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SUBNORMAL INDEPENDENT RANDOM VARIABLES AND LEVY'S PHENOMENON FOR ENTIRE FUNCTIONS

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Suppose that (Z_n) is a sequence of real independent subnormal random variables, i.e. such that there exists $D > 0$ satisfying following inequality for expectation $\mathbf{E}(e^{\lambda_0 Z_k}) \le e^{D\lambda_0^2}$ for any $k \in \mathbb{N}$ for all $\lambda_0 \in \mathbb{R}$. In this paper is proved that for random entire functions of the form $f(z,\omega) = \sum_{n=0}^{+\infty} Z_n(\omega) a_n z^n$ Levy's phenomenon holds.

1. Introduction. By the classical Wiman-Valiron theorem ([1]–[4]), for every non-constant entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and any $\varepsilon > 0$ there exist a set $E = E(f) \subset (1, +\infty)$ of finite logarithmic measure ($\int_E d\ln r < +\infty$) such that for all $r \in [r_0(\varepsilon); +\infty) \setminus E$ the inequality (*Wiman's inequality*)

$$M_f(r) \le \mu_f(r) \ln^{1/2+\varepsilon} \mu_f(r) \tag{1}$$

holds, where $M_f(r) = \max\{|f(z)| : \ |z| = r\}$, $\mu_f(r) = \max\{|a_n| r^n : \ n \ge 0\}$. Note that the constant $1/2$ cannot be replaced in general by a smaller number. Indeed, for entire function $f(z) = e^z$ we have ([3], p. 177) $M_f(r) \sim \sqrt{2\pi}\mu_f(r) \ln^{1/2} \mu_f(r)$ $(r \to +\infty)$.

In the class of entire functions f represented by gap power series of the form

$$f(z) = \sum_{k=0}^{+\infty} a_k z^{n_k}, \quad n_k \in \mathbb{Z}_+, \tag{2}$$

inequality (1) can be improved (for example see [5, 6]). In particular, from one result ([5]) obtained for entire Dirichlet series it follows that under the condition

$$(\exists \Delta \in (0; +\infty))(\exists \rho \in [1/2; 1])(\exists D > 0) : \quad |n(t) - \Delta t^\rho| \le D \quad (t \ge t_0), \tag{3}$$

(here $n(t) = \sum_{n_k \le t} 1$ is counting function of the sequence (n_k)), the inequality

$$M_f(r) \le \mu_f(r) \ln^{(2\rho-1)/2+\varepsilon} \mu_f(r), \tag{4}$$

holds for any $\varepsilon > 0$ and all $r \in [r_0(\varepsilon); +\infty) \setminus E_1$, where E_1 is a set of finite logarithmic measure (for $\rho = 1$ from inequlity (4) we get the classical Wiman's inequality). From other

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result ([6], see also [7]) obtained for entire Dirichlet series it follows that under condition (3) there exists an entire function f of the form (2) such that

$$\frac{M_f(r)}{\mu_f(r)\ln^{(2\rho-1)/2}\mu_f(r)} \to +\infty \quad (r \to +\infty). \tag{5}$$

From relation (5) for $\rho = 1$ it follows that there exists entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ such that

$$\frac{M_f(r)}{\mu_f(r)\ln^{1/2}\mu_f(r)} \to +\infty \quad (r \to +\infty).$$

On the other hand (see, for example, [8]–[11]) almost surely (a.s.) on the Steinhaus probability space $(\Omega, \mathcal{A}, P)$ exponent $1/2$ in inequality (1) can be replaced by $1/4$, and in inequality (4) (see [7]) a.s. exponent $(2\rho-1)/2$ can be replaced by $(2\rho-1)/4$ (*Levy's phenomenon*). Here $\Omega = [0;1]$, $\mathcal{A}$ is the σ-algebra of Borel's subsets of $[0;1]$ and P is the Lebesque measure (see [12, p. 9]). Note, that similar results for random entire functions of two complex variables we find in [13]–[15], and for random entire functions of several variables in [16, 17].

Let $\mathcal{N} = (n_k)$ be a sequence integer numbers such that $n_0 = 0$, $n_k < n_{k+1}$ $(k \geq 0)$, power series of the form (2) be an entire function, and $\big(X_n(\omega)\big)$ be a *multiplicative system* (MS), i.e. the sequence of real random variables on Steinhaus probability space such that

$$\mathbf{E}(X_{i_1}X_{i_2}\cdots X_{i_k}) = 0$$

for any $i_1 < i_2 < \ldots < i_k$, $k \geq 1$, where $\mathbf{E}\xi$ is the expectation of a random variable ξ, i.e. $\mathbf{E}\xi = \int_\Omega \xi(\omega)\mathbb{P}(d\omega)$. We denote

$$\mathcal{K}(f, \mathcal{Z}, \mathcal{N}) = \left\{ f(z,t) = \sum_{k=0}^{+\infty} a_k Z_k(t) z^{n_k} : t \in [0,1] \right\}, \tag{6}$$

where $\mathcal{Z} = (Z_k(t))$ is a sequence of complex-valued random variables.

