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ASYMPTOTIC BEHAVIOUR OF MEANS OF NONPOSITIVE $\mathcal{M}$-SUBHARMONIC FUNCTIONS

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We describe growth and decrease of pth means, $1 < p < \frac{2n-1}{2(n-1)}$, of nonpositive $\mathcal{M}$-subharmonic functions in the unit ball in $\mathbb{C}^n$ in terms of smoothness properties of a measure. As consequence we obtain a haracterization of asumptotic behaviour for means of Poisson integrals in the unit ball defined by a positive measure.

1. Introduction and main result. The purpose of this paper is to investigate the growth and decrease of pth means of subharmonic function, in terms of smoothness properties of the Riesz measure μ. For one-dimensional case this interplay was studied in [4] and it is based on a concept of the complete measure in the sense of Grishin (see [6, 3]) or related measure.

For $n \in \mathbb{N}$, let $\mathbb{C}^n$ denote the n-dimensional complex space with the inner product

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j\overline{w}_j, \text{ and } |z| = \sqrt{\langle z, z\rangle}, \ z, w \in \mathbb{C}^n.$$

Let B denote the unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ and $S = \{z \in \mathbb{C}^n : |z| = 1\}$ denote the unit sphere.

For $z, w \in B$, define the *involutive automorphism* φ_w of the unit ball B given by

$$\varphi_w(z) = \frac{w - P_w z - (1 - |w|^2)^{1/2} Q_w z}{1 - \langle z, w\rangle}$$

where $P_0 z = 0$, $P_w z = \frac{\langle z,w\rangle}{|w|^2} w$, $w \neq 0$, is the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by w and $Q_w = I - P_w$ ([8, 9]).

An upper semicontinuous function $u : B \to [-\infty, \infty)$, with $u \not\equiv -\infty$, is $\mathcal{M}$-*subharmonic* on B if

$$u(a) \leq \int_S u(\varphi_a(r\xi))d\sigma(\xi) \qquad (1)$$

for all $a \in B$ and all r sufficiently small, where $d\sigma$ is the Lebesgue measure on S normalized so that $\sigma(S) = 1$. A continuous function u for which equality holds in (1) is said to be $\mathcal{M}$-*harmonic* on B.

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The *invariant Laplacian* $\tilde{\Delta}$ on B is defined by

$$\tilde{\Delta} f(a) = \Delta(f \circ \varphi_a)(0),$$

where $f \in C^2(B)$, Δ is the ordinary Laplacian. It is known that $\tilde{\Delta}$ is invariant with respect to any holomorphic automorphism of B, i.e., $\tilde{\Delta}(f \circ \psi) = (\tilde{\Delta} f) \circ \psi$ for all $\psi \in \mathcal{M}$, the group of holomorphic automorphisms of B ([8, Chap.4], [9]).

We note that $u \in C^2$ is $\mathcal{M}$-subharmonic function if and only if $(\tilde{\Delta} u)(a) \geq 0$ for all $a \in B$, and $(\tilde{\Delta} u)(a) = 0$ if and only if u is $\mathcal{M}$-harmonic there.

The concept and the theory of $\mathcal{M}$-subharmonic function are due to David Ulrich ([14]).

The *Green's function* for the invariant Laplacian is defined by $G(z, w) = g(\varphi_w(z))$, where $g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1 - t^2)^{n-1} t^{-2n+1} dt$ ([7], [14], [9, Chap.6.2]).

If μ is a nonnegative Borel measure on B, the function G_μ defined by

$$G_\mu(z) = \int_B G(z, w) d\mu(w)$$

is called the *(invariant) Green potential* of μ, provided $G_\mu \not\equiv +\infty$. It is known that ([9, Chap.6.4]) the last condition is equivalent to

$$\int_B (1 - |w|^2)^n d\mu(w) < \infty. \tag{2}$$

Let u be a measurable function locally integrable on B. For $0 < p < \infty$ we define

$$m_p(r, u) = \left(\int_S |u(r\xi)|^p \, d\sigma(\xi) \right)^{\frac{1}{p}}, \ 0 < r < 1.$$

The class of twice continuously differentiable functions with compact support in B will be denoted by $C_0^2(B)$. For $\mathcal{M}$-subharmonic functions the following theorem holds.

Theorem A. ([9]) *If u is $\mathcal{M}$-subharmonic on B, then there exist a unique Borel measure μ_u on B such that*

$$\int_B \psi d\mu_u = \int_B u \tilde{\Delta} \psi d\tau \tag{3}$$

for all $\psi \in C_0^2(B)$, where τ is the invariant volume measure on B $\left(d\tau(z) = \frac{dA(z)}{(1-|z|^2)^{n+1}} \right)$, i.e. $d\mu_u = \tilde{\Delta} u d\tau$ in the sense of distributions.

