УДК 519.4

T. Banakh, I. Protasov, K. Protasova

RAMSEY-PRODUCT SUBSETS OF A GROUP

T. Banakh, I. Protasov, K. Protasova. *Ramsey-product subsets of a group*, Mat. Stud. **47** (2017), 145–149.

Given an infinite group G and a number vector $\overrightarrow{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ of finite length k, we say that a subset A of G is a *Ramsey $\overrightarrow{m}$-product set* if every infinite subset $X \subset G$ contains distinct elements $x_1, \dots, x_k \in X$ such that $x^{m_1}_{\sigma(1)} \dots x^{m_k}_{\sigma(k)} \in A$ for any permutation $\sigma \in S_k$. We use these subsets to characterize combinatorially some algebraically defined subsets of the Stone-Čech compactification βG of G.

All groups under consideration are supposed to be infinite; a countable set means a countably infinite set.

Let G be a group and let $\overrightarrow{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ be a number vector of length $k \in \mathbb{N}$. We say that a subset A of a group G is a *Ramsey $\overrightarrow{m}$-product subset* if every infinite subset X of G contains pairwise distinct elements $x_1, \dots, x_k \in X$ such that

$$x^{m_1}_{\sigma(1)}\, x^{m_2}_{\sigma(2)} \dots x^{m_k}_{\sigma(k)} \in A$$

for every permutation $\sigma \in S_k$.

Why Ramsey? The answer follows from the proof of (i) in

Proposition 1. *For a group G and a number vector $\overrightarrow{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ the following statements hold:*

(i) *a subset A of G is a Ramsey $\overrightarrow{m}$-product subset if and only if for every infinite subset X contains a countable subset Y such that $y_1^{m_1} \dots y_k^{m_k} \in A$ for any distinct elements $y_1, \dots, y_k \in Y$;*

(ii) *the family $\varphi_{\overrightarrow{m}}$ of all Ramsey $\overrightarrow{m}$-product subsets of G is a filter.*

Proof. (i) The "if" part is trivial. To prove the "only if" part, assume that A is a Ramsey $\overrightarrow{m}$-product subset in G and $X \subset G$ is an infinite set. Define the coloring $\chi : [X]^k \longrightarrow \{0, 1\}$ of the set $[X]^k = \{K \subset X : |K| = k\}$ by the rule: $\chi(\{x_1, \dots, x_k\}) = 1$ if and only if $x^{m_1}_{\sigma(1)} \dots x^{m_k}_{\sigma(k)} \in A$ for every $\sigma \in S_k$.

By the classical Ramsey theorem, there exists a countable subset Y of X such that the set $[Y]^k$ is χ-monochrome. Since A is a Ramsey $\overrightarrow{m}$-product subset, by the definition of χ, there exists $K \in [Y]^k$ such that $\chi(K) = 1$, which implies that $\chi([Y]^k) = \{1\}$.

(ii) We take $A, B \in \varphi_{\overrightarrow{m}}$ and prove that $A \cap B \in \varphi_{\overrightarrow{m}}$. For an infinite subset X, we choose Y given by (i). For B and Y, we choose corresponding $Z \subset Y$ and take distinct $z_1, \dots, z_k \in Z$. Then $z^{m_1}_{\sigma(1)} \dots z^{m_k}_{\sigma(k)} \in A \cap B$ for any $\sigma \in S_n$. □

2010 *Mathematics Subject Classification:* 05D10.
Keywords: Stone-Čech compactification; product of ultrafilters; Ramsey product subset.
doi:10.15330/ms.47.2.145-149

For some specific number vectors, Ramsey $\vec{m}$-product sets have been studied in the literature. In particular, Ramsey $(-1,1)$-product sets are exactly Δ_ω sets from [6], Ramsey $(-1,1)$-product sets containing the unit of the group are exactly ω-fat sets of Sipacheva [7]; Ramsey $(-1,1)$-product sets are similar to Δ^*-sets, studied in [1] (see [7, p.6]).

