Suppose that \((Z_n)\) is a sequence of real independent subnormal random variables, i.e. such that there exists \(D > 0\) satisfying following inequality for expectation 
\[
E(e^{k_0 Z_n}) \leq e^{D k_0^2}\n\]
for any \(k \in \mathbb{N}\) for all \(k_0 \in \mathbb{R}\). In this paper is proved that for random entire functions of the form 
\[
f(z, \omega) = \sum_{n=0}^{+\infty} Z_n(\omega) a_n z^n\n\]
Levy’s phenomenon holds.

1. Introduction. By the classical Wiman-Valiron theorem ([1]–[4]), for every non-constant entire function 
\[
f(z) = \sum_{n=0}^{+\infty} a_n z^n\n\]
and any \(\varepsilon > 0\) there exist a set \(E = E(f) \subset (1, +\infty)\) of finite logarithmic measure \(\left(\int_E d\ln r < +\infty\right)\) such that for all \(r \in [r_0(\varepsilon); +\infty) \setminus E\) the inequality (Wiman’s inequality) 
\[
M_f(r) \leq \mu_f(r) \ln^{1/2+\varepsilon} \mu_f(r) \tag{1}\n\]
holds, where \(M_f(r) = \max\{|f(z)| : |z| = r\}, \mu_f(r) = \max\{|a_n|r^n : n \geq 0\}\). Note that the constant \(1/2\) cannot be replaced in general by a smaller number. Indeed, for entire function 
\[
f(z) = e^z\n\]
we have ([3], p. 177) 
\[
M_f(r) \sim \sqrt{2\pi} \mu_f(r) \ln^{1/2} \mu_f(r) \quad (r \to +\infty).\n\]

In the class of entire functions \(f\) represented by gap power series of the form 
\[
f(z) = \sum_{k=0}^{+\infty} a_k z^{n_k}, \quad n_k \in \mathbb{Z}_+, \tag{2}\n\]
inequality (1) can be improved (for example see [5, 6]). In particular, from one result ([5]) obtained for entire Dirichlet series it follows that under the condition 
\[
(\exists \Delta \in (0; +\infty))(\exists \rho \in [1/2; 1])(\exists D > 0) : |n(t) - \Delta t^\rho| \leq D \quad (t \geq t_0), \tag{3}\n\]
(here \(n(t) = \sum_{n_k \leq t} 1\) is counting function of the sequence \((n_k)\)), the inequality 
\[
M_f(r) \leq \mu_f(r) \ln^{(2\rho-1)/2+\varepsilon} \mu_f(r), \tag{4}\n\]
holds for any \(\varepsilon > 0\) and all \(r \in [r_0(\varepsilon); +\infty) \setminus E_1\), where \(E_1\) is a set of finite logarithmic measure (for \(\rho = 1\) from inequality (4) we get the classical Wiman’s inequality). From other

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result ([6], see also [7]) obtained for entire Dirichlet series it follows that under condition (3) there exists an entire function \( f \) of the form (2) such that

\[
\frac{M_f(r)}{\mu_f(r) \ln^{(2p-1)/2} \mu_f(r)} \to +\infty \quad (r \to +\infty).
\]  

From relation (5) for \( \rho = 1 \) it follows that there exists entire function \( f(z) = \sum_{n=0}^{+\infty} a_n z^n \) such that

\[
\frac{M_f(r)}{\mu_f(r) \ln^{1/2} \mu_f(r)} \to +\infty \quad (r \to +\infty).
\]

On the other hand (see, for example, [8]–[11]) almost surely (a.s.) on the Steinhaus probability space \((\Omega, \mathcal{A}, P)\) exponent \(1/2\) in inequality (1) can be replaced by \(1/4\), and in inequality (4) (see [7]) a.s. exponent \((2\rho - 1)/2\) can be replaced by \((2\rho - 1)/4\) (Levy’s phenomenon). Here \( \Omega = [0; 1], \mathcal{A} \) is the \( \sigma \)-algebra of Borel’s subsets of \([0; 1] \) and \( P \) is the Lebesque measure (see [12, p. 9]). Note, that similar results for random entire functions of two complex variables we find in [13]–[15], and for random entire functions of several variables in [16, 17].

