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ON ASYMPTOTIC BEHAVIOR OF THE $p$TH MEANS OF THE GREEN POTENTIAL FOR $0 < p \leq 1$

1. Introduction and main result. For $n \in \mathbb{N}$, let $\mathbb{C}^n$ denote the $n$-dimensional complex space with the inner product

$$\langle z,w \rangle = \sum_{j=1}^{n} z_j \overline{w}_j, \quad z, w \in \mathbb{C}^n.$$ 

Let $B$ denote the unit ball $\{z \in \mathbb{C}^n: |z| < 1\}$ with the boundary $S = \{z \in \mathbb{C}^n: |z| = 1\}$, where $|z| = \sqrt{\langle z,z \rangle}$.

For $z, w \in B$, define the involutive automorphism $\varphi_w$ of the unit ball $B$ given by

$$\varphi_w(z) = \frac{w - P_w z - (1 - |w|^2)^{1/2} Q_w z}{1 - \langle z, w \rangle},$$

where $P_0 z = 0$, $P_w z = \frac{\langle z, w \rangle}{|w|^2} w$, $w \neq 0$, is the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by $w$ and $Q_w = I - P_w$. We note that ([10, p.11])

$$1 - |\varphi_w(z)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \langle z, w \rangle|^2}. \quad (1)$$

The invariant Laplacian $\tilde{\Delta}$ on $B$ is defined by

$$\tilde{\Delta} f(a) = \Delta(f \circ \varphi_a)(0),$$

where $f \in C^2(B)$, $\Delta = 4 \sum_{i=1}^{n} (\partial^2/\partial z_i \partial \overline{z}_i)$ is the ordinary Laplacian. The operator $\tilde{\Delta}$ is invariant with respect to any holomorphic automorphism of $B$, i.e., $\tilde{\Delta}(f \circ \psi) = (\tilde{\Delta} f) \circ \psi$ for all $\psi \in M$, the group of holomorphic automorphisms of $B$ ([8, Chap.4], [10]).

The Green’s function for the invariant Laplacian is defined by $G(z, w) = g(\varphi_w(z))$, where

$$g(z) = \frac{n+1}{2n} \int_{|z|^2}^{1} (1 - t^2)^{n-1} t^{-2n+1} dt \quad (10, \text{Chap.6.2}).$$

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If $\mu$ is a nonnegative Borel measure on $B$, the function $G_\mu$ defined by

$$G_\mu(z) = \int_B G(z, w) d\mu(w)$$

is called the (invariant) Green potential of $\mu$, provided $G_\mu \not\equiv +\infty$. It is known that ([10, Chap. 6.4]) the condition $G_\mu \not\equiv +\infty$ is equivalent to

$$\int_B (1 - |w|^2)^n d\mu(w) < \infty. \quad (2)$$

The Green potential is closely connected to the notion of an $M$-subharmonic function ([10, Chap. 3]). A function $u$ on $B$ is called $M$-harmonic if $u \in C^2(B)$ and $\tilde{\Delta} u = 0$. A function $u$ on $B$ is called $M$-subharmonic if it is upper semicontinuos and $\tilde{\Delta} u \geq 0$ in the sense of distributions. In particular, $-G_\mu$ is $M$-subharmonic. Note that in the case $n = 1$ the classes of $M$-subharmonic functions and subharmonic functions coincide.

Let $u$ be a measurable function locally integrable on $B$. For $0 < p < \infty$ we define

$$m_p(r, u) = \left( \int_S |u(r, \xi)|^p d\sigma(\xi) \right)^{1/p},$$

where $d\sigma$ is the Lebesgue measure on $S$ normalized so that $\sigma(S) = 1$.

The following Riesz Decomposition Theorem holds.

**Theorem A** ([11]). Suppose that $u$ is $M$-subharmonic in $B$ and

$$\sup_{1/2 \leq r < 1} m_1(r, u) < \infty.$$

Let $\mu$ be the Riesz measure of $u$ in $B$ with $d\mu(z) = \tilde{\Delta} u(z)(1 - |z|^2)^{-n-1} dV(z)$, where $V$ is the Lebesgue measure on $B$. Then there exists a signed Borel measure $\nu$ on $S$ such that for all $z \in B$

$$u(z) = P[\nu](z) - G_\mu(z) \quad (3)$$

where

$$P[\nu](z) = \int_S \frac{(1 - |z|^2)^n}{|1 - \langle z, \zeta \rangle|^{2n}} d\nu(\zeta)$$

is the Poisson-Stieltjes integral.

Growth of the integral $P[\nu](z)$ in the uniform metric is described in terms of smoothness properties of the measure $\nu$ in [1] for $n = 1$, and in [4] for arbitrary $n \in \mathbb{N}$. Growth of $m_p(r, P[\nu])$ for $n = 1$ and $p \geq 1$ is described in [15].