In [7] we find the following theorem.

Theorem 1 ([7]). *Let a sequence $\mathcal{N} = (n_k)$ satisfy condition* (3)*, f be a non-constant entire function of the form* (2)*, a sequence complex valued variables $\mathcal{Z} = (Z_k)$ be such that $\big(\operatorname{Re} Z_k(t)\big) \in MS$, $\big(\operatorname{Im} Z_k(t)\big) \in MS$ and $|Z_k(t)| = 1$ a.s. $(k \geq 0)$. Then for every $\varepsilon > 0$ a.s. in $\mathcal{K}(f, \mathcal{Z}, \mathcal{N})$ there exists a set $E := E(\varepsilon, t, f) \subset [1, +\infty)$ of finite logarithmic measure such that the inequality*

$$M_f(r,t) := \max\{|f(z,t)| : \ |z| = r\} \leq \mu_f(r)(\ln\mu_f(r))^{(2\rho-1)/4+\varepsilon} \tag{7}$$

holds for $r \in [1; +\infty) \setminus E$.

In the case $n_k \equiv k$ (i.e. $\mathcal{N} = \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$) Theorem 1 implies corresponding result from paper [10] (see also [11]), and when in addition we suppose that $\mathcal{Z} = \mathcal{R}$, $\mathcal{Z} = \mathcal{H}$ or $\mathcal{Z} = \mathcal{S}$, then we obtain corresponding results from [8], [9] and [18] (see also [19]), respectively, where $\mathcal{R} = \big(R_k(t)\big)$ is the *Rademacher sequence*, i.e. a sequence of independent random variables, such that $\mathbb{P}\{t\colon R_k(t) = -1\} = \mathbb{P}\{t\colon R_k(t) = 1\} = 0,5$ $(k \in \mathbb{N})$, and $\mathcal{H} = \big(H_k(t)\big)$ is the *Steinhaus sequence*, i.e. a sequence independent random variables $H_k(t) = \exp\{2\pi i\eta_k(t)\}$, where $\{\eta_k(t)\}$ is a sequence independent uniformly distributed on $[0;1]$ random variables,

$\mathcal{S} = \big(\exp\{2\pi i\theta_k \cdot t\}\big)$, where (θ_k) is the sequence of integers numbers such that $\theta_{k+1}/\theta_k \geq q > 2$, $k \geq 0$. We remark that $\big(\cos(2\pi\theta_k t)\big) \in MS$, $\big(\sin(2\pi\theta_k t)\big) \in MS$ in this case (in [18] $q > 1$).

In general, the exponent $(2\rho - 1)/4$ in inequality (7) cannot be replaced by a smaller number. It follows from such a statement.

Theorem 2 ([7]). *If a sequence $\mathcal{N} = (n_k)$ satisfies condition* (3), *a sequence of complex valued variables $\mathcal{Z} = (Z_k) \in MS$ and $|Z_k(t)| = 1$ a.s. $(k \geq 0)$, then there exists an entire function f of the form* (2) *such that*

$$\lim_{r\to+\infty} \frac{M_f(r,t)}{\mu_f(r)(\ln \mu_f(r))^{(2\rho-1)/4}} = +\infty$$

a.s. in $\mathcal{K}(f, \mathcal{Z}, \mathcal{N})$.

Note, that in the paper [9] it the following assertion is proved:
For entire function $f(z) = e^z$ and every $\varepsilon > 0$ the relation

$$\lim_{r\to+\infty} \frac{M_f(r,t)}{\mu_f(r)\ln^{1/4-\varepsilon}\mu_f(r)} = +\infty \tag{8}$$

holds a.s. in $\mathcal{K}(f, \mathcal{R}, \mathbb{Z}_+)$ and in $\mathcal{K}(f, \mathcal{H}, \mathbb{Z}_+)$. Theorem 2 (for $\rho = 1$ in condition (3)) implies that there exists entire function f such that relation (8) holds with $\varepsilon = 0$.