Let \( N = (n_k) \) be a sequence integer numbers such that \( n_0 = 0, n_k < n_{k+1} \) \((k \geq 0)\), power series of the form (2) be an entire function, and \( (X_n(\omega)) \) be a multiplicative system \((MS)\), i.e. the sequence of real random variables on Steinhaus probability space such that

\[
E(X_{i_1} X_{i_2} \cdots X_{i_k}) = 0
\]

for any \( i_1 < i_2 < \ldots < i_k, \ k \geq 1 \), where \( E\xi \) is the expectation of a random variable \( \xi \), i.e. \( E\xi = \int_{\Omega} \xi(\omega) dP(\omega) \). We denote

\[
\mathcal{K}(f, Z, N) = \left\{ f(z, t) = \sum_{k=0}^{+\infty} a_k Z_k(t) z^{n_k} : t \in [0, 1] \right\},
\]

where \( Z = (Z_k(t)) \) is a sequence of complex-valued random variables.

In [7] we find the following theorem.

**Theorem 1 ([7]).** Let a sequence \( N = (n_k) \) satisfy condition (3), \( f \) be a non-constant entire function of the form (2), a sequence complex valued variables \( Z = (Z_k) \) be such that \( (\Re Z_k(t)) \in MS, (\Im Z_k(t)) \in MS \) and \( |Z_k(t)| = 1 \) a.s. \((k \geq 0)\). Then for every \( \varepsilon > 0 \) a.s. in \( \mathcal{K}(f, Z, N) \) there exists a set \( E := E(\varepsilon, t, f) \subset [1, +\infty) \) of finite logarithmic measure such that the inequality

\[
M_f(r, t) := \max\{|f(z, t)| : |z| = r\} \leq \mu_f(r) (\ln \mu_f(r))^{(2\rho-1)/4+\varepsilon}
\]

holds for \( r \in [1; +\infty) \setminus E \).

In the case \( n_k \equiv k \) \((i.e. N = Z_+ := \mathbb{N} \cup \{0\})\) Theorem 1 implies corresponding result from paper [10] (see also [11]), and when in addition we suppose that \( Z = R, Z = H \) or \( Z = S \), then we obtain corresponding results from [8], [9] and [18] (see also [19]), respectively, where \( R = (R_k(t)) \) is the Rademacher sequence, i.e. a sequence of independent random variables, such that \( \mathbb{P}\{t: R_k(t) = -1\} = \mathbb{P}\{t: R_k(t) = 1\} = 0.5 \) \((k \in \mathbb{N})\), and \( H = (H_k(t)) \) is the Steinhaus sequence, i.e. a sequence independent random variables \( H_k(t) = \exp\{2\pi i \eta_k(t)\} \), where \( \{\eta_k(t)\} \) is a sequence independent uniformly distributed on \([0; 1]\) random variables,
\(S = \{ \exp\{2\pi i \theta_k \cdot t\} \},\) where \((\theta_k)\) is the sequence of integers numbers such that \(\theta_{k+1}/\theta_k \geq q > 2, k \geq 0.\) We remark that \((\cos(2\pi \theta_k t)) \in MS, \ (\sin(2\pi \theta_k t)) \in MS\) in this case (in \([18]\) \(q > 1).\)

In general, the exponent \((2\rho - 1)/4\) in inequality (7) cannot be replaced by a smaller number. It follows from such a statement.

**Theorem 2** ([7]). If a sequence \(N = (n_k)\) satisfies condition (3), a sequence of complex valued variables \(Z = (Z_k) \in MS\) and \(|Z_k(t)| = 1\) a.s. \((k \geq 0),\) then there exists an entire function \(f\) of the form (2) such that

\[
\lim_{r \to +\infty} \frac{M_f(r,t)}{\mu_f(r)(\ln \mu_f(r))^{(2\rho-1)/4}} = +\infty
\]

a.s. in \(K(f, Z, N).\)

Note, that in the paper \([9]\) it the following assertion is proved:

*For entire function \(f(z) = e^z\) and every \(\varepsilon > 0\) the relation

\[
\lim_{r \to +\infty} \frac{M_f(r,t)}{\mu_f(r) \ln^{1/4-\varepsilon} \mu_f(r)} = +\infty
\]

holds a.s. in \(K(f, R, Z_+}\) and in \(K(f, H, Z_+).\) Theorem 2 (for \(\rho = 1\) in condition (3)) implies that there exists entire function \(f\) such that relation (8) holds with \(\varepsilon = 0.\)

Remark, that in statements cited above (Theorem 1 from \([7]\) and others similarly results) the expectation of random variables is equal to zero. In connection with this prof. M. M. She-remeta posed the following question: _Can one obtain the sharper Wiman’s inequality for classes of random entire functions of the form \(f(z) = \sum_{k=0}^{+\infty} Z_k(t)a_k z^{n_k}\) and \(E Z_k = \alpha \neq 0\) \((k \geq 0)\)?_ Negative answer to this question one can find in \([10]\).

