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FOURIER PROBLEM FOR PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS OF NONLINEARITY AND TIME DELAY

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The Fourier problem for nonlinear parabolic equations with variable exponents of nonlinearity and time delay is considered. The existence and uniqueness of weak solutions of the problem are investigated. Also, its a priori estimates are obtained.

Introduction. In this paper we consider the Fourier problems for the nonlinear parabolic equations with variable exponents of nonlinearity and time depended delay. A typical example of the equations being studied here is

$$u_t - \sum_{i=1}^n \left(\widehat{a}_i(x, t) |u_{x_i}|^{p_i(x)-2} u_{x_i} \right)_{x_i} + \widehat{a}_0(x, t) |u|^{p_0(x)-2} u + \int_{t-\tau(t)}^t \widehat{c}(x, t, s) u(x, s) ds = f(x, t), \quad (1)$$

$(x, t) \in Q := \Omega \times (-\infty, 0)$, where Ω is a bounded domain in \mathbb{R}^n ($n \in \mathbb{N}$), $\widehat{a}_0, \dots, \widehat{a}_n$ are measurable positive functions on Q , p_0, \dots, p_n (the exponents of nonlinearity) are measurable bounded functions such that $p_0(x) > 2, \dots, p_n(x) > 2$ for a.e. $x \in \Omega$, τ is a nonnegative continuous function, \widehat{c} is a measurable bounded function, f is an integrable function, u is unknown function.

Fourier problems for evolution equations arise in modeling different nonstationary processes in nature, that started a long time ago and initial conditions do not affect on them in the actual time moment, but boundary conditions do affect them. Thus, we can assume that the initial time is $-\infty$, while 0 is the final time, and initial conditions can be replaced with the behaviour of the solution as time variable turns to $-\infty$. Fourier problem for evolution equations appears during modeling in many fields of science such as ecology, economics, physics, cybernetics, etc. (see, e.g., [6]–[8], [12]–[14], [30], [31], [33], [36]–[38]), and they have been widely studied. Good reference overview on problems without initial conditions for evolution equations without time delay can be found in [12].

Equation (1) in the case $Q = \Omega \times (0, T)$ or $Q = \Omega \times (-\infty, 0)$, where Ω is a domain in \mathbb{R}^n , $T > 0$, is an example of nonlinear parabolic equations with variable exponents of nonlinearity. These equations describe many physical processes such as electromagnetic fields, electrorheological fluids, image reconstruction processes, current flow in variable temperature

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field (see [27], [35]). For investigation of such equations generalized Lebesgue and Sobolev spaces are used. The mentioned spaces were firstly introduced in [32]. Properties of these spaces were studied in [32], [22], [26], [28], [18] e.a. The various problems for nonlinear differential equations of type (1) with $\widehat{c} = 0$ are actively being investigated (see, e.g., [27], [1], [2], [3] [6], [9], [13], [23], [29], and references therein).

Equations with time delay arise during modeling in population dynamics, in non-Newtonian filtration, heat flux, etc. ([24]). The equations of type (1) on finite time interval with both constant exponents of nonlinearity and time delay are studied in [4], [5], [25], [21], [17] and others. Good reference overview on such papers can be found in [25]. Note that in these papers the semigroups theory is used.

Partial differential equations with a variable delay are less studied, and we known only paper [11] and works of Chueshov and Rezounenko (in particular, [15], [34]), where equations of type (1) on finite time interval, with $p_0(x) = \dots = p_n(x) = 2$, $\tau = \tau(u)$, are considered. In [15], a certain abstract parabolic problem with the state dependent delay term of a rather general structure is investigated. In [34], a nonlinear partial functional differential equations with main linear elliptic operator and the non-local nonlinear term with delay is investigated. In [11] equations of type (1) on finite time interval with variable exponents of nonlinearity and time delay are investigated. For proving existence of solutions of problems considered in these papers the Galerkin approximation is used. Fourier problems for parabolic equations with constant time delay were investigated in [10], [19].

To the best of our knowledge, the Fourier problems for parabolic equations with variable exponents of nonlinearity and time depended delay are an untreated topic in the literature. These problems are considered in our paper. Existence and uniqueness of solution of the problem are proved. This paper can be considered as continuation of [11].

The paper is organized as follows. In Section 1, the main notations and functional spaces are introduced. The statement of the problem and formulation of the main result are given in Section 2. The main result is proved in Section 3.

1. Notation and auxiliary facts. Let n be a natural number, \mathbb{R}^n be the standard linear space of ordered collections $x = (x_1, \dots, x_n)$ of real numbers with the norm $|x| := (|x_1|^2 + \dots + |x_n|^2)^{1/2}$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with the piecewise smooth boundary $\partial\Omega$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 is the closure of an open set on $\partial\Omega$ (in particular, either $\Gamma_0 = \emptyset$ or $\Gamma_0 = \partial\Omega$), $\Gamma_1 := \partial\Omega \setminus \Gamma_0$, $\nu = (\nu_1, \dots, \nu_n)$ is the unit outward pointing normal vector on the $\partial\Omega$. Denote $S := (-\infty, 0]$, $Q := \Omega \times S$, $\Sigma_0 := \Gamma_0 \times S$, $\Sigma_1 := \Gamma_1 \times S$, $Q_{t_1, t_2} := \Omega \times (t_1, t_2)$ for arbitrary real t_1 and t_2 , $t_1 < t_2$.