Remark, that in statements cited above (Theorem 1 from [7] and others similarly results) the expectation of random variables is equal to zero. In connection with this prof. M. M. Sheremeta posed the following question: *Can one obtain the sharper Wiman's inequality for classes of random entire functions of the form $f(z) = \sum_{k=0}^{+\infty} Z_k(t)a_k z^{n_k}$ and $\mathbf{E}Z_k = \alpha \neq 0$ $(k \geq 0)$?* Negative answer to this question one can find in [10].

Also in these statements a sequence of random variables is almost surely uniformly bounded. In connection with this prof. O. B. Skaskiv posed the following **question**: *Does Levi's phenomenon hold in the case of unbounded random variables?*

In this paper we give partial positive answer for this question in the case of a sequence of independent subnormal random variables.

2. Auxiliary lemmas. For $r \geq 0$ and an entire function

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \tag{9}$$

denote by $\nu_f(r) = \max\{n\colon |a_n|r^n = \mu_f(r)\}$ the central index,

$$\mathfrak{M}_f(r) = \sum_{n=0}^{+\infty} |a_n|r^n,\ S_f^2(r) = \sum_{n=0}^{+\infty} |a_n|^2 r^{2n},\ S_N^2(r) = \sum_{n=0}^{N} |a_n|^2 r^{2n},\ \ln_2 x = \ln\ln x.$$

We need the following elementary statement (see also [20, 21]).

Proposition 1. *If a sequence of random variables $\big(Z_n(\omega)\big)$ satisfies the condition*

$$(\exists \alpha > 0)(\exists n_0 \in \mathbb{N})\colon \quad \sup\{\mathbf{E}|Z_n|^\alpha\colon\ n \geq n_0\} < +\infty, \tag{10}$$

then a.s.

$$(\exists N_1(\omega) \geq n_0)(\forall n > N_1(\omega))\colon \quad |Z_n(\omega)| \leq n^{1/\alpha}\ln^{2/\alpha} n.$$

Indeed, by Markov's inequality and condition (10) we have

$$\sum_{n=n_0}^{+\infty} \mathbb{P}\{\omega\colon\ |Z_n(\omega)|^\alpha \geq n\ln^2 n\} \leq \sum_{n=n_0}^{+\infty} \frac{\mathbf{E}|Z_n(\omega)|^\alpha}{n\ln^2 n} < +\infty.$$

Therefore, the First Lemma of Borel-Cantelli implies the statement of Proposition 1.

By condition (10) the radius of convergence of a series of form (6) $R(f_t) = +\infty$ a.s.

Also we need the following statement.

Lemma 1 ([22])**.** *For non-constant entire function $f(z)$ and every $\delta > 0$ there exists a set $E(\delta) \subset (1, +\infty)$ of finite logarithmic measure such that for all $r \in (1; +\infty)\backslash E$ we have*

$$\nu_f(r) \leq \ln \mu_f(r) \ln_2^{1+\delta} \mu_f(r), \tag{11}$$

$$(\forall n \in \mathbb{Z}_+)\colon |a_n|r^n \leq \mu_f(r) \exp\Big\{-\frac{k^2}{(|k| + \nu_f(r)) \ln^{1+\delta}(|k| + \nu_f(r))}\Big\}, \tag{12}$$

where $k = n - \nu_f(r)$.

Define

$$N(r) = \min\{n_0\colon (\forall n \geq n_0 \geq \ln \mu_f(r))|a_n|r^n < 1\},$$
$$N_\varepsilon(r) = N(re^\varepsilon) = \min\{n_0\colon (\forall n \geq n_0 \geq \ln \mu_f(re^\varepsilon))|a_n|r^n e^{n\varepsilon} < 1\} =$$
$$= \min\{n_0\colon (\forall n \geq n_0 \geq \ln \mu_f(re^\varepsilon))|a_n|r^n < e^{-n\varepsilon}\},\ \varepsilon = \frac{1}{N^\gamma(r)},\ \gamma > 0.$$

Remark that by the definition of $N_\varepsilon(r)$ we have $N_\varepsilon(r) \geq \ln \mu_f(r)$.

Similarly as in [23] one can prove such a statement.