In the case $n > 1$, sharp estimates of the growth rate of $m_p(r, G_\mu)$ for the whole class of Borel measures satisfying (2) are proved by M. Stoll in [9]. The case $n = 1$ is studied much more deeper, see e.g. [12, 13, 14]).

**Theorem B** ([9]). Let $G_\mu$ be the Green potential on $B$.

1. If $1 \leq p < \frac{2n-1}{2(n-1)}$, then

$$\lim_{r \to 1-} (1 - r^2)^{(n-1)/p} m_p(r, G_\mu) = 0. \quad (4)$$

2. If $n \geq 2$ and $\frac{2n-1}{2(n-1)} \leq p < \frac{2n-1}{2n-3}$, then

$$\liminf_{r \to 1-} (1 - r^2)^{(n-1)/p} m_p(r, G_\mu) = 0. \quad (5)$$
Theorem B gives the maximal growth rate of the $p$th mean of the Green potentials, but does not take into account particular properties of a measure $\mu$. It appears that smoothness properties of the so called complete measure (in the sense of Grishin [7, 2, 3]) or the related measure (see [6]) of a subharmonic function allow us to describe its growth. Here we just note that in the case when $n = 1$ and $u = -G_\mu$, the complete measure $\lambda = \lambda_u$ of $u$ is the weighted Riesz measure $d\lambda(z) = (1 - |z|) d\mu(z)$.

Define for $a, b \in \bar{B}$ the nonisotropic metric on $S$ by $d(a, b) = |1 - \langle a, b \rangle|^{1/2}$ ([8, Chap.5.1]).

For $\xi \in S$ and $\delta > 0$ we set $C(\xi, \delta) = \{z \in B: d(z, \xi) < \delta\}$, $D(\xi, \delta) = \{z \in B: d(z, \xi) < \delta\}$, $d\lambda(z) = (1 - |z|)^n d\mu(z)$.

The growth of $m_p(r, G_\mu)$ in terms of properties of the measure $\mu$ are described in [5] for $n > 1$. One dimensional analogue has been established earlier in [3] for all $p > 1$.

**Theorem C ([5])**. Let $n \in \mathbb{N}$, $1 < p < \frac{2n-1}{2(n-1)}$, $0 \leq \gamma < 2n$, and let $\mu$ be a Borel measure satisfying (2). Then

$$m_p(r, G_\mu) = O \left( (1 - r)^{\gamma - n} \right), \ r \uparrow 1$$

holds if and only if

$$\left( \int_S \lambda^p (C(\xi, \delta)) d\sigma(\xi) \right)^{\frac{1}{p}} = O \left( \delta^{\gamma} \right), \ 0 < \delta < 1.$$

In this paper we would like to consider the case $0 < p \leq 1$. For this interval one can obtain an analogue of necessity part of Theorem C.

**Theorem 1**. Let $n > 1$, $0 < p \leq 1$, $0 \leq \gamma < 2n$, and let $\mu$ be a Borel measure satisfying (2) and

$$m_p(r, G_\mu) = O \left( (1 - r)^{\gamma - n} \right), \ r \uparrow 1$$

hold. Then

$$\left( \int_S \lambda^p (C(\xi, \delta)) d\sigma(\xi) \right)^{\frac{1}{p}} = O \left( \delta^{\gamma} \right), \ 0 < \delta < 1.$$

**Proof.** The proof repeats that of necessity in Theorem C.

The following theorem is the main result of the paper.

**Theorem 2**. Let $n > 1$, $0 < p \leq 1$, $0 \leq \gamma < 2n$, and let $\mu$ be a Borel measure satisfying (2) and

$$\int_S \lambda (C(\xi, \delta)) d\sigma(\xi) = O \left( \delta^{\gamma} \right), \ 0 < \delta < 1,$$

hold. Then

$$m_p(r, G_\mu) = O \left( (1 - r)^{\gamma - n} \right), \ r \uparrow 1.$$

**Remark 1.** An example in Section 4 shows that estimate (11) is sharp for all $p \in (0, 1]$. As a corollary we obtain a criterion of the growth of $m_p(r, G_\mu)$ in terms of properties of the measure $\mu$ in the case $p = 1$. 


Corollary 1. Let $n > 1$, $0 \leq \gamma < 2n$, and let $\mu$ be a Borel measure satisfying (2). Then

$$\int_S \lambda(C(\xi, \delta)) \, d\sigma(\xi) = O(\delta^n), \ 0 < \delta < 1,$$

holds if and only if

$$m_1(r, G_\mu) = O((1 - r)^{\gamma - n}), \ r \uparrow 1.$$

Remark 2. Due to Proposition 1.10 ([5]) we always have

$$\int_S \lambda(C(\xi, \delta)) \, d\sigma(\xi) = o(\delta^n), \ \delta \downarrow 0.$$ 

This agrees with the relation $m_1(r, G_\mu) = o(1)$, $r \uparrow 1$ as it was shown by Ulrich ([11], see also [10]).