Let us introduce some functional spaces. Let either $G = \Omega$ or $G = Q_{t_1, t_2}$ ($-\infty < t_1 < t_2 < +\infty$), or $G = Q$. Suppose that $r \in L_\infty(\Omega)$, $r(x) \geq 1$ for a.e. $x \in \Omega$. Consider the linear space $L_{r(\cdot)}(G)$ of the measurable functions $v: G \rightarrow \mathbb{R}$ such that $\rho_{G,r}(v) < \infty$, where $\rho_{G,r}(v) := \int_\Omega |v(x)|^{r(x)} dx$ if $G = \Omega$, $\rho_{G,r}(v) := \iint_{Q_{t_1, t_2}} |v(x, t)|^{r(x)} dx dt$ if $G = Q_{t_1, t_2}$ ($-\infty < t_1 < t_2 < +\infty$), and $\rho_{G,r}(v) := \iint_Q |v(x, t)|^{r(x)} dx dt$ if $G = Q$. This is a Banach space with respect to the norm $\|v\|_{L_{r(\cdot)}(G)} := \inf\{\lambda > 0 \mid \rho_{G,r}(v/\lambda) \leq 1\}$ (see [26, p. 599]) and it is called a *generalized Lebesgue space*. Note that if $r(x) = r_0 = \text{const} \geq 1$ for a.e. $x \in \Omega$ then $\|\cdot\|_{L_{r(\cdot)}(G)}$ is equal to the standard norm $\|\cdot\|_{L_{r_0}(G)}$ of the Lebesgue space $L_{r_0}(G)$. Note also that the set $C(\overline{G})$ is dense in $L_{r(\cdot)}(G)$ (see [26, p. 603]). According to [26, p. 599], if $\text{ess inf}_{x \in \Omega} r(x) > 1$, then the space $L_{r(\cdot)}(G)$ is reflexive and the conjugate space $[L_{r(\cdot)}(G)]'$ equals $L_{r'(\cdot)}(G)$, where the function r' is defined by the equality $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$ for a.e. $x \in \Omega$. However, there

exist properties of standard Lebesgue spaces that is not valid for the generalized Lebesgue spaces. Thus the properties of the generalized Lebesgue spaces are not simple corollaries of the correspondent properties of the standard Lebesgue spaces (see, e.g., the mentioned papers).

We denote by $L_{r(\cdot),\text{loc}}(\overline{Q})$ the space of measurable functions $g: Q \rightarrow \mathbb{R}$ such that the restriction of g on Q_{t_1,t_2} belongs to $L_{r(\cdot)}(Q_{t_1,t_2})$ for each $t_1, t_2 \in S$. This space is complete locally convex with respect to the family of seminorms $\{\|\cdot\|_{L_{r(\cdot)}(Q_{t_1,t_2})} \mid t_1, t_2 \in S\}$. A sequence $\{g_m\}$ is said to be convergent strongly (resp., weakly) in $L_{r(\cdot),\text{loc}}(\overline{Q})$ provided the sequences of restrictions $\{g_m|_{Q_{t_1,t_2}}\}$ are convergent strongly (resp., weakly) in $L_{r(\cdot)}(Q_{t_1,t_2})$ for all $t_1, t_2 \in S$. Similarly we can define the space $L_{\infty,\text{loc}}(\overline{Q})$.

Consider the function $p = (p_0, p_1, \dots, p_n): \Omega \rightarrow \mathbb{R}^{n+1}$ such that following condition is satisfied:

(P) for every $i \in \{0, 1, \dots, n\}$, $p_i: \Omega \rightarrow \mathbb{R}$ is a measurable function such that

$$p_i^- := \operatorname{ess\,inf}_{x \in \Omega} p_i(x) \geq 2, \quad p_i^+ := \operatorname{ess\,sup}_{x \in \Omega} p_i(x) < +\infty.$$

Denote by $p' = (p'_0, \dots, p'_n): \Omega \rightarrow \mathbb{R}^{n+1}$ the vector-function such that $\frac{1}{p_i(x)} + \frac{1}{p'_i(x)} = 1$ for a.e. $x \in \Omega$ ($i = \overline{0, n}$). Clearly that $p'_i: \Omega \rightarrow \mathbb{R}$ is a measurable bounded function such that $\operatorname{ess\,inf}_{x \in \Omega} p'_i(x) \geq 1$ ($i = \overline{0, n}$).

Now let us give the definitions of the following functional spaces. First, denote by $W_{p(\cdot)}^1(\Omega)$ the generalized Sobolev space of the functions $v \in L_{p_0(\cdot)}(\Omega)$ such that $v_{x_i} \in L_{p_i(\cdot)}(\Omega)$ ($i = \overline{1, n}$). This is a Banach space with respect to the norm $\|v\|_{W_{p(\cdot)}^1(\Omega)} := \|v\|_{L_{p_0(\cdot)}(\Omega)} + \sum_{i=1}^n \|v_{x_i}\|_{L_{p_i(\cdot)}(\Omega)}$. Let $\widetilde{W}_{p(\cdot)}^1(\Omega)$ be the subspace of $W_{p(\cdot)}^1(\Omega)$ that is the closure of the space $\widetilde{C}^1(\overline{\Omega}) := \{v \in C^1(\overline{\Omega}) \mid v|_{\Gamma_0} = 0\}$ in $W_{p(\cdot)}^1(\Omega)$. Set $V_p(\Omega) := \widetilde{W}_{p(\cdot)}^1(\Omega)$.