Lemma 2. *For every $\delta > 0$ there exists a set $E(\delta) \subset (1, +\infty)$ of finite logarithmic measure such that for all $r \in (1; +\infty)\backslash E$*

$$N(r) < \ln^\rho \mu_f(r) \ln_2^{\rho+\delta} \mu_f(r).$$

Proof. Remark that if $n = k + \nu_f(r), k > 0$ then (12) implies that for some $\delta_0 > 0$ and $r \notin E$ we get

$$|a_n|r^n \leq \mu_f(r) \exp\Big\{-\frac{(n - \nu_f(r))^2}{n \ln^{1+\delta_0} n}\Big\}.$$

Choose $n_0(r) = 4 \ln \mu_f(r) \ln_2^{1+\delta_0} \mu_f(r)$. Then

$$\ln(|a_{n_0}|r^{n_0}) \leq \ln \mu_f(r) - \frac{9 \ln^2 \mu_f(r) \ln_2^{2+2\delta_0} \mu_f(r)}{4 \ln \mu_f(r) \ln_2^{1+\delta_0} \mu_f(r) \ln^{1+\delta_0}(4 \ln \mu_f(r) \ln_2^{1+\delta_0} \mu_f(r))} \leq$$
$$\leq \ln \mu_f(r) - \frac{9}{8} \ln \mu_f(r) < 0.$$

Therefore for $n > n_0(r)$ we get $|a_n|r^n < 1$. Finally, for $\delta = 2\rho\delta_0$ and $r \notin E$ we obtain

$$N(r) < 2\Delta(4 \ln \mu_f(r) \ln_2^{1+\delta_0} \mu_f(r))^\rho < \ln^\rho \mu_f(r) \ln_2^{\rho+\delta} \mu_f(r).$$

□

Lemma 3 ([24]). *Suppose that $L(r)$ is a positive increasing function of r for $r > r_0$. If $\gamma > 0$ and $|h| < L^{-\gamma}(r)$ then*

$$|L(re^h) - L(r)| < \gamma L(r)$$

for all r outside some set of finite logarithmic measure.

Remark that function $N(r)$ satisfies conditions of Lemma 2 and, therefore, for $r \to +\infty$ $(r \notin E)$ we get

$$N(r) \le N_\varepsilon(r) \le (1+\gamma)N(r) \le (1+\gamma)\ln^\rho \mu_f(r) \ln_2^{\rho+\delta} \mu_f(r) \le \ln^\rho \mu_f(r) \ln_2^{\rho+2\delta} \mu_f(r). \quad (13)$$

For an entire function $f(z)$ and sequence of a random variables $Z_n(t)$ we denote

$$g_n = g_n(r,\theta) = a_n r^n e^{in\theta} = q_n(r,\theta) + ip_n(r,\theta),$$

$$G = G(r,\theta,t) = Q(r,\theta,t) + iP(r,\theta,t) = \sum_{n=0}^{N} Z_n(t) g_n(r,\theta),$$

$$\|G\|_\infty = \max_{0\le\theta<2\pi} |G(r,\theta,t)|, \ \|Q\|_\infty = \max_{0\le\theta<2\pi} |\operatorname{Re} G(r,\theta,t)|, \quad \|P\|_\infty = \max_{0\le\theta<2\pi} |\operatorname{Im} G(r,\theta,t)|,$$

$$S_N = S_N(r) = \Big(\sum_{n=0}^{N} |g_n(r,\theta)|^2\Big)^{1/2} = \Big(\sum_{n=0}^{N} |a_n|^2 r^{2n}\Big)^{1/2}.$$

Lemma 4 ([12], p. 75). *If $Q(\theta) = \sum_{n=0}^{N} b_n \cos(n\theta + \theta_n)$, $N \ge 2$, $\theta_n \in \mathbb{R}$, then there exists a segment I such that its measure is equal to $1/N^2$ and for $\theta \in I$ we have*

$$|Q(\theta)| \ge \frac{1}{2} \max_{0\le\theta<2\pi} |Q(\theta)|.$$

The similar statement holds for $P(r,\theta) = \sum_{n=0}^{N} a_n r^n \sin(n\theta + \theta_n)$.

Lemma 5. *If $P(\theta) = \sum_{n=0}^{N} b_n \sin(n\theta + \theta_n)$, $N \ge 2$, $\theta_n \in \mathbb{R}$, then there exists a segment I such that its measure is equal to $1/N^2$ and for $\theta \in I$ we have*

$$|P(\theta)| \ge \frac{1}{2} \max_{0\le\theta<2\pi} |P(\theta)|.$$

It is enough consider $\theta_n + \frac{\pi}{2}$ instead of θ_n. If $\theta_n = \theta_n' + \frac{\pi}{2}$, then $P(r,\theta) = \sum_{n=0}^{N} a_n r^n \cos(n\theta + \theta_n')$. It remains to apply the inequality of Lemma 4.

Suppose that (Z_n) is a sequence of real independent subnormal random variables, i.e. such that there exists $D > 0$ such that for any $k \in \mathbb{N}$ and all $\lambda_0 \in \mathbb{R}$ we have

$$\mathbf{E}(e^{\lambda_0 Z_k}) \le e^{D\lambda_0^2}. \quad (14)$$

The class of such random variables is denoted by Ξ. Remark that any sequence of random variables $\{Z_n\} \in \Xi$ satisfies conditions of Proposition 1 with $\alpha = 2$ and random power series of form (6) is a. s. entire.