Now we present some examples and establish some topological properties of Ramsey $\vec{m}$-product sets.

Proposition 2. *For any totally bounded topological group G, any neighborhood $U \subset G$ of the unit e of G is a Ramsey $\vec{m}$-product set for any number vector $\vec{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ with $m_1 + \dots + m_k = 0$.*

Proof. By the continuity of the group operation, there exists an open neighborhood $V \subset G$ of e such that $V^{m_1} \cdots V^{m_k} \subset U$. Since the topological group G is totally bounded, we can additionally assume that the neighborhood V is invariant in the sense that $zV = Vz$ for any $z \in G$. By the total boundedness of G, there exists a finite set $F \subset G$ such that $G = FV$.

To prove that U is a Ramsey $\vec{m}$-product set, take any infinite set $A \subset G = FV$. By the Pigeonhole Principle, for some $z \in F$ the set $A \cap zV$ is infinite. We claim that $x_1^{m_1} \cdots x_k^{m_k} \in U$ for any points $x_1, \dots, x_k \in A \cap zV$. Taking into account that the set V is invariant, we conclude that

$$x_1^{m_1} \cdots x_k^{m_k} \in (zV)^{m_1} \cdots (zV)^{m_k} = z^{m_1 + \dots + m_k} V^{m_1} \cdots V^{m_k} \subset z^0 U = U.$$

□

For the vector $\vec{m} = (-1,1)$, Proposition 2 can be complemented by the following proposition proved in [6, Proposition 5]. We recall that a *quasi-topological group* is a group G endowed with a topology such that for any $a, b \in G$ and $\varepsilon \in \{-1, 1\}$ the map $G \to G$, $x \mapsto a x^\varepsilon b$, is continuous.

Proposition 3. *The closure $\bar{A}$ of any Ramsey $(-1,1)$-product set A in a quasi-topological group G is a neighborhood of the unit.*

On the other hand, Ramsey $\vec{m}$-product sets have the following density property.

Proposition 4. *Let $\vec{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ be a number vector and $s = m_1 + \dots + m_k$. For any Ramsey $\vec{m}$-product subset A of a group G, the set $\{x^s : x \in G\}$ is contained in the closure of A in any non-discrete group topology on G.*

Proof. To derive a contradiction, assume that for some non-discrete group topology τ on G, the closure $\bar{A}$ of A does not contain the power x^s of some element $x \in G$. By the continuity of the group operations, the element x has a neighborhood $U_x \in \tau$ such that $U_x^s \cap A = \varnothing$. Since the topology τ is not discrete, the set U_x is infinite. Moreover, the choice of U_x ensures that $x_1^{m_1} \cdots x_k^{m_k} \notin A$ for any elements $x_1, \dots, x_k \in U_x$, which means that A is not Ramsey $\vec{m}$-product. □

A group G is defined to be *s-divisible* for $s \in \mathbb{Z}$ if for every $g \in G$ there exists $x \in G$ such that $x^s = g$.

Corollary 1. *Let $\vec{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ be a number vector and G be an s-divisible group for $s = m_1 + \dots + m_k$. Then any Ramsey $\vec{m}$-product set $A \subset G$ is dense in any non-discrete group topology τ on G.*

Proposition 2 cannot be reversed as shown by the following example. We recall that a subset S of a group G is called *syndetic* if $G = FS$ for some finite subset $F \subset G$. A topological group G is totally bounded if and only if each neighborhood of the unit is syndetic.

Example 1. Let G be the Boolean group of all finite subsets of $\mathbb{Z}$, endowed with the group operation of symmetric difference (of finite sets). The set

$$A = G \setminus \big\{\{x, y\}\colon x, y \in \mathbb{Z},\ 0 \neq x - y \in \{z^3 \colon z \in \mathbb{Z}\}\big\}$$

has the following properties:

1. A is a Ramsey $\vec{m}$-product set for any vector $\vec{m} = (m_1, \dots, m_k) \in (2\mathbb{Z}+1)^k$ of length $k \geq 2$;
2. A does not contain the difference BB^{-1} of any syndetic set $B \subset G$;
3. A is not a neighborhood of zero in a totally bounded group topology of G.