Next, for arbitrary $t_1, t_2 \in \mathbb{R}$, we denote by $W_{p(\cdot)}^{1,0}(Q_{t_1,t_2})$ the set of functions $w \in L_{p_0(\cdot)}(Q_{t_1,t_2})$ such that $w_{x_i} \in L_{p_i(\cdot)}(Q_{t_1,t_2})$ for all $i \in \{1, \dots, n\}$. We define the norm $\|w\|_{W_{p(\cdot)}^{1,0}(Q_{t_1,t_2})} := \|w\|_{L_{p_0(\cdot)}(Q_{t_1,t_2})} + \sum_{i=1}^n \|w_{x_i}\|_{L_{p_i(\cdot)}(Q_{t_1,t_2})}$. Denote by $\widetilde{W}_{p(\cdot)}^{1,0}(Q_{t_1,t_2})$ the subspace of $W_{p(\cdot)}^{1,0}(Q_{t_1,t_2})$ consisting of functions v such that $v(\cdot, t) \in \widetilde{W}_{p(\cdot)}^1(\Omega)$ for a. e. $t \in [t_1, t_2]$.

Let us denote by $\widetilde{W}_{p(\cdot),\text{loc}}^{1,0}(\overline{Q})$ the linear space of measurable functions such that their restrictions on Q_{t_1,t_2} belong to $\widetilde{W}_{p(\cdot)}^{1,0}(Q_{t_1,t_2})$ for all $t_1, t_2 \in S$. This space is complete locally convex with respect to the family of seminorms $\{\|\cdot\|_{W_{p(\cdot)}^{1,0}(Q_{t_1,t_2})} \mid t_1, t_2 \in \mathbb{R}\}$.

Denote by $F_{p'(\cdot),\text{loc}}(Q)$ the space of vector-functions (f_0, f_1, \dots, f_n) such that for each $i \in \{0, 1, \dots, n\}$ $f_i \in L_{p'_i(\cdot),\text{loc}}(\overline{Q})$ ($i = \overline{1, n}$), and for each $i \in \{1, \dots, n\}$ $f_i = 0$ a.e. in some neighborhood of the surface Σ_1 .

Finally, define by $C_c^1(I)$, where I is an interval, a linear space defined on I continuously differentiable finite functions, moreover, if $I = (t_1, t_2)$, then we will write $C_c^1(t_1, t_2)$ instead of $C_c^1((t_1, t_2))$.

The following auxiliary result, which had been proved in [9], will be used in the sequel.

Lemma 1. *Suppose that $w \in \widetilde{W}_{p(\cdot)}^{1,0}(Q_{t_1,t_2})$, where $t_1, t_2 \in \mathbb{R}$ ($t_1 < t_2$), satisfies the following identity*

$$\int_{t_1}^{t_2} \int_{\Omega} \left\{ -w\psi\varphi' + (g_0\psi + \sum_{i=1}^n g_i\psi_{x_i})\varphi \right\} dxdt = 0, \quad \psi \in V_p(\Omega), \quad \varphi \in C_c^1(t_1, t_2), \quad (2)$$

for some $g_i \in L_{p'_i(\cdot)}(Q_{t_1, t_2})$ ($i = \overline{0, n}$).

Then $w \in C([t_1, t_2]; L_2(\Omega))$ and for any $\sigma_1, \sigma_2 \in [t_1, t_2]$ ($\sigma_1 < \sigma_2$), for every $\theta \in C^1([t_1, t_2])$ we have

$$\frac{1}{2}\theta(t) \int_{\Omega} |w(x, t)|^2 dx \Big|_{t=\sigma_1}^{t=\sigma_2} - \frac{1}{2} \int_{\sigma_1}^{\sigma_2} \int_{\Omega} |w|^2 \theta' dx dt + \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \{g_0 w + \sum_{i=1}^n g_i w_{x_i}\} \theta dx dt = 0. \quad (3)$$

Remark 1. Note that from $w \in L_2(Q)$ it follows that

$$\lim_{\sigma \rightarrow -\infty} \int_{\sigma-1}^{\sigma} \int_{\Omega} |w|^2 dx dt = 0. \quad (4)$$

If $w \in L_2(Q) \cap C(S, L_2(\Omega))$, then it follows that there exists sequence $\{t_k\}_{k=0}^{\infty} \subset S$ such that $t_k \rightarrow -\infty$ as $k \rightarrow +\infty$, and

$$\lim_{k \rightarrow +\infty} \int_{\Omega} |w(x, t_k)|^2 dx = 0. \quad (5)$$

2. Statement of the problem and main result. In this paper we consider the problem of finding a function $u: \overline{Q} \rightarrow \mathbb{R}$ satisfying (in some sense) the equation

$$\begin{aligned} u_t - \sum_{i=1}^n \frac{d}{dx_i} a_i(x, t, u, \nabla u) + a_0(x, t, u, \nabla u) + \int_{t-\tau(t)}^t c(x, t, s, u(x, s)) ds \\ = - \sum_{i=1}^n \frac{\partial}{\partial x_i} f_i(x, t) + f_0(x, t), \quad (x, t) \in Q, \end{aligned} \quad (6)$$

and the boundary conditions

$$u \Big|_{\Sigma_0} = 0, \quad \sum_{i=1}^n a_i(x, t, u, \nabla u) \nu_i \Big|_{\Sigma_1} = 0. \quad (7)$$

Here $\tau: S \rightarrow \mathbb{R}$ is a continuous bounded function such that $\tau(t) \geq 0$ for all $t \in S$, $\tau^+ := \sup_{t \in S} \tau(t)$, and $a_i: Q \times \mathbb{R}^{1+n} \rightarrow \mathbb{R}$, $c: Q \times S \times \mathbb{R} \rightarrow \mathbb{R}$, $f_i: Q \rightarrow \mathbb{R}$ ($i = \overline{0, n}$) are given real-valued functions.

