THE CAUCHY-RIEMANN EQUATIONS FOR A CLASS
OF (0,1)-FORMS IN $l^2$

1. Introduction. The study of local exactness of infinitely differentiable (0,1)-forms was the object of important work, in particular those of L. Lempert. This author gets local exactness in the space $l^1$ and on any space of Banach when the forms are real analytical ([1], [2]).

In Hilbert spaces few results are known, however we must mention an important result due to G. Coeuré: he gives an example of (0,1)-form $\omega$ of class $C^1$ in the unit ball of an infinite dimensional separable Hilbert space such that the equation $\partial f = \omega$ does not admit any local solution around 0, (see Mazet [3]). No other example is known with $\omega$ of the class $C^p(1 < p \leq \infty)$.

In this paper, we study the local exactness of $\partial$ in the Hilbert space $l^2$, for a particular class of (0,1)-forms of the type

$$\omega(z) = \sum_i z_i \omega^i(z)d\tau_i, \quad z = (z_i) \text{ in } l^2$$

under the following assumptions $(H_1)$:

i) Each function $\omega^i$ is indefinitely differentiable on the closed unit ball of $l^2$ denoted $\overline{B}$, and of the form

$$\omega^i(z) = \sum_k \omega^i_k(z^k)$$

where the series (1) is supposed to be absolutely convergent, and where $\mathbb{N} = \bigcup I_k$ is a partition of $\mathbb{N}$, with $z^k$ standing for the projection of $z$ on $\mathbb{C}^{I_k}$, and $\omega^i_k$ being a function of class $C^\infty$ on the closed unit ball of $\mathbb{C}^{I_k}$ provided with the norm of $l^2$.

ii) For all $k$, card $I_k$ noted $|k|$ is finite.

The used method is based on the expansion in Fourier series of the indefinitely differentiable functions $f$ on the closed unit ball of $\mathbb{C}^N$. In ([5], Theorem 2.1) we show that such

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functions admit necessarily a Fourier series expansion of the form
\[ f(z) = \sum_{(\alpha,\beta) \in (\mathbb{N} \times \mathbb{N})^N} z^\alpha \overline{z}^\beta f_{\alpha,\beta}(|z|^2), \]
with \( z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N} \)
and \(|z|^2 = (|z_1|^2, \ldots, |z_N|^2)\). This allows us to study the local exactness of \( \overline{\partial} \) for a restricted class of forms \( \omega \) which respond moreover to the additional assumption \((H_2)\):
\[
\omega^i_k = \sum_{\alpha \in \mathbb{N}^{|i|}} (z^k)^{\alpha} \omega^i_{\alpha,k}(|z|^2), \quad \text{for all } i \text{ and } k.
\]

In [5] the following results was proved.

**Theorem A.** Let \( \omega \) be a closed \((0,1)\)-form of the class \( C^\infty \) on \( \overline{B} \) of the type \( \omega(z) = \sum z_i \omega^i(z) d\overline{z}_i \) and verifying the assumptions \((H_1)\) and \((H_2)\). If there exists a positive integer \( M \) such that the coefficients \( \omega^i_{\alpha,k} \) are null for all \(|\alpha| > M\), all \( k \) and all \( i \) in \( I_k \), then the series \( F_k \) and \( F \) converge and define indefinitely differentiable functions on \( \overline{B} \).

**Theorem B.** Let \( \omega \) be a closed \((0,1)\)-form of class \( C^\infty \) on \( \overline{B} \) according to the type
\[
\omega(z) = \sum z_i \omega^i(z) d\overline{z}_i
\]
and verifying the assumptions \((H_1)\) and \((H_2)\). We assume furthermore that the sequence \(|\{k\}|\) is bounded and that the derivatives \( D^p \omega^i \) are uniformly bounded in \( i \) on the unit ball of \( l^2 \) for \( 0 \leq p \leq 2 \). Then there exists a real number \( r > 0 \) and a function \( F \) of class \( C^\infty \) on the ball with radius \( r \) such that
\[
\overline{\partial} F = \omega \quad \text{and} \quad |F(z)| \leq C \|z\|^2 \sup_{i, 0 \leq p \leq 2} \|D^p \omega^i\|_\infty \text{ for } \|z\| < r
\]
where \( C \) is a constant and \( D \) designates the differentiation operator.