Also in these statements a sequence of random variables is almost surely uniformly bounded. In connection with this prof. O. B. Skaskiv posed the following question: _Does Levi’s phenomenon hold in the case of unbounded random variables?_

In this paper we give partial positive answer for this question in the case of a sequence of independent subnormal random variables.

2. **Auxiliary lemmas.** For \(r \geq 0\) and an entire function

\[
f(z) = \sum_{n=0}^{+\infty} a_n z^n
\]

denote by \(\nu_f(r) = \max\{n: |a_n|r^n = \mu_f(r)\}\) the central index,

\[
\mathfrak{M}_f(r) = \sum_{n=0}^{+\infty} |a_n|r^n, \ S_f^2(r) = \sum_{n=0}^{+\infty} |a_n|^2r^{2n}, \ S_N^2(r) = \sum_{n=0}^{N} |a_n|^2r^{2n}, \ \ln_2 x = \ln \ln x.
\]

We need the following elementary statement (see also \([20, 21]\)).

**Proposition 1.** If a sequence of random variables \((Z_n(\omega))\) satisfies the condition

\[
(\exists \alpha > 0)(\exists n_0 \in \mathbb{N}): \sup\{E|Z_n|^\alpha: n \geq n_0\} < +\infty,
\]

then a.s.

\[
(\exists N_1(\omega) \geq n_0)(\forall n > N_1(\omega)): |Z_n(\omega)| \leq n^{1/\alpha} \ln^{2/\alpha} n.
\]
Indeed, by Markov’s inequality and condition (10) we have

\[
\sum_{n=n_0}^{+\infty} P\{\omega: |Z_n(\omega)|^\alpha \geq n \ln^2 n\} \leq \sum_{n=n_0}^{+\infty} \frac{E|Z_n(\omega)|^\alpha}{n \ln^2 n} < +\infty.
\]

Therefore, the First Lemma of Borel-Cantelli implies the statement of Proposition 1.

By condition (10) the radius of convergence of a series of form (6) \( R(f_t) = +\infty \) a.s.

Also we need the following statement.

**Lemma 1** ([22]). For non-constant entire function \( f(z) \) and every \( \delta > 0 \) there exists a set \( E(\delta) \subset (1, +\infty) \) of finite logarithmic measure such that for all \( r \in (1; +\infty) \setminus E \) we have

\[
\nu_f(r) \leq \ln \mu_f(r) \ln_2^{1+\delta} \mu_f(r), \tag{11}
\]

\[
(\forall n \in \mathbb{Z}_+): |a_n|r^n \leq \mu_f(r) \exp\left\{-\frac{k^2}{\ln \nu_f(r)} \left(\frac{\ln 2}{\ln \nu_f(r)}\right)^{1+\delta}\right\}, \tag{12}
\]

where \( k = n - \nu_f(r) \).

Define

\[
N(r) = \min\{n_0: (\forall n \geq n_0 \geq \ln \mu_f(r))|a_n|r^n < 1\},
\]

\[
N_\varepsilon(r) = N(re^\varepsilon) = \min\{n_0: (\forall n \geq n_0 \geq \ln \mu_f(re^\varepsilon))|a_n|r^n e^{-\varepsilon} < 1\} = \min\{n_0: (\forall n \geq n_0 \geq \ln \mu_f(re^\varepsilon))|a_n|r^n < e^{-\varepsilon}\}, \quad \varepsilon = \frac{1}{\ln 2}, \quad \gamma > 0.
\]

Remark that by the definition of \( N_\varepsilon(r) \) we have \( N_\varepsilon(r) \geq \ln \mu_f(r) \).

Similarly as in [23] one can prove such a statement.