We introduce the following classes of the initial data.

Let p satisfy condition (\mathcal{P}) . Define $\mathcal{A}(p)$ to be the set of the collections (a_0, a_1, \dots, a_n) of the real-value functions $a_i: Q \times \mathbb{R}^{1+n} \rightarrow \mathbb{R}$ ($i = \overline{0, n}$) satisfying the following assumptions:

(\mathcal{A}_1) for every $i \in \{0, 1, \dots, n\}$, a_i is the Caratheodory function (i.e., $a_i(x, t, \cdot, \cdot): \mathbb{R}^{1+n} \rightarrow \mathbb{R}$ is a continuous for a.e. $(x, t) \in Q$, and $a_i(\cdot, \cdot, \rho, \xi): Q \rightarrow \mathbb{R}$ is a measurable for every $(\rho, \xi) \in \mathbb{R}^{1+n}$), and $a_i(x, t, 0, 0) = 0$ for a.e. $(x, t) \in Q$;

(\mathcal{A}_2) for every $i \in \{0, 1, \dots, n\}$, for a.e. $(x, t) \in Q$, and for every $(\rho, \xi) \in \mathbb{R}^{1+n}$ the following estimate is valid

$$|a_i(x, t, \rho, \xi)| \leq C_1 \left(\sum_{j=1}^n |\xi_j|^{p_j(x)/p'_i(x)} + |\rho|^{p_0(x)/p'_i(x)} \right) + h_i(x, t),$$

where C_1 is a positive constant, and $h_i \in L_{p'_i(\cdot)}(Q)$;

(\mathcal{A}_3) for a.e. $(x, t) \in Q$ and for every $(\rho_1, \xi^1), (\rho_2, \xi^2) \in \mathbb{R}^{1+n}$ the following inequality holds

$$\begin{aligned} \sum_{i=1}^n (a_i(x, t, \rho_1, \xi^1) - a_i(x, t, \rho_2, \xi^2))(\xi_i^1 - \xi_i^2) + (a_0(x, t, \rho_1, \xi^1) - a_0(x, t, \rho_2, \xi^2))(\rho_1 - \rho_2) \geq \\ \geq K_1 \left(\sum_{i=1}^n |\xi_i^1 - \xi_i^2|^{p_i(x)} + |\rho_1 - \rho_2|^{p_0(x)} \right) + K_2 |\rho_1 - \rho_2|^2, \end{aligned} \quad (8)$$

where K_1, K_2 are positive constants.

Define \mathcal{C} to be the set of the functions $c(x, t, s, \rho)$, $(x, t, s, \rho) \in Q \times S \times \mathbb{R}$, satisfying the following assumptions:

(\mathcal{C}_1) c is a Caratheodory function (i.e., $c(x, t, s, \cdot): \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function for a.e. $(x, t, s) \in Q \times S$, and $c(\cdot, \cdot, \cdot, \rho): Q \times S \rightarrow \mathbb{R}$ is a measurable function for every $\rho \in \mathbb{R}$), in addition, $c(x, t, s, 0) = 0$ for a.e. $(x, t, s) \in Q \times S$;

(\mathcal{C}_2) there exists a constant $L > 0$ (depending on c) such that for a.e. $(x, t, s) \in Q \times S$ and for every $\rho_1, \rho_2 \in \mathbb{R}$ the inequality

$$|c(x, t, s, \rho_1) - c(x, t, s, \rho_2)| \leq L|\rho_1 - \rho_2| \quad (9)$$

holds.

Remark 2. From the condition $c(x, t, s, 0) = 0$ (see (\mathcal{C}_1)) and (\mathcal{C}_2) follows that for a.e. $(x, t, s) \in Q \times S$, and for every $\rho \in \mathbb{R}$ the following estimate is valid

$$|c(x, t, s, \rho)| \leq L|\rho|. \quad (10)$$

Now we can give a definition of the weak solution of problem (6), (7).

Definition 1. Let p satisfy condition (\mathcal{P}) , $(a_0, a_1, \dots, a_n) \in \mathcal{A}(p)$, $c \in \mathcal{C}$, $(f_0, f_1, \dots, f_n) \in F_{p'(\cdot), \text{loc}}(\overline{Q})$. The function $u \in \widetilde{W}_{p(\cdot), \text{loc}}^{1,0}(\overline{Q}) \cap L_2(Q) \cap C(S; L_2(\Omega))$ is called a *weak solution* of problem (6), (7), if the integral equality

$$\begin{aligned} \iint_Q \left\{ \sum_{i=1}^n a_i(x, t, u, \nabla u) v_{x_i} \varphi + a_0(x, t, u, \nabla u) v \varphi + v \varphi \int_{t-\tau(t)}^t c(x, t, s, u(x, s)) ds - \right. \\ \left. - uv \varphi' \right\} dx dt = \iint_Q \left\{ \sum_{i=1}^n f_i v_{x_i} \varphi + f_0 v \varphi \right\} dx dt \end{aligned} \quad (11)$$

holds for every $v \in V_p(\Omega)$ and $\varphi \in C_c^1(-\infty, 0)$.

