q_1q_2}\int_\beta^{\beta+q_1}\int_\alpha^{\alpha+q_2}\int_{\alpha^*}^{\beta^*}\ln|f(r_ke^{i\varphi})|\,d\varphi d\alpha^*d\beta^*=$$
$$=r_k^\rho\Big(\int_\alpha^\beta h(\varphi)\,d\varphi+O(q_1)+O(q_2)\Big)+O\Big(\frac{r_k^\rho}{\delta(r_k)}\Big),\quad k\to\infty. \tag{19}$$

Substituting (17)–(19) into (14) we obtain

$$\frac{1}{q_1q_2}\int_\beta^{\beta+q_1}\int_\alpha^{\alpha+q_2}N(r_k,\alpha^*,\beta^*)\,d\alpha^*d\beta^*=$$
$$=\frac{r_k^\rho}{2\pi\rho^2}\Big(h'(\beta)-h'(\alpha)+\rho^2\int_\alpha^\beta h(\varphi)\,d\varphi\Big)+$$
$$+O\Big(\frac{r_k^\rho}{q_1\delta(r_k)}\Big)+O\Big(\frac{r_k^\rho}{q_2\delta(r_k)}\Big)+O\Big(r_k^\rho(|q_1|+|q_2|)\Big)=$$
$$=\frac{r_k^\rho}{2\pi\rho}\Big(\tilde{s}(\beta)-\tilde{s}(\alpha)+O(q_1)+O(q_2)+O\Big(\frac{1}{q_1\delta(r_k)}\Big)+O\Big(\frac{1}{q_1\delta(r_k)}\Big)\Big),\quad k\to\infty.$$

Choosing $q_2=-q_1=(\delta(r_k))^{-1/2}$ we deduce

$$N(r_k,\alpha,\beta)\le\delta(r_k)\int\limits_{\beta-(\delta(r_k))^{-1/2}}^{\beta}\ \int\limits_{\alpha}^{\alpha+(\delta(r_k))^{-1/2}}N(r_k,\alpha^*,\beta^*)\,d\alpha^*d\beta^*=$$

$$= \frac{r_k^\rho}{2\pi\rho}\Big(s(\alpha,\beta) + O((\delta(r_k)^{-1/2})\Big).$$

The choice $-q_2 = q_1 = (\delta(r_k))^{-1/2}$ yields the same lower estimate. Therefore

$$N(r_k,\alpha,\beta) = \frac{s(\alpha,\beta) r_k^\rho}{2\pi\rho^2} + O\Big(\frac{r_k^\rho}{\sqrt{\delta(r_k)}}\Big), \quad k\to\infty.$$

Let $r \in [r_k, r_{k+1})$. Then

$$N(r,\alpha,\beta) \le N(r_{k+1},\alpha,\beta) \le \frac{s(\alpha,\beta)\Big(r\Big(1+\frac{1}{\delta(r)}\Big)\Big)^\rho}{2\pi\rho^2} + O\Big(\frac{r^\rho}{\sqrt{\delta(r)}}\Big) =$$
$$= \frac{s(\alpha,\beta) r^\rho}{2\pi\rho^2} + O\Big(\frac{r^\rho}{\sqrt{\delta(r)}}\Big).$$

Lemma 3 and (7) is proved. □

We continue the proof of the theorem. Let $n(r) = n(r,\alpha,\beta)$, $N(r) = N(r,\alpha,\beta)$. We use the following known estimate ([7])

$$n(r)\log\frac{R}{r} \le N(R) - N(r) \le n(R)\log\frac{R}{r}, \quad 1 < r < R. \tag{20}$$