For $Z \in \Xi$ we have ([12, Exercise 7.8, p.81]) for any $k \in \mathbb{N}$: $\mathbf{E}(Z_k) = 0$ and

$$\sup_{k\in\mathbb{N}} \mathbf{E}(Z_k^2) = \sup_{k\in\mathbb{N}} \mathbf{D}(Z_k) \le 2D, \quad (15)$$

where $\mathbf{D}(Z_k) := \mathbf{E}(Z_k^2) - (\mathbf{E}Z_k)^2$ is the variance of random variable Z_k.

We prove the following analogue of the Salem-Zygmund theorem ([12], [25]).

Lemma 6. *Let $Z \in \Xi$, $N = N_\varepsilon(r)$. Then there exist an absolute constant $C > 0$ and set E of finite logarithmic measure such that*

$$\mathbb{P}\{\|G\|_\infty \geq C S_N \ln_2 S_N \sqrt{\ln N}\} \leq \frac{2}{N^2}, \quad r \to +\infty \ (r \notin E). \tag{16}$$

Proof. Using condition (14) we get

$$\mathbf{E}(e^{\lambda Q(r,\theta,t)}) = \mathbf{E}\Big(e^{\lambda \sum\limits_{n=0}^{N} Z_n q_n(r,\theta)}\Big) = \mathbf{E}\Big(\prod_{n=0}^{N} e^{\lambda Z_n q_n(r,\theta)}\Big) = \prod_{n=0}^{N} \mathbf{E} e^{\lambda Z_n q_n(r,\theta)}.$$

By Lemma 3 there exists a set $I = I(\omega)$ such that $m(I) \geq \frac{1}{N^2}$ and for $\theta \in I$ we have either

$$Q(r,\theta) \geq \frac{\|Q\|_\infty}{2} \quad \text{or} \quad -Q(r,\theta) \geq \frac{\|Q\|_\infty}{2}.$$

Then

$$\begin{aligned}\mathbf{E}(e^{\lambda\|Q\|_\infty/2}) &\leq N^2 \mathbf{E}\Big(\int_I (e^{\lambda Q(r,\theta)} + e^{-\lambda Q(r,\theta)}) d\theta\Big) \leq N^2 \mathbf{E}\Big(\int_0^{2\pi} (e^{\lambda Q(r,\theta)} + e^{-\lambda Q(r,\theta)}) d\theta\Big) \leq \\ &\leq N^2 \int_0^{2\pi} (\mathbf{E}(e^{\lambda Q(r,\theta)}) + \mathbf{E}(e^{-\lambda Q(r,\theta)})) d\theta \leq \\ &\leq N^2 \int_0^{2\pi} \Big(\prod_{n=0}^{N} \mathbf{E} e^{\lambda Z_n q_n(r,\theta)} + \prod_{n=0}^{N} \mathbf{E} e^{-\lambda Z_n q_n(r,\theta)}\Big) d\theta. \end{aligned} \tag{17}$$

Let us choose $N = N_\delta(r)$ and

$$\lambda = \frac{3\sqrt{\ln N}}{\sqrt{2D} S_N \ln_2 S_N}.$$

For any $k \in \mathbb{N}$ there exists $D > 0$ such that for all $\lambda_0 \in \mathbb{R}$ we have $\mathbf{E}(e^{\lambda_0 Z_k}) \leq e^{D\lambda_0^2}$. Therefore, from (17) we obtain

$$\mathbf{E}(e^{\lambda\|Q\|_\infty/2}) \leq 2N^2 \prod_{n=0}^{N} e^{D\lambda^2 |q_n(r,\theta)|^2} = 2N^2 \prod_{n=0}^{N} e^{D\lambda^2 |a_n|^2 r^{2n}} = 2N^2 e^{D\lambda^2 S_N^2},$$

$$\mathbf{E}(e^{\lambda\|Q\|_\infty/2 - D\lambda^2 S_N^2}) \leq 2N^4 \cdot \frac{1}{N^2},$$

i.e.

$$\mathbf{E}\Big(\exp\Big\{\frac{\lambda}{2}\Big(\|Q\|_\infty - 2D\lambda S_N^2 - \frac{2}{\lambda}\ln(2N^4)\Big)\Big\}\Big) \leq \frac{1}{N^2}.$$

From this inequality for $N \geq 4$ follows

$$\mathbf{E}\Big(\exp\Big\{\frac{\lambda}{2}\Big(\|Q\|_\infty - 2D\lambda S_N^2 - \frac{9}{\lambda}\ln N\Big)\Big\}\Big) \leq \frac{1}{N^2}.$$