The main result of our paper is the following statement.

Theorem 1. Let p satisfy condition (\mathcal{P}) , $(a_0, a_1, \dots, a_n) \in \mathcal{A}(p)$, $c \in \mathcal{C}$, $(f_0, f_1, \dots, f_n) \in L_{p_0(\cdot)}(Q) \times L_{p_1(\cdot)}(Q) \times \dots \times L_{p_n(\cdot)}(Q)$, and

$$K_2 - 2L\tau^+ > 0. \quad (12)$$

Then problem (6), (7) has a unique weak solution, and it belongs to $\widetilde{W}_{p(\cdot)}^{1,0}(Q)$, and it satisfies the following estimate

$$\begin{aligned} \sup_{t \in S} \int_{\Omega} |u(x, t)|^2 dx + \iint_Q \left\{ \sum_{i=1}^n |u_{x_i}(x, t)|^{p_i(x)} + |u(x, t)|^{p_0(x)} + |u(x, t)|^2 \right\} dx dt \leq \\ \leq C_2 \iint_Q \left\{ \sum_{i=0}^n |f_i(x, t)|^{p'_i(x)} \right\} dx dt, \end{aligned} \quad (13)$$

where $C_2 > 0$ is a constant depending only on K_1, K_2, L, τ^+ and p_i^- ($i = \overline{0, n}$).

3. Proof of the main result. For a function $w: Q \rightarrow \mathbb{R}$ we denote

$$\begin{aligned} a_j(w)(x, t) &:= a_j(x, t, w(x, t), \nabla w(x, t)), \quad (x, t) \in Q, \quad j = \overline{0, n}, \\ c(w)(x, t, s) &:= c(x, t, s, w(x, s)), \quad (x, t, s) \in Q \times S, \quad \partial_i w = w_{x_i}, \quad i = \overline{1, n}, \quad \partial_0 w = w. \end{aligned}$$

Let us prove Theorem 1 in three steps: firstly, we prove the uniqueness of solution of problem (6), (7), later, its existence and, finally, we show correctness of estimate (13).

First step (uniqueness of solution). Assume the opposite. Let u_1 and u_2 be two different weak solutions of the problem. Denote $w := u_1 - u_2$. Consider the difference between (11) with $u = u_2$ and (11) with $u = u_1$, we obtain

$$\begin{aligned} - \iint_Q w v \varphi' dx dt + \iint_Q \left[\sum_{i=0}^n (a_i(u_1) - a_i(u_2)) \partial_i v + \right. \\ \left. + v \int_{t-\tau(t)}^t (c(u_1) - c(u_2)) ds \right] \varphi dx dt = 0 \quad \forall v \in V_p(\Omega), \quad \forall \varphi \in C_c^1(-\infty, 0). \end{aligned} \quad (14)$$

According to Lemma 1, with $\theta(t) = 1$, $t \in \mathbb{R}$, from equality (14) we get

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |w(x, \sigma_2)|^2 dx - \frac{1}{2} \int_{\Omega} |w(x, \sigma_1)|^2 dx + \iint_{\sigma_1}^{\sigma_2} \int_{\Omega} \left[\sum_{i=0}^n (a_i(u_1) - a_i(u_2)) (\partial_i u_1 - \partial_i u_2) + \right. \\ \left. + w \int_{t-\tau(t)}^t (c(u_1) - c(u_2)) ds \right] dx dt = 0, \end{aligned} \quad (15)$$

where $\sigma_1, \sigma_2 \in S$ ($\sigma_1 < \sigma_2$) are arbitrary.

From condition (\mathcal{A}_3) for a.e. $(x, t) \in Q$ we have

$$\sum_{i=0}^n (a_i(u_1) - a_i(u_2)) (\partial_i u_1 - \partial_i u_2) \geq K_1 \sum_{i=0}^n |\partial_i u_1 - \partial_i u_2|^{p_i(x)} + K_2 |u_1 - u_2|^2. \quad (16)$$

Consider the last term from equality (15). Using condition (\mathcal{C}_2) , Fubini Theorem and

Hölder's inequality, for a.e. $x \in \Omega$ we obtain

$$\begin{aligned} \left| \int_{\sigma_1}^{\sigma_2} w(x, t) \left(\int_{t-\tau(t)}^t (c(u_1)(x, t, s) - c(u_2)(x, t, s)) ds \right) dt \right| &\leq L \int_{\sigma_1}^{\sigma_2} |w(x, t)| \left(\int_{t-\tau^+}^t |w(x, s)| ds \right) dt \leq \\ &\leq L\sqrt{\tau^+} \left(\int_{\sigma_1}^{\sigma_2} |w(x, t)|^2 dt \right)^{1/2} \left(\int_{\sigma_1}^{\sigma_2} \left(\int_{t-\tau^+}^t |w(x, s)|^2 ds \right) dt \right)^{1/2}. \end{aligned} \quad (17)$$

Now consider the second integral in right-hand side of inequality (17). Changing order of integration and assuming $w(x, t) = 0$ for $x \in \Omega$, $t > 0$, for a.e. $x \in \Omega$ we have

$$\int_{\sigma_1}^{\sigma_2} \left(\int_{t-\tau^+}^t |w(x, s)|^2 ds \right) dt \leq \int_{\sigma_1-\tau^+}^{\sigma_2} |w(x, s)|^2 ds \int_s^{s+\tau^+} dt = \tau^+ \left(\int_{\sigma_1}^{\sigma_2} |w(x, s)|^2 ds + \int_{\sigma_1-\tau^+}^{\sigma_1} |w(x, s)|^2 ds \right). \quad (18)$$