If u is $\mathcal{M}$-subharmonic on B, the unique Borel measure μ_u satisfying (3) is called the *Riesz measure* of u.

If $z \in B$ and $\xi \in S$, then

$$\mathcal{P}(z, \xi) = \left\{ \frac{1 - |z|^2}{|1 - \langle z, \xi \rangle|^2} \right\}^n \tag{4}$$

is called the *Poisson kernel* of B.

If μ is a complex Borel measure on S and $z \in B$, then

$$\mathcal{P}[\mu](z) = \int_S \mathcal{P}(z, \xi) d\mu(\xi) \tag{5}$$

is called the *Poisson integral*.

Remark 1. It is known ([9, Prop. 5.10]) that for every (nonnegative) $\mathcal{M}$-harmonic function F on B, there exists a nonnegative Borel measure ν on S such that $F(z) = \mathcal{P}[\nu](z)$.

An $\mathcal{M}$-subharmonic function u on B has an *$\mathcal{M}$-harmonic majorant* on B if there exists an $\mathcal{M}$-harmonic function h on B such that $u(z) \le h(z)$ for all $z \in B$. Furthermore, if there exists an $\mathcal{M}$-harmonic function H satisfying $u(z) \le H(z)$, for all $z \in B$, and $H(z) \le h(z)$ for any $\mathcal{M}$-harmonic majorant h of u, then H is called the *least $\mathcal{M}$-harmonic majorant* of u, and will be denoted by H_u.

Theorem B. ***(Riesz Decomposition Theorem, [14, Th.2.16])*** *Suppose $u \not\equiv -\infty$ is $\mathcal{M}$-subharmonic on B and has an $\mathcal{M}$-harmonic majorant on B. Then*

$$u(z) = H_u(z) - \int_B G(z, w) d\mu_u(w), \tag{6}$$

where μ_u is the Riesz measure of u and H_u is the least $\mathcal{M}$-harmonic majorant u.

Remark 2. If $u \le 0,\ u \not\equiv -\infty$ is $\mathcal{M}$-subharmonic on B, then $v \equiv 0$ is an $\mathcal{M}$-harmonic majorant. Therefore, for H_u in representation (6), we have $H_u(z) \le 0,\ z \in B$. And according to Remark 1

$$H_u(z) = -\mathcal{P}[\nu](z),\ z \in B$$

where ν is a nonnegative Borel measure on S.

In [1] it was described the growth of pth means of the invariant Green potential in the unit ball in $\mathbb{C}^n$ in terms of smoothness properties of a measure. For the whole class of Borel measure satisfying (2) the growth rate of $m_p(r, G_\mu)$ was studied by Stoll in [10], [11]. And in the real case such research was published in [5, 12, 13].

Define for $a, b \in \bar{B}$ the *nonisotropic metric* on S by $d(a, b) = |1 - \langle a, b\rangle|^{1/2}$ ([8, Chap.5.1]). For $\xi \in S$ and $\delta > 0$ we set $C(\xi, \delta) = \{z \in B : d(z, \xi) < \delta^{1/2}\}$.

Theorem C. *([1]) Let $n > 1$, $1 < p < \frac{2n-1}{2(n-1)}$, $0 \le \gamma < 2n$, and let μ be a Borel measure satisfying (2). Then*

$$m_p\left(r, G_\mu\right) = O\left((1-r)^{\gamma - n}\right),\ r \uparrow 1 \tag{7}$$

holds if and only if

$$\left(\int_S \lambda^p\left(C(\xi, \delta)\right) d\sigma(\xi)\right)^{\frac{1}{p}} = O\left(\delta^\gamma\right),\ 0 < \delta < 1. \tag{8}$$

By using Theorem C we can get a generalization which describe the growth of pth means of $\mathcal{M}$-subharmonic function, which has representation (6), in terms of properties of the measure μ.

Let us define

$$d\lambda(w) = \frac{4n^2}{n+1} d\nu(w) + (1 - |w|^2)^n d\mu_u(w) \tag{9}$$

for $w \in \bar{B}$.

Theorem 1. *Let u be a nonpositive $\mathcal{M}$-subharmonic function in B, $u \not\equiv -\infty$, $1 < p < \frac{2n-1}{2(n-1)}$, $0 \le \gamma < 2n$ and u has an $\mathcal{M}$-harmonic majorant on B. Then*

$$m_p\left(r, u\right) = O\left((1-r)^{\gamma - n}\right),\ r \uparrow 1 \tag{10}$$

holds if and only if

$$\left(\int_S \lambda^p\left(C(\xi,\delta)\right)d\sigma(\xi)\right)^{\frac{1}{p}} = O\left(\delta^{\gamma}\right),\ 0<\delta<1. \tag{11}$$

For $\mathcal{M}$-harmonic function the following statement is true.