**Lemma 2.** For every \( \delta > 0 \) there exists a set \( E(\delta) \subset (1, +\infty) \) of finite logarithmic measure such that for all \( r \in (1; +\infty) \setminus E \)

\[
N(r) < \ln^\rho \mu_f(r) \ln_2^{\rho+\delta} \mu_f(r).
\]

**Proof.** Remark that if \( n = k + \nu_f(r), k > 0 \) then (12) implies that for some \( \delta_0 > 0 \) and \( r \not\in E \) we get

\[
|a_n|r^n \leq \mu_f(r) \exp\left\{-\frac{(n - \nu_f(r))^2}{n \ln^1 n}\right\}.
\]

Choose \( n_0(r) = 4 \ln \mu_f(r) \ln_2^{1+\delta_0} \mu_f(r) \). Then

\[
\ln(|a_{n_0}|r^{n_0}) \leq \ln \mu_f(r) - \frac{9 \ln^2 \mu_f(r) \ln_2^{2+2\delta_0} \mu_f(r)}{4 \ln \mu_f(r) \ln_2^{1+\delta_0} \mu_f(r) (4 \ln \mu_f(r) \ln_2^{1+\delta_0} \mu_f(r))} \leq \ln \mu_f(r) - \frac{9}{8} \ln \mu_f(r) < 0.
\]

Therefore for \( n > n_0(r) \) we get \( |a_n|r^n < 1 \). Finally, for \( \delta = 2\rho\delta_0 \) and \( r \not\in E \) we obtain

\[
N(r) < 2\Delta(4 \ln \mu_f(r) \ln_2^{1+\delta_0} \mu_f(r))^\rho < \ln^\rho \mu_f(r) \ln_2^{\rho+\delta} \mu_f(r).
\]

\( \square \)
Lemma 3 ([24]). Suppose that $L(r)$ is a positive increasing function of $r$ for $r > r_0$. If $\gamma > 0$ and $|h| < L^{-\gamma}(r)$ then
\[ |L(re^h) - L(r)| < \gamma L(r) \]
for all $r$ outside some set of finite logarithmic measure.

Remark that function $N(r)$ satisfies conditions of Lemma 2 and, therefore, for $r \to +\infty$ ($r \not\in E$) we get
\[ N(r) \leq N_e(r) \leq (1 + \gamma)N(r) \leq (1 + \gamma)\ln^p \mu_f(r) \ln^{p+\delta} \mu_f(r) \leq \ln^p \mu_f(r) \ln^{p+2\delta} \mu_f(r). \quad (13) \]

For an entire function $f(z)$ and sequence of a random variables $Z_n(t)$ we denote
\[ g_n = g_n(r, \theta) = a_n r^n e^{in\theta} = q_n(r, \theta) + ip_n(r, \theta), \]
\[ G = G(r, \theta, t) = Q(r, \theta, t) + iP(r, \theta, t) = \sum_{n=0}^{N} Z_n(t) g_n(r, \theta), \]
\[ \|G\|_{\infty} = \max_{0 \leq \theta < 2\pi} |G(r, \theta, t)|, \quad \|Q\|_{\infty} = \max_{0 \leq \theta < 2\pi} |\text{Re} G(r, \theta, t)|, \quad \|P\|_{\infty} = \max_{0 \leq \theta < 2\pi} |\text{Im} G(r, \theta, t)|, \]
\[ S_N = S_N(r) = \left( \sum_{n=0}^{N} |g_n(r, \theta)|^2 \right)^{1/2} = \left( \sum_{n=0}^{N} |a_n|^2 r^{2n} \right)^{1/2}. \]

Lemma 4 ([12], p. 75). If $Q(\theta) = \sum_{n=0}^{N} b_n \cos(n\theta + \theta_n)$, $N \geq 2$, $\theta_n \in \mathbb{R}$, then there exists a segment $I$ such that its measure is equal to $1/N^2$ and for $\theta \in I$ we have
\[ |Q(\theta)| \geq \frac{1}{2} \max_{0 \leq \theta < 2\pi} |Q(\theta)|. \]

The similar statement holds for $P(r, \theta) = \sum_{n=0}^{N} a_n r^n \sin(n\theta + \theta_n)$.

Lemma 5. If $P(\theta) = \sum_{n=0}^{N} b_n \sin(n\theta + \theta_n)$, $N \geq 2$, $\theta_n \in \mathbb{R}$, then there exists a segment $I$ such that its measure is equal to $1/N^2$ and for $\theta \in I$ we have
\[ |P(\theta)| \geq \frac{1}{2} \max_{0 \leq \theta < 2\pi} |P(\theta)|. \]

It is enough consider $\theta_n + \frac{\pi}{2}$ instead of $\theta_n$. If $\theta_n = \theta'_n + \frac{\pi}{2}$, then $P(r, \theta) = \sum_{n=0}^{N} a_n r^n \cos(n\theta + \theta'_n)$. It remains to apply the inequality of Lemma 4.