Here we study the local exactness of \( \overline{\partial} \) when the sequence \(|\{k\}|\) is not bounded.

If \( z = (z_i) \) is a finite or infinite sequence of numbers, we denote by \( \#z \) the number of nonzero entries \( z_i \). For every integers \( 0 \leq n \leq N \), let \( \mathbb{N}_n^N \) be the set of all multiindices \( \alpha \in \mathbb{N}^N \) such that \( \#\alpha = n \). In section 3, we establish the following result which generalize Theorem B.

**Theorem 1.** Let \( \omega \) be a closed \((0,1)\)-form of the class \( C^\infty \) on the closed unit ball of \( l^2 \) according to the type \( \omega(z) = \sum z_i \omega^i(z) d\overline{z}_i \) and verifying the assumptions \((H_1)\) and \((H_2)\). Let \( (n_k) \) be a sequence of integers such that \( 1 \leq n_k \leq |k| \) for all \( k \), and \( \liminf_{k \to +\infty} \frac{n_k}{|k|} > 0 \). We assume furthermore that for every \( k \), the coefficients \( \omega^i_{\alpha,k} \) are null if \( \alpha \in \mathbb{N}_n^N \) for all \( n < n_k \), and all \( i \) in \( I_k \), and that the derivatives \( D^p \omega^i \) are uniformly bounded in \( i \) on the unit ball of \( l^2 \) for \( 0 \leq p \leq 2 \). Then there exist a real number \( r > 0 \) and a function \( F \) of the class \( C^\infty \) on the ball with radius \( r \) such that
\[
\overline{\partial} F = \omega \quad \text{and} \quad |F(z)| \leq C \sup_{i, 0 \leq p \leq 2} \|D^p \omega^i\|_\infty \text{ for } \|z\| < r,
\]
where \( C \) is a constant and \( D \) designates the differentiation operator.
2. Preliminaries.

2.1. Notations. In this work our main concern will be the Hilbert space $l^2$, and so, unless indicated otherwise, $||\cdot||$ will denote the $l^2$-norm on $l^2$ or on $\mathbb{C}^N$; if $z = (z_i) \in l^2$ or $\mathbb{C}^N$, $||z|| = \sum |z_i|^2$. $B(r)$ and $B_N(r)$ will denote the ball $||z|| < r$ in $l^2$ and $\mathbb{C}^N$ respectively. When $r = 1$, we simply write $B$ and $B_N$ for $B(1)$ and $B_N(1)$, respectively. We shall make extensive use of multi-indices. A multi-index $\alpha = (\alpha_i)_{i=1}^\infty$ for us is a sequence of integers $\alpha_i \geq 0$ with $\alpha_i = 0$ for $i$ sufficiently large. The length of $\alpha$ is $|\alpha| = \sum_{i=1}^\infty \alpha_i$. We let $\alpha! = \prod_{i=1}^\infty \alpha_i!$, where the usual convention $0! = 1$.

If $z$ and $w$ are in $\mathbb{C}^N$, the following notations will be used in the sequel: $z_i' = (z_1, z_2, \ldots, z_i)$; $z_i'' = (z_i, z_{i+1}, \ldots, z_N)$ ($i = 1, 2, \ldots, N$). When $\alpha$ is a multi-index of $\mathbb{N}^N$, we simply write $z^{\alpha_i'}$ for $(z_i')^{\alpha_i'}$, $|z| = (|z_1|^2, \ldots, |z_N|^2)$, $zw = (z_1w_1, \ldots, z_Nw_N)$. If $x$ is a vector of $\mathbb{R}_+^N$, then $\sqrt{x} = (\sqrt{x_1}, \ldots, \sqrt{x_N})$.