Substituting (18) in (17), the last term from obtained above relations chain instead of the first one, and using Cauchy inequality: $\sqrt{ab} \leq a + b$ ($a \geq 0, b \geq 0$), we obtain

$$\begin{aligned} \left| \int_{\sigma_1}^{\sigma_2} w(x, t) \left(\int_{t-\tau(t)}^t (c(u_1)(x, t, s) - c(u_1)(x, t, s)) ds \right) dt \right| &\leq \\ &\leq L\tau^+ \left(2 \int_{\sigma_1}^{\sigma_2} |w(x, t)|^2 dt + \int_{\sigma_1-\tau^+}^{\sigma_1} |w(x, t)|^2 dt \right). \end{aligned} \quad (19)$$

Using (16), (19), from (15) we obtain

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |w(x, \sigma_2)|^2 dx + K_1 \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left(\sum_{i=0}^n |\partial_i u_1 - \partial_i u_2|^{p_i(x)} \right) dx + (K_2 - 2L\tau^+) \int_{\sigma_1}^{\sigma_2} \int_{\Omega} |w(x, t)|^2 dx dt &\leq \\ &\leq \frac{1}{2} \int_{\Omega} |w(x, \sigma_1)|^2 dx + L\tau^+ \int_{\sigma_1-\tau^+}^{\sigma_1} \int_{\Omega} |w(x, t)|^2 dx dt. \end{aligned}$$

Then, taking into account (12), we have

$$\int_{\Omega} |w(x, \sigma_2)|^2 dx \leq \int_{\Omega} |w(x, \sigma_1)|^2 dx + 2L\tau^+ \int_{\sigma_1-\tau^+}^{\sigma_1} \int_{\Omega} |w(x, t)|^2 dx dt. \quad (20)$$

Since $w \in L_2(Q) \cap C(S, L_2(\Omega))$, according to Remark 1, there exists a sequence $\{\sigma_{1,k}\}_{k=0}^{\infty} \subset S$, such that $\sigma_{1,k} \xrightarrow[k \rightarrow +\infty]{} -\infty$ and $\int_{\Omega} |w(x, \sigma_{1,k})|^2 dx + 2L\tau^+ \int_{\sigma_{1,k}-\tau^+}^{\sigma_{1,k}} \int_{\Omega} |w(x, t)|^2 dx dt \xrightarrow[k \rightarrow +\infty]{} 0$.

Let us fix an arbitrary σ_2 in (20), and take in (20) $\sigma_{1,k}$ for $k \in \mathbb{N}$ such that $\sigma_{1,k} < \sigma_2$ instead of σ_1 . Tending k to $+\infty$, we get the equality $\int_{\Omega} |w(x, \sigma_2)|^2 dx = 0$. Since $\sigma_2 \in S$ is arbitrary, we obtain $w(x, t) = 0$ for a.e. $(x, t) \in Q$, that contradicts to our assumption. Therefore, a weak solution of problem (6), (7) is unique.

Second step (existence of solution). For each $m \in \mathbb{N}$ denote $Q_m = \Omega \times (-m, 0]$, $\tau_m = \min_{-m \leq t \leq 0} \{t - \tau(t)\}$. It is clear that $\tau_m \leq -m$. Denote $f_{i,m}(\cdot, t) := \begin{cases} f_i(\cdot, t), & -m < t \leq 0; \\ 0, & t \leq -m, \end{cases}$ and consider the problem: to find a function $u_m \in \widetilde{W}_{p(\cdot)}^{1,0}(Q_m) \cap C([- \tau_m, 0]; L_2(\Omega))$, which satisfies the initial condition

$$u_m(x, t) = 0, \quad (x, t) \in \overline{\Omega} \times [\tau_m, -m] \quad (21)$$

(if $\tau_m = -m$ then $[\tau_m, -m] := \{-m\}$) and equation (6) in Q_m in the sense of integral equality

$$\begin{aligned} & \iint_{Q_m} \left\{ \sum_{i=1}^n a_i(x, t, u_m, \nabla u_m) v_{x_i} \varphi + a_0(x, t, u_m, \nabla u_m) v \varphi + v \varphi \int_{t-\tau(t)}^t c(x, t, s, u_m(x, s)) ds - \right. \\ & \left. - u_m v \varphi' \right\} dx dt = \iint_{Q_m} \left\{ \sum_{i=1}^n f_{i,m} v_{x_i} \varphi + f_{0,m} v \varphi \right\} dx dt, \quad v \in V_p, \quad \varphi \in C_c^1(-m, 0). \end{aligned} \quad (22)$$

Existence and uniqueness of solution of this problem follows from results of the paper [11]. For each $m \in \mathbb{N}$ we extend u_m by 0 on all $\Omega \times \mathbb{R}$ and denote this extension as u_m .