Suppose that $(Z_n)$ is a sequence of real independent subnormal random variables, i.e. such that there exists $D > 0$ such that for any $k \in \mathbb{N}$ and all $\lambda_0 \in \mathbb{R}$ we have
\[ \mathbb{E}(e^{\lambda_0 Z_k}) \leq e^{D\lambda_0^2}. \quad (14) \]

The class of such random variables is denoted by $\Xi$. Remark that any sequence of random variables $\{Z_n\} \in \Xi$ satisfies conditions of Proposition 1 with $\alpha = 2$ and random power series of form (6) is a. s. entire.

For $Z \in \Xi$ we have ([12, Exercise 7.8, p.81]) for any $k \in \mathbb{N}$: $\mathbb{E}(Z_k) = 0$ and
\[ \sup_{k \in \mathbb{N}} \mathbb{E}(Z_k^2) = \sup_{k \in \mathbb{N}} D(Z_k) \leq 2D, \quad (15) \]
where $D(Z_k) := \mathbb{E}(Z_k^2) - (\mathbb{E}Z_k)^2$ is the variance of random variable $Z_k$.

We prove the following analogue of the Salem-Zygmund theorem ([12], [25]).
Lemma 6. Let \( Z \in \Xi, \; N = N_\varepsilon(r) \). Then there exist an absolute constant \( C > 0 \) and set \( E \) of finite logarithmic measure such that

\[
P\{ \|G\|_\infty \geq CS_N \ln_2 S_N \sqrt{\ln N} \} \leq \frac{2}{N^2}, \; r \to +\infty (r \notin E).
\]  

Proof. Using condition (14) we get

\[
E(e^{\lambda Q(r,\theta,t)}) = E\left( e^{\sum_{n=0}^N Z_n q_n(r,\theta)} \right) = E\left( \prod_{n=0}^N e^{Z_n q_n(r,\theta)} \right) = \prod_{n=0}^N e^{Z_n q_n(r,\theta)}.
\]

By Lemma 3 there exists a set \( I = I(\omega) \) such that \( m(I) \geq \frac{1}{N^2} \) and for \( \theta \in I \) we have either

\[
Q(r, \theta) \geq \frac{\|Q\|_\infty}{2} \quad \text{or} \quad -Q(r, \theta) \geq \frac{\|Q\|_\infty}{2}.
\]

Then

\[
E(e^{\|Q\|_\infty/2}) \leq N^2 E\left( \int_0^{2\pi} (e^{\lambda Q(r,\theta)} + e^{-\lambda Q(r,\theta)}) \, d\theta \right) \leq N^2 E\left( \int_0^{2\pi} (e^{\lambda Q(r,\theta)} + e^{-\lambda Q(r,\theta)}) \, d\theta \right)
\]

\[
\leq N^2 \int_0^{2\pi} \left( \prod_{n=0}^N e^{Z_n q_n(r,\theta)} + \prod_{n=0}^N e^{-Z_n q_n(r,\theta)} \right) \, d\theta.
\]  

(17)

Let us choose \( N = N_\delta(r) \) and

\[
\lambda = \frac{3\sqrt{\ln N}}{\sqrt{2DS_N \ln_2 S_N}}.
\]

For any \( k \in \mathbb{N} \) there exists \( D > 0 \) such that for all \( \lambda_0 \in \mathbb{R} \) we have \( E(e^{\lambda_0 Z_k}) \leq e^{D\lambda_0^2} \). Therefore, from (17) we obtain

\[
E(e^{\|Q\|_\infty/2}) \leq 2N^2 \prod_{n=0}^N e^{2D\lambda^2 |q_n(r,\theta)|^2} = 2N^2 \prod_{n=0}^N e^{2D\lambda^2 |a_n|^2r^{2n}} = 2N^2 e^{2D\lambda^2 S_N^2},
\]

\[
E(e^{\lambda \|Q\|_\infty/2 - D\lambda^2 S_N^2}) \leq 2N^4 \cdot \frac{1}{N^2},
\]

i.e.

\[
E\left( \exp\left\{ \frac{\lambda}{2} \left( \|Q\|_\infty - 2D\lambda S_N^2 - \frac{2}{\lambda} \ln(2N^4) \right) \right\} \right) \leq \frac{1}{N^2}.
\]

From this inequality for \( N \geq 4 \) follows

\[
E\left( \exp\left\{ \frac{\lambda}{2} \left( \|Q\|_\infty - 2D\lambda S_N^2 - \frac{9}{\lambda} \ln N \right) \right\} \right) \leq \frac{1}{N^2}.
\]