If $f$ is in $C^\infty(B_N)$, in the sense of Frechet, then for each $p \in \mathbb{N}$, we put

$$ ||D^p f|| = \sup_{z \in B_N} ||D^p f(z)||, $$

where $||D^p f(z)||$ denotes the norm of the $p$th differential operator.

2.2. A series in infinitely many variables. If $z = (z_i)_{i=1}^\infty$ is in the unit ball of $l^2$, we put

$$ P_n(z) = \sum_{\alpha \geq n} |\alpha|^{1/2} \alpha^{\alpha/\alpha!} z^\alpha. $$

Lemma 1. Given $1 \leq n \leq N$, and $\epsilon \in ]0, 1/2[$, there is a real number $\rho > 0$ and a constant $C > 0$ such that if $z \in B_N(\rho)$, then

$$ |P_n(z)| \leq C (e^{1/2\epsilon})^n C_N^n. $$

$C$ depends only on $\rho$ and $\epsilon$ but not on $N$.

Proof. Let us consider in $\mathbb{C}$ the entire function $g(z) = \sum_{\alpha \geq 1} \frac{z^\alpha}{\alpha^{1/2}}$. For every $\epsilon > 0$, we have

$$ |g(z)| \leq \sum_{\alpha \geq 1} \frac{|z|^{1/2}}{\epsilon^{\alpha/2}}. $$

Using the Cauchy-Schwarz inequality, we obtain

$$ |g(z)| \leq \left( \sum_{\alpha \geq 1} \frac{|z|^{2\alpha}}{\epsilon^\alpha} \right)^{1/2} \left( \sum_{\alpha \geq 1} \epsilon^\alpha \right)^{1/2}. $$

Let $1 \leq n \leq N$, $q \in \mathbb{N}^*$, and let $z \in \mathbb{C}^N$, we have

$$ \sum_{\alpha \in \mathbb{N}^k_{|\alpha| = q}} \frac{z^\alpha}{\alpha^{1/2}} = \sum_{1 \leq i_1 < \ldots < i_n \leq N} \sum_{\alpha_{i_1} \ldots \alpha_{i_n} > 0} \frac{z_{i_1}^{\alpha_{i_1}}}{\alpha_{i_1}^{1/2}} \ldots \frac{z_{i_n}^{\alpha_{i_n}}}{\alpha_{i_n}^{1/2}}. $$
For any $1 \leq i_1 < \ldots < i_n \leq N$, we observe that the second sum in the right hand of (2) is the homogeneous component of degree $q$ of the product $g(z_{i_1}) \ldots g(z_{i_n})$. It follows, when $z \in B_N(\sqrt{q})$, the majorization
\[
|\sum_{\alpha_{i_1}, \ldots, \alpha_n > 0} \frac{z_{i_1}^{\alpha_{i_1}} \ldots z_{i_n}^{\alpha_n}}{\alpha_{i_1}^{\alpha_{i_1}/2} \ldots \alpha_n^{\alpha_n/2}}| < \exp \left( \frac{q}{2\epsilon} \right) \left( \sqrt{\frac{\epsilon}{1-\epsilon}} \right)^n.
\]

By homothety on the ball of radius $\rho$, we get
\[
|\sum_{\alpha_{i_1}, \ldots, \alpha_n > 0} \frac{z_{i_1}^{\alpha_{i_1}} \ldots z_{i_n}^{\alpha_n}}{\alpha_{i_1}^{\alpha_{i_1}/2} \ldots \alpha_n^{\alpha_n/2}}| < \left( \frac{\rho}{\sqrt{q}} \right)^q \exp \left( \frac{q}{2\epsilon} \right) \left( \sqrt{\frac{\epsilon}{1-\epsilon}} \right)^n
\]
and therefore, if $\rho$ is sufficiently small, we get
\[
|P_n(z)| \leq \sum_{q \geq n} \sum_{\alpha \in \mathbb{N}^N_{|\alpha|=q}} \frac{q^{q/2}z^\alpha}{\alpha^{\alpha/2}} \leq \left( \frac{e^{1/2e} \rho}{1 - \rho e^{1/2e}} \right)^n C_n^N.
\]

\[\square\]

2.3. Fourier series expansion.

**Theorem 2.** If $f$ is in $C^\infty(\overline{B}_N)$, then it admits the Fourier series expansion
\[
f(z) = \sum_{\alpha \in \mathbb{Z}^N} z^\alpha f_\alpha(|z|^2),
\]
where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_N^{\alpha_N}$ with $z_i^{\alpha_i} = (z_i)^{\alpha_i}$ if $\alpha_i < 0$.