Now we shall get estimates of u_m for each $m \in \mathbb{N}$. Firstly, note that for each $m \in \mathbb{N}$ function u_m belongs to $\widetilde{W}_{p(\cdot)}^{1,0}(Q) \cap L_2(Q) \cap C(S; L_2(\Omega))$ and satisfies integral equality (11) with $f_{i,m}$ instead of f_i , i.e., the following equality is correct

$$\begin{aligned} & \iint_Q \left\{ \sum_{i=1}^n a_i(x, t, u_m, \nabla u_m) v_{x_i} \varphi + a_0(x, t, u_m, \nabla u_m) v \varphi + v \varphi \int_{t-\tau(t)}^t c(x, t, s, u_m(x, s)) ds - \right. \\ & \left. - u_m v \varphi' \right\} dx dt = \iint_Q \left\{ \sum_{i=1}^n f_{i,m} v_{x_i} \varphi + f_{0,m} v \varphi \right\} dx dt, \quad v \in V_p(\Omega), \quad \varphi \in C_c^1(-\infty, 0). \end{aligned} \quad (23)$$

Further we need Young's inequality in the form

$$ab \leq \varepsilon |a|^{r(x)} + \varepsilon^{-\frac{1}{r(x)-1}} |b|^{r'(x)} \quad \text{for a.e. } x \in \Omega, \quad a, b \in \mathbb{R}, \quad 0 < \varepsilon \leq 1, \quad (24)$$

where $r \in L^\infty(\Omega)$, $r(x) > 1$, $r'(x) := r(x)/(r(x) - 1)$ for a.e. $x \in \Omega$, $r^- := \operatorname{ess\,inf}_{x \in \Omega} r(x) > 1$.

Applying Lemma 1, with $\theta(t) = 2$, $\sigma_1, \sigma_2 \subset S$, ($\sigma_1 < \sigma_2$) to equality (23), we obtain

$$\begin{aligned} & \int_{\Omega} |u_m(x, \sigma_2)|^2 dx - \int_{\Omega} |u_m(x, \sigma_1)|^2 dx + 2 \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left[\sum_{i=0}^n a_i(u_m) \partial_i u_m \right] dx dt + \\ & + 2 \int_{\sigma_1}^{\sigma_2} \int_{\Omega} u_m \left[\int_{t-\tau(t)}^t c(u_m)(x, t, s) ds \right] dx dt = 2 \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left\{ \sum_{i=0}^n f_{i,m} \partial_i u_m \right\} dx dt. \end{aligned} \quad (25)$$

Applying inequality (24), the following estimation can be obtained

$$\int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left\{ \sum_{i=0}^n f_{i,m} \partial_i u_m \right\} dx dt \leq \varepsilon \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left\{ \sum_{i=0}^n |\partial_i u_m(x, t)|^{p_i(x)} \right\} dx dt +$$

$$+ \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left\{ \sum_{i=0}^n \varepsilon^{-\frac{1}{p_i^- - 1}} |f_i(x, t)|^{p_i'(x)} \right\} dx dt. \quad (26)$$

Analogically as inequality (19) has been obtained (using (10) instead of condition (\mathcal{C}_2)), we can get

$$\left| \int_{\sigma_1}^{\sigma_2} u_m(x, t) \int_{t-\tau(t)}^t c(u_m)(x, t, s) ds dt \right| \leq L\tau^+ \left(2 \int_{\sigma_1}^{\sigma_2} |u_m(x, t)|^2 dt + \int_{\sigma_1-\tau^+}^{\sigma_1} |u_m(x, t)|^2 dt \right). \quad (27)$$

From conditions (\mathcal{A}_1) , (\mathcal{A}_3) we have

$$\int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left[\sum_{i=0}^n a_i(u_m) \partial_i u_m \right] dx dt \geq \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left[K_1 \sum_{i=0}^n |\partial_i u|^{p_i(x)} + K_2 |u|^2 \right] dx dt. \quad (28)$$

From inequality (25), using estimates (26)–(28) and taking $\sigma_1 < -m$, we obtain

$$\begin{aligned} & \int_{\Omega} |u_m(x, \sigma_2)|^2 dx + 2(K_1 - \varepsilon) \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left\{ \sum_{i=0}^n |\partial_i u_m(x, t)|^{p_i(x)} \right\} dx dt + \\ & + 2(K_2 - 2L\tau^+) \int_{\sigma_1}^{\sigma_2} \int_{\Omega} |u_m(x, t)|^2 dx dt \leq 2 \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left\{ \sum_{i=0}^n \varepsilon^{-\frac{1}{p_i^- - 1}} |f_{i,m}(x, t)|^{p_i'(x)} \right\} dx dt. \end{aligned} \quad (29)$$

Taking $\varepsilon = \frac{1}{2} \min\{K_1, 1\}$ and using (12), from (29) we have

$$\begin{aligned} & \sup_{\sigma \in S} \int_{\Omega} |u_m(x, \sigma)|^2 dx + C_4 \iint_Q \left\{ \sum_{i=0}^n |\partial_i u_m(x, t)|^{p_i(x)} + |u_m(x, t)|^2 \right\} dx dt \leq \\ & \leq C_5 \iint_Q \left\{ \sum_{i=0}^n |f_{i,m}(x, t)|^{p_i'(x)} \right\} dx dt, \end{aligned} \quad (30)$$

where C_4 and C_5 are positive constants depending only on K_1, K_2, L, τ^+ and p_i^- ($i = \overline{0, n}$).

According to definition of $f_{i,m}$ from (30) we have

$$\begin{aligned} & \sup_{\sigma \in S} \int_{\Omega} |u_m(x, \sigma)|^2 dx + \iint_Q \left\{ \sum_{i=0}^n |\partial_i u_m(x, t)|^{p_i(x)} + |u_m(x, t)|^2 \right\} dx dt \leq \\ & \leq C_6 \iint_Q \left\{ \sum_{i=0}^n |f_i(x, t)|^{p_i'(x)} \right\} dx dt, \end{aligned} \quad (31)$$

where $C_6 > 0$ is a positive constant, depending only on τ^+, K_1, K_2, L and p_i ($i = \overline{0, n}$).