Proof. 1. Given an infinite set $B \subset G$ and $k \geq 2$, we should find distinct elements $x_1, \dots, x_k \in B$ such that $x^{m_1}_{\sigma(1)} \dots x^{m_k}_{\sigma(k)} \in A$ for any permutation $\sigma \in S_k$. Taking into account that the group G is Boolean and $(m_1, \dots, m_k) \in (2\mathbb{Z}+1)^k$, we conclude that $x^{m_1}_{\sigma(1)} \dots x^{m_k}_{\sigma(k)} = x_1 \cdots x_k$ for any points $x_1, \dots, x_k \in G$ and any permutation $\sigma \in S_k$. So, it suffices to find distinct points $x_1, \dots, x_k \in B$ such that $x_1 \cdots x_k \in A$.

For an element $b \in B$ by $|b|$ we denote the cardinality of b (as a finite subset of $\mathbb{Z}$). Two cases are possible: if the set $\{|b| \colon b \in B\}$ is infinite, then we can fix any distinct elements $x_1, \dots, x_{k-1} \in B$ and then choose an element $x_k \in B \setminus \{x_1, \dots, x_{k-1}\}$ such that $|x_k| > 3 + |x_1 \cdots x_{i-1}|$. Then the set $x := x_1 \cdots x_k$ has cardinality $|x| \geq |x_k| - |x_1 \cdots x_{k-1}| \geq 3$ and hence $x \in A$.

If the set $\{|b| \colon b \in B\}$ is finite, then by the classical Sunflower System Lemma [2] of Erdős and Rado, there exist a set $z \in G$ and a sequence $x_0, \dots, x_k$ of pairwise distinct elements of B such that $x_i \cap x_j = z \neq x_i$ for any distinct indices $i, j \leq k$. Then the set $x := x_1 \cdots x_k$ contains the union $\bigcup_{i=1}^k (x_k \setminus z)$ and hence has cardinality $\geq k$. If $k \geq 3$, then $x \in A$.

If $k = 2$, then we shall show that at least one of the sets x_0x_1, x_0, x_2 or x_1x_2 belongs to A. Assuming that these three sets do not belong to A, we conclude that $x_0 = z \cup \{a\}$, $x_1 = z \cup \{b\}$ and $x_2 = z \cup \{c\}$ where a, b, c are distinct integer numbers such that $a - b$, $b - c$ and $a - c$ belong to the set $\{n^3 \colon n \in \mathbb{Z} \setminus \{0\}\}$. Since $(a - b) + (b - c) = a - c$, this contradicts the Fermat Theorem (saying that the equality $x^3 + y^3 = z^3$ has no solutuions in non-zero integer numbers).

2,3. The second statement is proved in Example 4 of [7] and the third statement trivially follows from the second statement. □

Now we endow G with the discrete topology and identify the Stone-Čech compactification βG of G with the set of all ultrafilters on G. The family $\{\bar{A} \colon A \subseteq G\}$, where $\bar{A} = \{p \in \beta G \colon A \in p\}$, forms the base for the topology of βG. Given a filter φ on G, we denote $\bar{\varphi} = \bigcap\{\bar{A} \colon A \in \varphi\}$, so φ defines the closed subset $\bar{\varphi}$ of βG, and every non-empty closed subset of βG can be defined in this way.

We use the standard extension [4, Section 4.1] of the multiplication on G to the semigroup multiplication on βG. Given two ultrafilters $p, q \in \beta G$, we choose $P \in p$ and, for each $x \in P$, pick $Q_x \in q$. Then $\bigcup_{x \in P} xQ_x \in pq$ and the family of all these subsets forms a base of the product pq. We note that the set G^*, $G^* = \beta G \backslash G$ of all free ultrafilters is a closed subsemigroup of βG.