By Markov’s inequality

\[
P\left\{ \|Q\|_\infty \geq 2D\lambda S_N^2 + \frac{9}{\lambda} \ln N \right\} = P\left\{ \frac{\lambda}{2} \left( \|Q\|_\infty - 2D\lambda S_N^2 - \frac{9}{\lambda} \ln N \right) \geq 0 \right\} =
\]
\[ \begin{align*}
&= \mathbb{P} \left\{ \exp \left( \frac{\lambda}{2} \left( \|Q\|_\infty - 2D\lambda S^2_N - \frac{9}{\lambda} \ln N \right) \right) \geq 1 \right\} \leq \\
&\leq \mathbb{E} \left( \exp \left( \frac{\lambda}{2} \left( \|Q\|_\infty - 2D\lambda S^2_N - \frac{9}{\lambda} \ln N \right) \right) \right) \leq \frac{1}{N^2}.
\end{align*} \]

Finally,
\[ \mathbb{P} \left\{ \|Q\|_\infty \geq 2D \frac{3\sqrt{\ln N}}{\sqrt{2DS_N \ln_2 S_N}} S^2_N + \frac{9\sqrt{2DS_N \ln_2 S_N}}{3\sqrt{\ln N}} \ln N \right\} \leq \frac{1}{N^2}, \]
\[ \mathbb{P} \left\{ \|Q\|_\infty \geq 3\sqrt{2D} \frac{S_N}{\ln_2 S_N} + 3\sqrt{2DS_N \ln_2 S_N \sqrt{\ln N}} \right\} \leq \frac{1}{N^2}, \]
\[ \mathbb{P} \left\{ \|Q\|_\infty \geq 5\sqrt{DS_N \ln_2 S_N \sqrt{\ln N}} \right\} \leq \frac{1}{N^2}. \]

Similarly we obtain
\[ \mathbb{P} \{ \|P\|_\infty \geq 5\sqrt{DS_N \ln_2 S_N \sqrt{\ln N}} \} \leq \frac{1}{N^2} \]
and
\[ \mathbb{P} \{ \|G\|_\infty \geq 10\sqrt{DS_N \ln_2 S_N \sqrt{\ln N}} \} \leq \frac{2}{N^2}. \]

\[ \square \]

**Lemma 7.** Let \( Z \in \Xi, \ N = N_\varepsilon(r). \) There exist an absolute constant \( C > 0 \) and a set \( E \) of finite logarithmic measure such that
\[ \mathbb{P} \{ t: M_f(r, t) \geq CS_N(r) \ln_2 S_N(r) \sqrt{\ln N} \} \leq \frac{3}{N^2}, \ r \to +\infty (r \not\in E). \quad (18) \]

**Proof.** Let us choose \( \varepsilon = \frac{1}{N(r)}. \) For \( n \in N_\varepsilon(r) \) we consider events \( B_n = \{ t: |Z_n(t)| \geq n^2 \}. \) Then probabilities of these events we can estimate using Markov’s inequality and (15). We obtain
\[ \mathbb{P}(B_n) = \mathbb{P}\{t: |Z_n(t)|^2 \geq n^4\} \leq \frac{\mathbb{D}Z_n}{n^4} \leq \frac{2D}{n^4}, \]
\[ \sum_{n=N_\varepsilon(r)}^{+\infty} \mathbb{P}(B_n) \leq 2D \sum_{n=N_\varepsilon(r)}^{+\infty} \frac{1}{n^4} \leq \frac{4D}{3N^3 \varepsilon(r)}, \ r \to +\infty. \]

Let \( B = \bigcup_{n=N_\varepsilon(r)}^{+\infty} B_n. \) Then \( \mathbb{P}(B) \leq \frac{2D}{3N^3 \varepsilon(r)}, \ r \to +\infty. \) For \( t \notin B \) we have using (13)
\[ \begin{align*}
&\max_{0 \leq \theta < 2\pi} \left| \sum_{n=N_\varepsilon(r)}^{+\infty} Z_n a_n r^n e^{in\theta} \right| \leq \sum_{n=N_\varepsilon(r)}^{+\infty} |Z_n| a_n |r^n| \leq \sum_{n=N_\varepsilon(r)}^{+\infty} n^4 e^{-n\varepsilon} \leq \]
\[ \leq CN_\varepsilon^5 (r) \leq \ln^5 \mu_f (r) \ln_2^6 \mu_f (r) < \ln^6 \mu_f (r) < S_N (r), \ r \to +\infty, (r \notin E). \]