Let us show that $\{u_m\}$ is a Cauchy sequence. Taking arbitrary $k, l \in \mathbb{N}$ such that $k < l$ and considering difference between u_k and u_l , similarly as estimate (30), for any $\sigma \in S$ such

that $-k \leq \sigma \leq 0$ one can obtain

$$\begin{aligned} \int_{\Omega} |u_k(x, \sigma) - u_l(x, \sigma)|^2 dx + \int_{-l}^{\sigma} \int_{\Omega} \left\{ \sum_{i=0}^n |\partial_i u_k - \partial_i u_l|^{p_i(x)} + |u_k - u_l|^2 \right\} dx dt \leq \\ \leq C_7 \int_{-l}^{-k} \int_{\Omega} \sum_{i=0}^n |f_{i,k}(x, t) - f_{i,l}(x, t)|^{p_i(x)} dx dt, \end{aligned} \quad (32)$$

where C_7 is a positive constant independent of k, l .

The condition $f_i \in L_{p_i(\cdot)}(Q)$ ($i = \overline{0, n}$) implies that the right-hand side of inequality (32) tends to zero when k and l tend to $+\infty$. This means that the sequence $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in the space $\widetilde{W}_{p(\cdot)}^{1,0}(Q) \cap L_2(Q) \cap C(S; L_2(\Omega))$. Consequently, there exists the function $u \in \widetilde{W}_{p(\cdot)}^{1,0}(Q) \cap L_2(Q) \cap C(S; L_2(\Omega))$ such that

$$u_m \xrightarrow{m \rightarrow \infty} u \quad \text{strongly in } \widetilde{W}_{p(\cdot)}^{1,0}(Q) \cap L_2(Q) \cap C(S; L_2(\Omega)). \quad (33)$$

From condition (C_2) , Fubini Theorem, Cauchy-Schwarz inequality and (33) we have

$$\begin{aligned} \iint_Q \left| \int_{t-\tau(t)}^t c(u_m)(x, t, s) ds - \int_{t-\tau(t)}^t c(u)(x, t, s) ds \right|^2 dx dt \leq \tau^+ \int_{-\infty}^0 \int_{\Omega} \left(\int_{t-\tau^+}^t |c(u_m)(x, t, s) - \right. \\ \left. - c(u)(x, t, s)|^2 ds \right) dx dt \leq L^2 \tau^+ \int_{\Omega} \left(\int_{-\infty}^0 \int_{t-\tau^+}^t |u_m(x, s) - u(x, s)|^2 ds dt \right) dx \leq \\ \leq L^2 \tau^+ \int_{\Omega} \int_{-\infty}^0 \int_s^{s+\tau^+} |u_m(x, s) - u(x, s)|^2 dt ds dx = L^2 \tau^{+2} \iint_Q |u_m(x, t) - u(x, t)|^2 dt dx \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

Thus, we obtain

$$\int_{t-\tau(t)}^t c(u_m) ds \xrightarrow{m \rightarrow \infty} \int_{t-\tau(t)}^t c(u) ds \quad \text{strongly in } L_2(Q). \quad (34)$$

From condition (A_2) , and estimate (31) we have that the following estimate is correct

$$\iint_Q |a_i(u_m)|^{p_i(x)} dx dt \leq C_{10} \iint_Q \left(\sum_{i=0}^n |\partial_i u_m|^{p_i(x)} + |h_i|^{p_i(x)} \right) dx dt \leq C_{11}, \quad (35)$$

where C_{10} and C_{11} are positive constants independent of m .

Hence, from (35) we obtain that $\{a_i(u_m)\}_{m=1}^{\infty}$ is bounded in $L_{p_i(\cdot)}(Q)$ for each $m \in \{0, \dots, n\}$. This and (33) yield that there exists a subsequence of $\{u_m\}_{m=1}^{\infty}$ (denoted also by $\{u_m\}_{m=1}^{\infty}$) and functions $\chi_i \in L_{p_i(\cdot)}(Q)$ ($i = \overline{0, n}$) such that

$$\partial_i u_m \xrightarrow{m \rightarrow \infty} \partial_i u, \quad \text{a.e. on } Q, \quad i = \overline{0, n}, \quad (36)$$

$$a_i(u_m) \xrightarrow{m \rightarrow \infty} \chi_i \quad \text{weakly in } L_{p_i(\cdot)}(Q), \quad i = \overline{0, n}. \quad (37)$$

Condition (\mathcal{A}_1) and (36) yield $a_i(u_m) \xrightarrow{m \rightarrow \infty} a_i(u)$ a.e. on Q , $i = \overline{0, n}$. From this relation and (37), according to [30, Lemma 1.3], it follows that

$$a_i(u_m) \xrightarrow{m \rightarrow \infty} a_i(u) \quad \text{weakly in } L_{p_i(\cdot)}(Q), \quad i = \overline{0, n}. \quad (38)$$

Let us show that the function u is a weak solution of problem (6), (7). For this purpose, we let $m \rightarrow \infty$ in identity (23), taking into account (33), (38), (34) and the definition of the functions $f_{i,m}$ ($i = \overline{0, n}$). As a result we obtain identity (11). Hence, we have proven that u is a weak solution of problem (6), (7).

Third step (correctness of the estimate). From (31) and (33) it follows that estimate (13) is valid. Hence, we have proved Theorem 1.

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