Corollary 1. *Let $u=\mathcal{P}[\nu](z)$ be an $\mathcal{M}$-harmonic function in B, where ν is a nonnegative Borel measure on S, $p>1$ and $0\le\gamma<2n$. Then*

$$m_p(r,u) = O\left((1-r)^{\gamma-n}\right),\ r\uparrow 1 \tag{12}$$

holds if and only if

$$\left(\int_S \nu^p\left(C(\xi,\delta)\right)d\sigma(\xi)\right)^{\frac{1}{p}} = O\left(\delta^{\gamma}\right),\ 0<\delta<1. \tag{13}$$

Note, that the growth of the integral $\mathcal{P}[\nu](z)$ in the uniform metric is described in terms of smoothness properties of the measure ν in [2] for arbitrary $n\in\mathbb{N}$.

2. Auxiliary results.

Lemma A. ([9]) *Let $0<\delta<\frac{1}{2}$ be fixed. Then g satisfies the following two inequalities:*

$$g(z)\ge\frac{n+1}{4n^2}(1-|z|^2)^n,\ z\in B, \tag{14}$$

$$g(z)\le c(\delta)(1-|z|^2)^n,\ z\in B, |z|\ge\delta, \tag{15}$$

where $c(\delta)$ is a positive constant. Furthermore, if $n>1$ then

$$g(z)\asymp|z|^{-2n+2},\qquad |z|\le\delta.$$

Let us define the kernel

$$K(z,w)=\begin{cases}\frac{G(z,w)}{(1-|w|^2)^n}, & \text{if } w\in B,\ z\in B;\\ \frac{n+1}{4n^2}\mathcal{P}(z,\xi), & \text{if } w\in S,\ z\in B.\end{cases}$$

We have the following properties of $K(z,w)$.

Proposition 1. *For $z, w=\rho\xi\in\bar{B}$ the following hold:*

a) For $w\in\{w:\ |\varphi_w(z)|\ge\frac{1}{4}\}$ the inequality

$$0\le K(z,w)\le c\frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}, \tag{16}$$

holds for some $c>0$.

b) $\lim\limits_{\rho\to 1-}\frac{G(z,\rho\xi)}{(1-\rho^2)^n}=\frac{n+1}{4n^2}\mathcal{P}(z,\xi)$ uniformly in $\xi\in S$.

c)

$$|K(z,w)|\ge\frac{n+1}{4n^2}\frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}},\ z\in B,\ w\in\bar{B}. \tag{17}$$

Proof. a) From (15) we get

$$0 \le K(z,\rho\xi) = \frac{g(\varphi_w(z))}{(1-\rho^2)^n} \le c\frac{(1-|\varphi_w(z)|^2)^n}{(1-\rho^2)^n}.$$

It is easily shown that $\varphi_w(z)$ satisfies ([9])

$$1-|\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z,w\rangle|^2}, \tag{18}$$

since $0 \le K(z,\rho\xi) \le c\frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}$.

b)

$$\lim_{\rho\to 1-}\frac{G(z,\rho\xi)}{(1-\rho^2)^n} = \lim_{\rho\to 1-}\frac{g(\varphi_w(z))}{(1-\rho^2)^n} = \lim_{\rho\to 1-}\frac{1}{(1-\rho^2)^n}\frac{n+1}{2n}\int_{|\varphi_{\rho\xi}(z)|}^1 (1-t^2)^{n-1}t^{-2n+1}dt.$$

By using of L'Hospital's rule we get

$$\begin{aligned}\lim_{\rho\to 1-}\frac{G(z,\rho\xi)}{(1-\rho^2)^n} &= \frac{n+1}{2n}\lim_{\rho\to 1-}\frac{(1-|\varphi_{\rho\xi}(z)|^2)^{n-1}|\varphi_{\rho\xi}(z)|^{-2n+1}}{n(1-\rho^2)^{n-1}2\rho}\frac{d}{d\rho}|\varphi_{\rho\xi}(z)| \\ &= \frac{n+1}{8n^2}\lim_{\rho\to 1-}\frac{(1-|\varphi_{\rho\xi}(z)|^2)^{n-1}}{(1-\rho^2)^{n-1}}\frac{d}{d\rho}|\varphi_{\rho\xi}(z)|^2 \\ &= -\frac{n+1}{8n^2}\lim_{\rho\to 1-}\frac{(1-|z|^2)^{n-1}}{|1-\langle z,\rho\xi\rangle|^{2(n-1)}}\frac{d}{d\rho}\frac{(1-\rho^2)(1-|z|^2)}{|1-\langle z,\rho\xi\rangle|^2}.\end{aligned}$$