By Markov's inequality

$$\mathbb{P}\Big\{\|Q\|_\infty \geq 2D\lambda S_N^2 + \frac{9}{\lambda}\ln N\Big\} = \mathbb{P}\Big\{\frac{\lambda}{2}\Big(\|Q\|_\infty - 2D\lambda S_N^2 - \frac{9}{\lambda}\ln N\Big) \geq 0\Big\} =$$

$$= \mathbb{P}\Big\{\exp\Big\{\frac{\lambda}{2}\Big(\|Q\|_\infty - 2D\lambda S_N^2 - \frac{9}{\lambda}\ln N\Big)\Big\}\geq 1\Big\}\leq$$
$$\leq \mathbf{E}\Big(\exp\Big\{\frac{\lambda}{2}\Big(\|Q\|_\infty - 2D\lambda S_N^2 - \frac{9}{\lambda}\ln N\Big)\Big\}\Big)\leq \frac{1}{N^2}.$$

Finally,

$$\mathbb{P}\Big\{\|Q\|_\infty \geq 2D\frac{3\sqrt{\ln N}}{\sqrt{2D}S_N\ln_2 S_N}S_N^2 + \frac{9\sqrt{2D}S_N\ln_2 S_N}{3\sqrt{\ln N}}\ln N\Big\}\leq \frac{1}{N^2},$$
$$\mathbb{P}\Big\{\|Q\|_\infty \geq 3\sqrt{2D}\frac{S_N}{\ln_2 S_N} + 3\sqrt{2D}S_N\ln_2 S_N\sqrt{\ln N}\Big\}\leq \frac{1}{N^2},$$
$$\mathbb{P}\Big\{\|Q\|_\infty \geq 5\sqrt{D}S_N\ln_2 S_N\sqrt{\ln N}\Big\}\leq \frac{1}{N^2}.$$

Similarly we obtain

$$\mathbb{P}\{\|P\|_\infty \geq 5\sqrt{D}S_N\ln_2 S_N\sqrt{\ln N}\} \leq \frac{1}{N^2}$$

and

$$\mathbb{P}\{\|G\|_\infty \geq 10\sqrt{D}S_N\ln_2 S_N\sqrt{\ln N}\} \leq \frac{2}{N^2}.$$

□

Lemma 7. *Let $Z \in \Xi$, $N = N_\varepsilon(r)$. There exist an absolute constant $C > 0$ and a set E of finite logarithmic measure such that*

$$\mathbb{P}\{t\colon M_f(r,t) \geq CS_N(r)\ln_2 S_N(r)\sqrt{\ln N}\} \leq \frac{3}{N^2}, \quad r \to +\infty \ (r \notin E). \tag{18}$$

Proof. Let us choose $\varepsilon = \frac{1}{N(r)}$. For $n \in N_\varepsilon(r)$ we consider events $B_n = \{t\colon |Z_n(t)| \geq n^2\}$. Then probabilities of these events we can estimate using Markov's inequality and (15). We obtain

$$\mathbb{P}(B_n) = \mathbb{P}\{t\colon |Z_n(t)|^2 \geq n^4\} \leq \frac{\mathbf{D}Z_n}{n^4} \leq \frac{2D}{n^4},$$
$$\sum_{n=N_\varepsilon(r)}^{+\infty}\mathbb{P}(B_n) \leq 2D\sum_{n=N_\varepsilon(r)}^{+\infty}\frac{1}{n^4} \leq \frac{4D}{3N_\varepsilon^3(r)}, \ r \to +\infty.$$

Let $B = \bigcup_{n=N_\varepsilon(r)}^{+\infty} B_n$. Then $\mathbb{P}(B) \leq \frac{2D}{3N_\varepsilon^3(r)}$, $r \to +\infty$. For $t \notin B$ we have using (13)

$$\max_{0\leq\theta<2\pi}\Big|\sum_{n=N_\varepsilon(r)}^{+\infty} Z_n a_n r^n e^{in\theta}\Big|\leq \sum_{n=N_\varepsilon(r)}^{+\infty}|Z_n||a_n|r^n \leq \sum_{n=N_\varepsilon(r)}^{+\infty} n^4 e^{-n\varepsilon} \leq$$
$$\leq CN_\varepsilon^5(r) \leq \ln^5\mu_f(r)\ln_2^6\mu_f(r) < \ln^6\mu_f(r) < S_N(r), \ r \to +\infty, \ (r \notin E).$$