The series (3) is normally convergent with its derivatives on $\overline{B}_N$; the coefficients $f_\alpha$ are $C^\infty$ on $\{x \in \mathbb{R}_+^N; x_1 + \ldots + x_N < 1\}$, that satisfy
\[
\forall \alpha \in \mathbb{Z}^N : z^\alpha f_\alpha(|z|^2) = \int_{[0,2\pi]^N} f(z e^{i\theta}) e^{-i(\alpha \cdot \theta)} \frac{d\theta}{(2\pi)^N},
\]
where $(\alpha \cdot \theta) = \sum_{i=1}^N \alpha_i \theta_i$, $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_N})$, and $\frac{d\theta}{(2\pi)^N} = \frac{d\theta_1}{2\pi} \ldots \frac{d\theta_N}{2\pi}$.

For the proof see ([5], Theorem 2.1).

3. Exactness of a class of $(0,1)$-forms. We study the local exactness of $\partial$ in the Hilbert space $l^2$ for a particular class of $(0,1)$-forms of the type
\[
\omega(z) = \sum_i z_i \omega^i(z) d\overline{z}_i, \quad z = (z_i) \text{ in } l^2
\]
under the following assumptions ($H_1$):

i) Each function $\omega^i$ is indefinitely differentiable on the closed unit ball of $l^2$, and it takes the form
\[
\omega^i(z) = \sum_k \omega^i_k(z^k)
\]
(4)
where the series \((4)\) is supposed to be absolutely convergent, and where \(\mathbb{N} = \bigcup I_k\) is a partition of \(\mathbb{N}\), with \(z^k\) standing for the projection of \(z\) on \(\mathbb{C}^l\), and \(\omega^k\) being a function of class \(C^\infty\) on the closed unit ball of \(\mathbb{C}^l\) provided with the norm of \(l^2\).

ii) For all \(k\), \(\text{card } I_k\) noted \(|k|\) is finite.

According to Theorem 2, for all \(i\) and \(k\) the function \(\omega^k_i\) admits a Fourier series expansion in the form \(\omega^k_i(z^k) = \sum_{\alpha \in \mathbb{N}^{|k|}} (z^k)\omega^k_{\alpha,k}(|z^k|^2)\), where the coefficients \(\omega^k_{\alpha,k}\) are functions of class \(C^\infty\), on the closed unit ball of \(\mathbb{C}^l\).

In what follows, we make the assumption \((H_2)\):

\[
\omega^k_i(z^k) = \sum_{\alpha \in \mathbb{N}^{|k|}} (z^k)\omega^k_{\alpha,i}(|z^k|^2) \quad \text{for all } i \text{ and } k \text{ in } \mathbb{N}.
\]

Following ([5], Theorem 3.2), \(\omega\) is \(\partial\)-closed if and only if the form \(\Phi_{\alpha,k} = \sum_{i \in I_k} \omega^k_{\alpha,i} dt_i\) is \(d\)-closed in the closed unit ball of \(\mathbb{R}^{|k|}\) for each \(\alpha\) and \(k\). So we are led to integrate the \((0,1)\)-form \(\hat{\omega} = \sum_k \hat{\omega}_k\) such that

\[
\hat{\omega}_k(z^k) = \sum_{i \in I_k} z_i \left[ \sum_{\alpha \in \mathbb{N}^{|k|}} (z^k)\alpha \frac{\partial \Omega_{\alpha,k}}{\partial t_i}(|z^k|^2) \right] d\zeta_i,
\]

where \(\Omega_{\alpha,k}\) is an anti-derivative of the form \(\Phi_{\alpha,k}\).