Therefore,
\[ \mathbb{P} \left\{ t: \max_{0 \leq \theta < 2\pi} \left| \sum_{n=N}^{+\infty} Z_n a_n r^n e^{in\theta} \right| \geq S_N \right\} \leq \frac{1}{N^2}, \ N = N_\varepsilon(r). \]

By (16) we have
\[ \mathbb{P} \{ \|G\|_\infty \geq CS_N \ln_2 S_N \sqrt{\ln N} \} \leq \frac{2}{N^2}, \ r \to +\infty (r \notin E). \]
From two previous inequalities we deduce that

\[
P\left\{ \frac{1}{t} \max_{0 \leq \theta < 2\pi} \left| \sum_{n=0}^{+\infty} Z_n a_n r^n e^{i\theta} \right| \geq 2CS_N \ln_2 S_N \sqrt{\ln N} \right\} \leq \frac{3}{N^2}, \quad N = N_e(r).
\]

Also we need the following lemma.

**Lemma 8** ([11], see also [10]). Let \( l(r) \) be a continuous increasing to \(+\infty\) function on \((1; +\infty), E \subset (1; +\infty)\) be a set such that its complement contains an unbounded open set. Then there is an infinite sequence \( 1 < r_1 \leq \ldots \leq r_n \to +\infty (n \to +\infty) \) such that

1. \( (\forall n \in \mathbb{N}) : r_n \notin E; \)
2. \( (\forall n \in \mathbb{N}) : \ln l(r_n) \geq \frac{n}{7}; \)
3. if \( (r_n; r_{n+1}) \cap E \neq (r_n, r_{n+1}), \) then \( l(r_{n+1}) \leq el(r_n); \)
4. the set of indices, for which (3) holds, is unbounded.

**3. Main result.**

**Theorem 3.** Let \( Z \in \Xi. \) Then there exists a set \( E(\delta) \) of finite logarithmic measure such that for all \( r \in (r_0(t), +\infty) \setminus E \) almost surely in \( K(f, \Xi) \) we have

\[
M_f(r, t) \leq \mu_f(r) \ln^{(2\rho-1)/4} \mu_f(r) \ln^{3/2+\delta} \mu_f(r).
\]  

**Proof.** Choose \( k(r) = \mu_f(r), \) a set \( E \) and a sequence \( \{r_k\} \) from Lemma 7. Let

\[
F_k = \{ t : M_f(r_k, t) \geq CS_{N_e(r_k)}(r_k) \ln_2 S_{N_e(r_k)}(r_k) \sqrt{\ln N_e(r_k)} \}.
\]

By Lemma 7 and by the definition of \( N_e(r) \) we get

\[
\sum_{k=1}^{+\infty} P(F_k) \leq \sum_{k=1}^{+\infty} \frac{1}{N_e^2(r_k)} \leq \sum_{k=1}^{+\infty} \frac{1}{\ln^2 \mu_f(r_k)} \leq \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty.
\]

Then by Borel-Cantelli’s lemma for almost all \( t \in [0, 1] \) for \( k \geq k_0(t) \) we obtain

\[
M_f(r_k, t) < CS_{N_e(r_k)}(r_k) \ln_2 S_{N_e(r_k)}(r_k) \sqrt{\ln N_e(r_k)}.
\]

Using inequalities \( S_{N_e(r)}(r) \leq M_f(r) \mu_f(r) \) and \( N_e(r) \leq \ln^\theta \mu_f(r) \ln^{3/2+\delta} \mu_f(r), (r \notin E) \) we get

\[
M_f(r_k, t) < C\sqrt{M_f(r_k) \mu_f(r_k) \ln_2 (M_f(r_k) \mu_f(r_k))} \sqrt{2 \ln_2 \mu_f(r_k)} <
\]

\[
< C\mu_f(r_k) \ln^{(2\rho-1)/4} \mu_f(r_k) \cdot 3 \ln_2 \mu_f(r_k) \sqrt{2 \ln_2 \mu_f(r_k)} <
\]

\[
< 5C\mu_f(r_k) \ln^{(2\rho-1)/4} \mu_f(r_k) \ln^{3/2} \mu_f(r_k).
\]