Therefore,

$$\mathbb{P}\Big\{t\colon \max_{0\leq\theta<2\pi}\Big|\sum_{n=N}^{+\infty} Z_n a_n r^n e^{in\theta}\Big|\geq S_N\Big\}\leq \frac{1}{N^2}, \ N = N_\varepsilon(r).$$

By (16) we have

$$\mathbb{P}\{\|G\|_\infty \geq CS_N\ln_2 S_N\sqrt{\ln N}\} \leq \frac{2}{N^2}, \quad r \to +\infty \ (r \notin E).$$

From two previous inequalities we deduce that

$$\mathbb{P}\Big\{t\colon \max_{0\le\theta<2\pi}\Big|\sum_{n=0}^{+\infty} Z_n a_n r^n e^{in\theta}\Big|\ge 2CS_N \ln_2 S_N\sqrt{\ln N}\Big\}\le \frac{3}{N^2},\ N=N_\varepsilon(r).$$

□

Also we need the following lemma.

Lemma 8 ([11], see also [10]). *Let $l(r)$ be a continuous increasing to $+\infty$ function on $(1;+\infty)$, $E\subset(1;+\infty)$ be a set such that its complement contains an unbounded open set. Then there is an infinite sequence $1<r_1\le ...\le r_n\to+\infty$ $(n\to+\infty)$ such that*

(1) $(\forall n\in\mathbb{N}):\ r_n\notin E$;

(2) $(\forall n\in\mathbb{N}):\ \ln l(r_n)\ge \frac{n}{2}$;

(3) if $(r_n;r_{n+1})\cap E\ne(r_n,r_{n+1})$, *then* $l(r_{n+1})\le el(r_n)$;

(4) the set of indices, for which (3) holds, is unbounded.

3. Main result.

Theorem 3. *Let $Z\in\Xi$. Then there exists a set $E(\delta)$ of finite logarithmic measure such that for all $r\in(r_0(t),+\infty)\backslash E$ almost surely in $\mathcal{K}(f,\mathcal{Z})$ we have*

$$M_f(r,t)\le \mu_f(r)\ln^{(2\rho-1)/4}\mu_f(r)\ln_2^{3/2+\delta}\mu_f(r). \tag{19}$$

Proof. Choose $k(r)=\mu_f(r)$, a set E and a sequence $\{r_k\}$ from Lemma 7. Let

$$F_k=\{t\colon M_f(r_k,t)\ge CS_{N_\varepsilon(r_k)}(r_k)\ln_2 S_{N_\varepsilon(r_k)}(r_k)\sqrt{\ln N_\varepsilon(r_k)}\}.$$

By Lemma 7 and by the definition of $N_\varepsilon(r)$ we get

$$\sum_{k=1}^{+\infty}P(F_k)\le\sum_{k=1}^{+\infty}\frac{1}{N_\varepsilon^2(r_k)}\le\sum_{k=1}^{+\infty}\frac{1}{\ln^2\mu_f(r_k)}\le\sum_{k=1}^{+\infty}\frac{1}{k^2}<+\infty.$$

Then by Borel-Cantelli's lemma for almost all $t\in[0,1]$ for $k\ge k_0(t)$ we obtain

$$M_f(r_k,t)<CS_{N_\varepsilon(r_k)}(r_k)\ln_2 S_{N_\varepsilon(r_k)}(r_k)\sqrt{\ln N_\varepsilon(r_k)}.$$

Using inequalites $S_{N_\varepsilon(r)}(r)\le\mathfrak{M}_f(r)\mu_f(r)$ and $N_\varepsilon(r)\le\ln^\rho\mu_f(r)\ln_2^{\rho+\delta}\mu_f(r),(r\notin E)$ we get

$$M_f(r_k,t)<C\sqrt{\mathfrak{M}_f(r_k)\mu_f(r_k)}\ln_2(\mathfrak{M}_f(r_k)\mu_f(r_k))\sqrt{2\ln_2\mu_f(r_k)}<$$
$$<C\mu_f(r_k)\ln^{(2\rho-1)/4}\mu_f(r_k)\cdot 3\ln_2\mu_f(r_k)\sqrt{2\ln_2\mu_f(r_k)}<$$
$$<5C\mu_f(r_k)\ln^{(2\rho-1)/4}\mu_f(r_k)\ln_2^{3/2}\mu_f(r_k).$$

Let $r\ge r_{k_0(t)}$ be an arbitrary number outside set the E, $r\in(r_p,r_{p+1})$. By Lemma 8 $\mu_f(r_{p+1})\le e\mu_f(r_p)\le e\mu_f(r)$. Therefore for almost all $t\in[0;1]$ and $r\ge r_0(t)$ outside a set of finite logarithmic measure E we have

$$M_f(r,t)\le M_f(r_{p+1},t)<5C\mu_f(r_{p+1})\ln^{(2\rho-1)/4}\mu_f(r_{p+1})\ln_2^{3/2}\mu_f(r_{p+1})<$$
$$<5Ce\mu_f(r)\ln^{(2\rho-1)/4}(e\mu_f(r))\ln_2^{3/2}(e\mu_f(r))<\mu_f(r)\ln^{(2\rho-1)/4}\mu_f(r)\ln_2^{3/2+\delta}\mu_f(r).$$

□

In the case of complex random variables we get such a statement.