Each \(\hat{\omega}_k\) is a \(\partial\)-closed \((0,1)\)-form of class \(C^\infty\) on the closed unit ball of \(\mathbb{C}^l\).

Let \(F_k(z^k) = \sum_{\alpha \in \mathbb{N}^{|k|}} (z^k)\alpha \Omega_{\alpha,k}(|z^k|^2)\). Then \(F = \sum_{k=1}^{\infty} F_k\) is a formal solution of the equation \(\partial F = \hat{\omega}\) and according to ([3], Appendix 3, Lemma 5) the problem is reduced to the existence of a real number \(r \in [0,1]\), and for every \(\alpha\) and \(k\) an anti-derivatives \(\Omega_{\alpha,k}\) such that the series \(F_k\) converge and satisfies an estimate independent of \(k\) on the ball of radius \(r\) of \(\mathbb{C}^l\). We give a positive response for two particular cases.

i) The polynomial case.

**Theorem 3.** Let \(\omega\) be a closed \((0,1)\)-form of class \(C^\infty\) on \(\overline{B}\) of the type \(\omega(z) = \sum z_i \omega^i(z) d\zeta_i\) and verifying the assumptions \((H_1)\) and \((H_2)\). If there exists a positive integer \(M\) such that the coefficients \(\omega^i_{\alpha,k}\) are null for all \(|\alpha| > M\), all \(k\) and all \(i\) in \(I_k\), then the series \(F_k\) and \(F\) converge and define indefinitely differentiable functions on \(\overline{B}\).

For the proof see ([5], Theorem 4.1).

ii) Non-polynomial case. Let \((n_k)\) be a sequence of integers such that \(1 \leq n_k \leq |k|\) for all \(k\), and \(\liminf \frac{n_k}{|k|} > 0\). If we suppose that for every \(k\), the coefficients \(\omega^i_{\alpha,k}\) are null if \(\alpha \in \mathbb{N}^{|k|}\) for all \(n < n_k\), and all \(i\) in \(I_k\), then we shall prove that there exists \(r > 0\) such that for \(k\) sufficiently large the series \(F_k\) and \(F\) converge and define indefinitely differentiable functions on \(B(r)\).

The proof of Theorem 1 is a direct consequence of the forthcoming proposition.

**Proposition 1.** Let \(\omega\) be a closed \((0,1)\)-form of the class \(C^\infty\) on the closed unit ball of \(l^2\) according to the type \(\omega(z) = \sum z_i \omega^i(z) d\zeta_i\) and verifying the assumptions \((H_1)\) and \((H_2)\). Let \((n_k)\) be a sequence of integers such that \(1 \leq n_k \leq |k|\) for all \(k\), and \(\liminf \frac{n_k}{|k|} > 0\). We suppose moreover that for each \(k\), the coefficients \(\omega^i_{\alpha,k}\) are null if \(\alpha \in \mathbb{N}^{|k|}\) for all \(n < n_k\), and
all $i$ in $I_k$, and that the derivatives $D^p\omega^j$ are uniformly bounded in $i$ on the unit ball of $l^2$ for $0 \leq p \leq 2$. Then there exists $r > 0$ and $\lambda > 0$ such that for $k$ sufficiently large and $z \in B(r)$, the series $F_k$ converge and defines a $\overline{\partial}$-antiderivatives of $\omega_k$ of class $C^\infty$ on $\overline{B}_k$ for which

$$|F_k(z^k)| \leq C \left( \|z^k\|^2 + (2r^\lambda |k|) \right) \sup_{0 \leq p \leq 2} \|D^p\omega\|_\infty$$

where $C$ is a constant independent of $k$.

**Proof.** We recall that $\Phi_{a,k}$ designates the closed form in $\mathbb{R}^{[k]}$ defined by $\Phi_{a,k} = \sum_{i \in I_k} \omega_{\alpha,k}^i dt_i$. Its anti-derivative is given by $\Omega_{a,k}(|z|^2) = \int_\gamma \Phi_{a,k}$, and the path $\gamma$ defined below joins the point $|z|^2$ to a fixed point of the closed unit ball of $\mathbb{R}^{[k]}$.