By taking the derivative we get

$$\lim_{\rho\to 1-}\frac{G(z,\rho\xi)}{(1-\rho^2)^n} = -\frac{n+1}{8n^2}\lim_{\rho\to 1-}\frac{(1-|z|^2)^n}{|1-\langle z,\rho\xi\rangle|^{2(n-1)}}$$
$$\times\frac{-2\rho|1-\rho\langle z,\xi\rangle|^2-(1-\rho^2)(-2\langle z,\xi\rangle+2\rho|\langle z,\xi\rangle|^2)}{|1-\langle z,\rho\xi\rangle|^4} = \frac{n+1}{4n^2}\frac{(1-|z|^2)^n}{|1-\langle z,\xi\rangle|^{2n}}.$$

c) From (14) and (18) it follows that

$$K(z,\rho\xi) = \frac{g(\varphi_w(z))}{(1-\rho^2)^n} \ge \frac{n+1}{4n^2}\frac{(1-|\varphi_w(z)|^2)^n}{(1-\rho^2)^n} = \frac{n+1}{4n^2}\frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}.$$

□

Then representation (6) for $\mathcal{M}$-subharmonic functions can be rewritten as

$$u(z) = -\int_{\bar{B}} K(z,w)d\lambda(w),$$

it follows from Proposition 1, Theorem B and Remark 1.

3. Proof of Theorem 1. *Sufficiency.* Denote

$$B^*\Big(z,\frac{1}{4}\Big) = \Big\{w\in B : |\varphi_w(z)| < \frac{1}{4}\Big\}.$$

Let us estimate the absolute values of

$$u_1(z) := \int\limits_{B^*\left(z,\frac{1}{4}\right)} K(z,w)d\lambda(w) \quad \text{and} \quad u_2(z) := \int\limits_{B\setminus B^*\left(z,\frac{1}{4}\right)} K(z,w)d\lambda(w).$$

We start with u_1. In this case $d\lambda(w) = (1-|w|^2)^n d\mu_u(w)$ and proof literally repeats proof of Theorem 1.5 ([1]), so we get

$$\int_S |u_1(r\xi)|^p d\sigma(\xi) \le c_2(1-r)^{p(\gamma-n)}. \tag{19}$$

Let us estimate

$$u_2(z) = -\int_B K(z,w)d\tilde{\lambda}(w)$$

where $d\tilde{\lambda}(w) = \frac{4n^2}{n+1}d\nu(w) + (1-|w|)^n \chi_{B\setminus B^*\left(z,\frac{1}{4}\right)}(w)d\mu(w)$, χ_E is the characteristic function of a set E. We may assume that $|z| \ge \frac{1}{2}$.

By (16) we get that

$$|u_2(z)| \le c\int_B \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\tilde{\lambda}(w) \le c\int_B \frac{(1+|w|)^n(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\tilde{\lambda}(w).$$

Further proof literally repeats that of Theorem 1.5 ([1]). So

$$\int_S |u_2(r\xi)|^p d\sigma(\xi) \le \frac{c_3(n,p,\gamma)}{(1-r)^{p(n-\gamma)}}.$$

The latter inequality together with (19) completes the proof of the sufficiency.
Necessity. By (17)

$$|u(z)| \ge \int_B K(z,w)d\lambda(w) \ge \frac{n+1}{4n^2}\int_B \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\lambda(w)$$
$$\ge \frac{n+1}{4n^2}\int_{C(\xi,1-r)} \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\lambda(w).$$

Further we argue as in the proof of Theorem 1.5 ([1]). Since for $w \in C(\xi,1-r)$ $|1-\langle z,w\rangle| \le 2(1-|z|)$, we have

$$|u(z)| \ge \frac{n+1}{4^{n+1}n^2}\frac{\lambda(C(\xi,1-r))}{(1-r)^n}.$$

From the assumption of the theorem it follows that

$$\left(\frac{n+1}{2^{2(n+1)}n^2}\right)^p \int_S \frac{\lambda^p(C(\xi,1-r))}{(1-r)^{np}}d\sigma(\xi) \le \int_S |u(r\xi)|^p d\sigma(\xi) \le c_4^p(1-r)^{p(\gamma-n)}.$$

Thus

$$\int_S \lambda^p(C(\xi,1-r))d\sigma(\xi) \le c_4^p(1-r)^{p\gamma}, \quad 0<r<1.$$

Remark 3. Note that the assumption $1 < p < \frac{2n-1}{2(n-1)}$ is used only to estimate $u_1(z)$. For $\mathcal{M}$-harmonic functions we have $u_1 \equiv 0$, so in Corollary 1 we get $p > 1$.

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