We choose $R = r\Big(1+\frac{1}{\sqrt[4]{\delta(r)}}\Big)$. Then

$$n(r) \le \frac{N\Big(r\Big(1+\frac{1}{\sqrt[4]{\delta(r)}}\Big)\Big) - N(r)}{\log\Big(1+\frac{1}{\sqrt[4]{\delta(r)}}\Big)} =$$
$$= \frac{s(\alpha,\beta)}{2\pi\rho^2}\frac{r^\rho\Big(1+\frac{1}{\sqrt[4]{\delta(r)}}\Big)^\rho + O\Big(\frac{R^\rho}{\sqrt{\delta(R)}}\Big) - r^\rho + O\Big(\frac{r^\rho}{\sqrt{\delta(r)}}\Big)}{\frac{1}{\sqrt[4]{\delta(r)}} + O\Big(\frac{1}{\sqrt{\delta(r)}}\Big)} =$$
$$= \frac{\frac{s(\alpha,\beta)}{2\pi\rho^2}\Big(\frac{\rho}{\sqrt[4]{\delta(r)}}r^\rho + O\Big(\frac{r^\rho}{\sqrt{\delta(r)}}\Big)\Big)}{\frac{1}{\sqrt[4]{\delta(r)}} + O\Big(\frac{1}{\sqrt{\delta(r)}}\Big)} = \frac{\frac{s(\alpha,\beta)r^\rho}{2\pi\rho}\Big(1+O\Big(\frac{1}{\sqrt[4]{\delta(r)}}\Big)\Big)}{1+O\Big(\frac{1}{\sqrt[4]{\delta(r)}}\Big)} =$$
$$= \frac{s(\alpha,\beta)r^\rho}{2\pi\rho}\Big(1+O\Big(\frac{1}{\sqrt[4]{\delta(r)}}\Big)\Big), \quad r\to\infty.$$

Analogously,

$$n(R) \ge \frac{N(R)-N(r)}{\log\frac{R}{r}} = \frac{s(\alpha,\beta)}{2\pi\rho^2}\frac{R^\rho + O\Big(\frac{R^\rho}{\sqrt{\delta(R)}}\Big) - \frac{R^\rho}{(1+\frac{1}{\sqrt[4]{\delta(r)}})^\rho}}{\frac{1}{\sqrt[4]{\delta(r)}} + O\Big(\frac{1}{\sqrt{\delta(r)}}\Big)} =$$
$$= \frac{s(\alpha,\beta)}{2\pi\rho^2}\frac{R^\rho + O\Big(\frac{R^\rho}{\sqrt{\delta(R)}}\Big) - R^\rho\Big(1-\frac{\rho}{\sqrt[4]{\delta(r)}} + O\Big(\frac{1}{\sqrt{\delta(r)}}\Big)\Big)}{\frac{1}{\sqrt[4]{\delta(r)}} + O\Big(\frac{1}{\sqrt{\delta(r)}}\Big)} =$$

$$= \frac{s(\alpha,\beta)R^\rho}{2\pi\rho} \frac{\frac{1}{\sqrt[4]{\delta(r)}} + O\Big(\frac{1}{\sqrt{\delta(r)}}\Big) + O\Big(\frac{1}{\sqrt{\delta(R)}}\Big)}{\frac{1}{\sqrt[4]{\delta(r)}} + O\Big(\frac{1}{\sqrt{\delta(r)}}\Big)} =$$

$$= \frac{s(\alpha,\beta)R^\rho}{2\pi\rho}\Big(1+O\Big(\frac{1}{\sqrt[4]{\delta(r)}}\Big)\Big) = \frac{s(\alpha,\beta)R^\rho}{2\pi\rho}\Big(1+O\Big(\frac{1}{\sqrt[4]{\delta(R)}}\Big)\Big).$$

The latter estimates give the desired asymptotics for the $n(r,\alpha,\beta)$. □

3. Further results. Theorem 1 allows the following generalizations.

1. One may assume that the asymptotic

$$\log|f(re^{i\varphi})| = r^{\rho(r)}h(\varphi) + O\Big(\frac{r^{\rho(r)}}{\delta(r)}\Big), \quad r\to\infty$$

holds for some proximate order $\rho(r)$ instead of a constant function ρ, $\rho(r)\to\rho\in(0,\infty)\setminus\mathbb{N}$, satisfying the additional hypothesis

$$\int_0^r t^{\rho(t)-1}\,dt = \frac{r^{\rho(r)}}{\rho}\Big(1+O\Big(\frac{1}{\sqrt{\delta(r)}}\Big)\Big), \quad r\to\infty$$

and

$$r^{\rho(r(1+\frac{1}{\delta(r)}))} = r^{\rho(r)}\Big(1+O\Big(\frac{1}{\sqrt{\delta(r)}}\Big)\Big), \quad r\to\infty,$$

which are needed in the proof of Lemma 3.

2. The assumption $f(0)\neq 0$ is technical. If it is not fulfilled, then (7) holds with

$$\int_0^r \frac{n(t,\alpha,\beta)-n(0,\alpha,\beta)}{t}\,dt + n(0,\alpha,\beta)\log r$$

instead of $N(r,\alpha,\beta)$.

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