Corollary 1. *Let* $\operatorname{Re} Z \in \Xi$, $\operatorname{Im} Z \in \Xi$. *Then there exists a set* $E(\delta)$ *of finite logarithmic measure such that for all* $r \in (r_0(t), +\infty)\backslash E$ *almost surely in* $\mathcal{K}(f, \mathcal{Z})$ *we have*

$$M_f(r,t) \leq \mu_f(r) \ln^{(2\rho-1)/4} \mu_f(r) \ln_2^{3/2+\delta} \mu_f(r). \tag{20}$$

4. Some examples. There exists $Z \notin \Xi$ such that $\mathbf{E}Z_n = 0$, $\sup_n \mathbf{D}Z_n = +\infty$ and inequality (19) does not hold. It follows from the following statement.

Theorem 4. *For any* $\alpha > 0$ *there exist a sequence of real independent random variables satisfying for all* $n \in \mathbb{Z}_+$

$$\mathbf{E}Z_n = 0,\ \sup_n \mathbf{D}Z_n = +\infty,$$

entire function $f(z)$ *and a constant* $C > 0$ *such that almost surely in* $K(f, Z)$

$$M_f(r,t) = \max\{|f(z,t)| \colon |z| = r\} \geq C\mu_f(r) \ln^{1/4+\alpha} \mu_f(r),\ r > r_0(t).$$

Proof. We choose

$$f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n^\alpha n!},\ g(z) = e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$$

and a sequence of independent random variables (Z_n) such that

$$\mathbb{P}\{t \colon Z_n(t) = -n^\alpha\} = \mathbb{P}\{t \colon Z_n(t) = n^\alpha\} = \frac{1}{2}.$$

Then

$$\mathbf{E}Z_n = -n^\alpha \frac{1}{2} + n^\alpha \frac{1}{2} = 0,\ \mathbf{D}Z_n = n^{2\alpha}\frac{1}{2} + n^{2\alpha}\frac{1}{2} = n^{2\alpha},\ \sup_n \mathbf{D}Z_n = +\infty.$$

Denote

$$f(z,t) = \sum_{n=1}^{+\infty} Z_n(t) \frac{z^n}{n^\alpha n!} = \sum_{n=1}^{+\infty} R_n(t) \frac{z^n}{n!} = g(z,t),$$
$$M_f(r,t) = \max\{|f(z,t)| \colon |z| \leq r\} = \max\{|g(z,t)| \colon |z| \leq r\} = M_g(r,t),$$

where $\{R_n(t)\}$ is a sequence of the Rademacher random variables. By Theorem 2 for $\rho = 1$ we conclude that for $g(z,t)$ and some $C > 0$

$$M_f(r,t) = M_g(r,t) \geq C\mu_g(r) \ln^{1/4} \mu_g(r),\ r \to +\infty.$$

Remark that

$$\mu_g(r) = \max_{n \in \mathbb{Z}_+} \left\{\frac{r^n}{n!}\right\} = \max_{n \in \mathbb{Z}_+} \left\{n^\alpha \frac{r^n}{n^\alpha n!}\right\} \geq \nu_f^\alpha(r) \mu_f(r)$$

and $\nu_f(r) > r/2 = \ln M_g(r)/2$, $r \to +\infty$. Therefore

$$\mu_g(r) > \frac{1}{2^\alpha} M_f(r) \ln^\alpha M_g(r) > \frac{1}{2^\alpha} M_f(r) \ln^\alpha \mu_g(r) > \frac{1}{2^\alpha} M_f(r) \ln^\alpha \mu_f(r).$$

Finally, almost surely in $K(f, Z)$ we get

$$M_f(r,t) > C\mu_g(r) \ln^{1/4} \mu_g(r) > C_1 M_f(r) \ln^\alpha \mu_f(r) \ln^{1/4}(M_f(r) \ln^\alpha \mu_f(r)) >$$
$$> C_1 \mu_f(r) \ln^{1/4+\alpha} \mu_f(r).$$

□

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