Let \( r \geq r_{k_0(t)} \) be an arbitrary number outside set the \( E, r \in (r_p, r_{p+1}). \) By Lemma 8 \( \mu_f(r_{p+1}) \leq e \mu_f(r_p) \leq e \mu_f(r). \) Therefore for almost all \( t \in [0; 1] \) and \( r \geq r_0(t) \) outside a set of finite logarithmic measure \( E \) we have

\[
M_f(r, t) \leq M_f(r_{p+1}, t) < 5C\mu_f(r_{p+1}) \ln^{(2\rho-1)/4} \mu_f(r_{p+1}) \ln^{3/2} \mu_f(r_{p+1}) <
\]

\[
< 5Ce \mu_f(r) \ln^{(2\rho-1)/4} (e \mu_f(r)) \ln^{3/2} (e \mu_f(r)) < \mu_f(r) \ln^{(2\rho-1)/4} \mu_f(r) \ln^{3/2+\delta} \mu_f(r).
\]
In the case of complex random variables we get such a statement.

**Corollary 1.** Let $\Re Z \in \Xi$, $\Im Z \in \Xi$. Then there exists a set $E(\delta)$ of finite logarithmic measure such that for all $r \in (r_0(t), +\infty) \setminus E$ almost surely in $K(f, Z)$ we have

$$M_f(r, t) \leq \mu_f(r) \ln^{(2n-1)/4} \mu_f(r) \ln^{3/2+\delta} \mu_f(r).$$

(20)

4. **Some examples.** There exists $Z \not\in \Xi$ such that $EZ_n = 0$, $\sup_n DZ_n = +\infty$ and inequality (19) does not hold. It follows from the following statement.

**Theorem 4.** For any $\alpha > 0$ there exist a sequence of real independent random variables satisfying for all $n \in \mathbb{Z}_+$

$$EZ_n = 0, \sup_n DZ_n = +\infty,$$

entire function $f(z)$ and a constant $C > 0$ such that almost surely in $K(f, Z)$

$$M_f(r, t) = \max\{|f(z, t)|: |z| = r\} \geq C \mu_f(r) \ln^{1/4+\alpha} \mu_f(r), \ r > r_0(t).$$

**Proof.** We choose

$$f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n^{2\alpha} n!}, \ g(z) = e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$$

and a sequence of independent random variables $(Z_n)$ such that

$$\mathbb{P}\{t: Z_n(t) = -n\alpha\} = \mathbb{P}\{t: Z_n(t) = n\alpha\} = \frac{1}{2}.$$

Then

$$EZ_n = -n\alpha \frac{1}{2} + n\alpha \frac{1}{2} = 0, \ DZ_n = n^{2\alpha} \frac{1}{2} + n^{2\alpha} \frac{1}{2} = n^{2\alpha}, \ \sup_n DZ_n = +\infty.$$

Denote

$$f(z, t) = \sum_{n=1}^{+\infty} Z_n(t) \frac{z^n}{n^{2\alpha} n!} = \sum_{n=1}^{+\infty} R_n(t) \frac{z^n}{n!} = g(z, t),$$

$$M_f(r, t) = \max\{|f(z, t)|: |z| \leq r\} = \max\{|g(z, t)|: |z| \leq r\} = M_g(r, t),$$

where $\{R_n(t)\}$ is a sequence of the Rademacher random variables. By Theorem 2 for $\rho = 1$ we conclude that for $g(z, t)$ and some $C > 0$

$$M_f(r, t) = M_g(r, t) \geq C \mu_g(r) \ln^{1/4} \mu_g(r), \ r \to +\infty.$$

Remark that

$$\mu_g(r) = \max_{n \in \mathbb{Z}_+} \left\{ \frac{r^n}{n!} \right\} = \max_{n \in \mathbb{Z}_+} \left\{ n^{\alpha} \frac{r^n}{n^{2\alpha} n!} \right\} \geq \nu_f(r) \mu_f(r)$$

and $\nu_f(r) > r/2 = \ln M_f(r)/2$, $r \to +\infty$. Therefore

$$\mu_{g}(r) > \frac{1}{2\alpha} M_f(r) \ln^{\alpha} M_g(r) > \frac{1}{2\alpha} M_f(r) \ln^{\alpha} \mu_{g}(r) > \frac{1}{2\alpha} M_f(r) \ln^{\alpha} \mu_{f}(r).$$

Finally, almost surely in $K(f, Z)$ we get

$$M_f(r, t) > C \mu_{g}(r) \ln^{1/4-\alpha} \mu_{g}(r) > C_1 M_f(r) \ln^{\alpha} \mu_f(r) \ln^{1/4} (M_f(r) \ln^{\alpha} \mu_f(r)) >$$

$$> C_1 \mu_f(r) \ln^{1/4+\alpha} \mu_f(r).$$

□
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