We also recall that we can take the function $F_k$ for a $\overline{\partial}$-antiderivatives of $\omega_k$ conditioned by its series convergence.

Given $0 < r < 1$ and $z \in \overline{B(r)}$, the path $\gamma$ is the union in $\mathbb{R}^{[k]}$ of the adjacent segments $[M^m, M^m+1]$ $(m = 0, \ldots, |k|)$, defined by $M^0 = |z|^2$, and for $m \in \{1, 2, \ldots, |k| + 1\}$,

$$M_i^m = \begin{cases} \frac{|a_i|}{r^2} & \text{if } i < m \\ \left(\frac{|z_i|^2}{r}\right)^2 & \text{if } i \geq m \end{cases}$$

Let

$$F_k^m(z^k) = \sum_{n \geq n_k} \sum_{\alpha \in \mathbb{N}^{[k]}} (z^k)^\alpha \int_{M^m}^{M^{m+1}} \Phi_{\alpha,k}, \quad m = 0, \ldots, |k|.$$ 

Since $F_k = \sum_{m=0}^{|k|} F_k^m$, it will be enough to prove that, for all $m = 0, \ldots, |k|$, the series $F_k^m$ converges and satisfies an estimate independent of $k$.

Let us start with the case $m = 0$.

$$F_k^0(z^k) = \sum_{n \geq n_k} \sum_{\alpha \in \mathbb{N}^{[k]}} \int_1^{1/r^2} (z^k)^\alpha \sum_{i \in I_k} \omega_{\alpha,i}^k (u|z|^2) \cdot |z_i|^2 du$$

$$= \sum_{q \geq n_k} \int_1^{1/r^2} \int_0^{2\pi} \frac{1}{(\sqrt{u})^{q+1}} (\tilde{\omega}(\sqrt{u}, e^{i\theta}, z^k)) e^{-iq\theta} d\theta du.$$

By making two integrations by parts relatively to $\theta$ in each of the above sum, we obtain

$$|F_k^0(z^k)| \leq C \|z^k\|^2 \sup_{p \leq 2} \|D^p\tilde{\omega}\|_\infty,$$

where $C$ is a constant independent of $k$. Now, let us consider the series $F_k^m$ for $m \geq 1$. We can write

$$F_k^m(z^k) = \sum_{\alpha \in \mathbb{N}^{[k]}} \int_0^{2\pi} \int_{M^m}^{M^{m+1}} \frac{|\alpha|}{\alpha^{\frac{1}{2}} (t_m |\alpha|)^{\frac{1}{2}}} \omega^m \left( \sqrt{t_m} e^{i\theta} \right) e^{-i|\alpha|^{\varepsilon} \frac{d\theta}{2\pi}} dt_m.$$ 

An easy computation shows that

$$|z_{m}^{\alpha_m}| \int_{M^m}^{M^{m+1}} \frac{dt_m}{(\sqrt{t_m})^{\alpha_m}} \leq C r^{\alpha_m},$$
where $C$ is a constant.

If we choose $r$ sufficiently small, then an application of Lemma 1 shows that

$$|F^m_k(z^k)| \leq C2^{|k|} \sup_i \|\omega^i\|_\infty r^{nk},$$

for every $z \in \overline{B(r)}$ and $m \geq 1$.

Since $\liminf_{k \to +\infty} \frac{n_k}{|k|} > 0$, there exists a real number $\lambda > 0$ such that $n_k \geq \lambda |k|$ for $k$ sufficiently large, hence we are led to the majorization

$$\sum_{m=1}^{|k|} |F^m_k(z^k)| \leq C(2r^\lambda)^{|k|} \sup_i \|\omega^i\|_\infty$$

(7)

for all $z \in \overline{B(r)}$, where $r$ is chosen sufficiently small, and $C$ is a constant independent of $k$. Now, (6) and (7) implies the required estimate (5).

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\square
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