For $t \in \mathbb{Z}$ and $q \in G^*$ we denote by $q^\wedge t$ the ultrafilter with the base $\{x^t : x \in Q\}$, $Q \in q$. Warning: $q^\wedge t$ and q^t are different things. Certainly, $q^\wedge t = q^t$ only if $t \in \{-1, 0, 1\}$.

In notations of Proposition 1, we state

Proposition 5. *An ultrafilter $p \in G^*$ belongs to the set $cl\{(q^\wedge m_1) \dots (q^\wedge m_k) : q \in G^*\}$ if and only if for every $P \in p$ there exists an injective sequence $(x_n)_{n\in\omega}$ in G such that*

$$\{x_{n_1}^{m_1}\, x_{n_2}^{m_2}\, \cdots\, x_{n_k}^{m_k} : \ n_1 < n_2 < \ldots < n_k < \omega\} \ \subseteq \ P.$$

Proof. The "if" part follows directly from the definition of multiplication of the ultrafilters: take an arbitrary $q \in G^*$ such that $\{x_n : n \in \omega\} \in q$.

The "only if" part is evident for $k = 1$. We prove it only for $k = 2$. We take $q \in G^*$ such that $P \in (q^\wedge m_1)(q^\wedge m_2)$, and choose $Q \in q$, $\{Q_x : Q_x \in q, x \in Q\}$ such that

$$x^{m_1} \cdot \{y^{m_2} : \quad y \in Q_x\} \subseteq P$$

for each $x \in Q$. Then the desired sequence $(x_n)_{n\in\omega}$ can be chosen inductively from elements of Q. □

In the case $k = 2$, $\overrightarrow{m} = (-1, 1)$, the following theorem were proved in [6].

Theorem 1. *For every group G and any number vector $\overrightarrow{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$, we have*

$$\overline{\varphi}_{\overrightarrow{m}} \ = \ cl\{(q^\wedge m_1) \ \cdots \ (q^\wedge m_k) : \ q \in \ G^*\}.$$

Proof. We assume that there is $p \in \ cl\{(q^\wedge m_1) \ \cdots \ (q^\wedge m_k) : \ q \in \ G^*\}$ such that $p \notin \varphi_{\overrightarrow{m}}$, choose $P \in p$ such that $G \setminus P \in \varphi_{\overrightarrow{m}}$ and let $(x_n)_{n\in\omega}$ be the sequence given for P by Proposition 2. We put $X = \{x_n : n \in \omega\}$ and note that every infinite subset Y of X contradicts Proposition 1.

On the other hand, we take an arbitrary $A \in \varphi_{\overrightarrow{m}}$ and an infinite X, use Proposition 1(i) to choose corresponding Y and apply Proposition 4. □

Question 1. *Let G be a group and $\overrightarrow{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$, $\overrightarrow{n} = (n_1, \dots, n_l) \in \mathbb{Z}^l$ be two number vectors. How one can detect whether*

(1) $cl\{(q^\wedge m_1) \ \cdots \ (q^\wedge m_k) : \ q \in \ G^*\} \ \cap \ cl\{(r^\wedge n_1) \ \cdots \ (r^\wedge n_l) : \ r \in \ G^*\} \ \neq \ \varnothing$?

Evidently, (1) holds if the equation

(2) $(x^\wedge m_1) \ \cdots \ (x^\wedge m_k) \ = \ (y^\wedge n_1) \ \cdots \ (y^\wedge n_l)$

has solutions $x, y \in G^*$. In the case of Abelian groups (2) turns into the equality

(3) $m_1 x \ + \cdots + \ m_k x \ = \ n_1 y + \ \cdots \ + \ n_l y.$

The equations (3) were studied in many paper with combinatorial, topological or purely aesthetic motivations, we mention only [3], [5].

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Ivan Franko National University of L'viv
t.o.banakh@gmail.com
Faculty of Computer Science and Cybernetics, Kyiv University
i.v.protasov@gmail.com, ksuha@freenet.com.ua

Received 5.04.2017