2. Some properties of the Green’s function. The following lemma gives some basic properties of $g$ which will be needed later.

Lemma A ([10]). Let $0 < \delta < \frac{1}{2}$ be fixed. Then $g$ satisfies the following two inequalities:

$$g(z) \geq \frac{n + 1}{4n^2}(1 - |z|^2)^n, \ z \in B,$$

$$g(z) \leq c(\delta)(1 - |z|^2)^n, \ z \in B, |z| \geq \delta,$$

where $c(\delta)$ is a positive constant. Furthermore, if $n > 1$ then

$$g(z) \asymp |z|^{-2n+2}, \ |z| \leq \delta.$$

We need an estimate of $p$-means of the Green’s function for $0 < p \leq 1$. Analogues estimates for $p > 1$ are established by Stoll ([9, Lemma 5]). His proof does not work for $p \leq 1$, though we use some ideas and notation from [9].

For fixed $\delta, 0 < \delta < 1/2$, denote $B^*(z, \delta) = \{w \in B: |\varphi_w(z)| < \delta\}$ and for $0 < r < 1$ denote

$$E(r) = \bigcup_{t \in S} B^*(rt, \delta).$$

Lemma 1. Let $0 < p \leq 1$, $n \in \mathbb{N}$. Then there exists $r_0 \in (0, 1)$ such that for all $r \in (r_0, 1)$ and $w \in E(r)$

$$m_p(G(\cdot, w), r) \asymp (1 - r^2)^{n/p}, \ \text{if} \ p \in (0, 1] \setminus \left\{\frac{1}{2(n - 1)}\right\},$$

$$m_p(G(\cdot, w), r) = O\left((1 - r^2)^{n/p}\left(\ln \frac{1}{1 - r}\right)^{1/p}\right), \ \text{if} \ p = \frac{1}{2(n - 1)}, \ n > 1.$$

Proof. Let $w \in E(r), |w| = \rho$. Since $\sigma$ is invariant under the group of unitary transformations of $\mathbb{C}^n$,

$$\int_S g(\varphi_w(rt))^pd\sigma(t) = \int_S g(\varphi_{we}(rt))^pd\sigma(t) = \int_S g(\varphi_{re}(\rho t))^pd\sigma(t),$$

where $e = (1, 0, \ldots, 0) \in \mathbb{C}^n$.

For $0 < r, \rho < 1$, and fixed $\delta \in (0, \frac{1}{2}]$, let $N^p = \{t \in S: \rho t \in B^*(re, \delta)\}$. 


For $t \in S \setminus N^p_r$, we have ([9, p. 491])

$$\int_S g(\varphi_{re}(\rho t))p\sigma(t) \leq c(1 - \rho^2)^p(1 - r^2)^{-n(p-1)} \leq c(1 - r^2)^n. \quad (16)$$

Also, for $c > 0$, let $Q^c_r = \{ se^{i\theta} : 0 < 1 - s < c(1 - r^2), |\theta| < c(1 - r^2) \}$ and $Q^c_r = \{ t = (t_1, \ldots, t_n) \in S : t_1 \in Q^c_r \}$.

By the definition of $N^p_r$, one has $|\varphi_{re}(\rho t)| < \delta$ for $t \in N^p_r$. Hence by (15) and (1)

$$g(\varphi_{re}(\rho t)) \approx |\varphi_{re}(\rho t)|^{-2(n-1)} = c_1 \frac{|1 - r\rho t_1|^{2(n-1)}}{(1 - r^2)(1 - \rho^2))^{n-1}}, \quad (17)$$

where $c_1 = c_1(n)$.

It is known that ([9, Lemma 3]) there exist $c_2 = c_2(\delta)$ and $r(\delta)$ such that $N^p_r \subset Q^c_r$ for all $\rho$ with $\rho e \in B^*(re, \delta)$, and all $r > r(\delta)$. Moreover, one can choose $r_0 \in (0, 1)$ such that the inclusion holds for all $r \in (r_0, 1)$ and $0 < \delta \leq \frac{1}{2}$.

By (1), $\rho t \in B^*(re, \delta)$ if and only if $(1 - r^2)(1 - \rho^2) > (1 - \delta^2)(1 - r\rho t_1)^2$, i.e.

$$|1 - r\rho t_1|^2 \leq \frac{1}{1 - \delta^2}(1 - r^2)(1 - \rho^2) \leq \frac{4}{3}(1 - r^2)(1 - \rho^2).$$

Since $t \in N^p_r$, we can apply the previous inequality to deduce

$$\int_{N^p_r} g(\varphi_{re}(\rho t))p\sigma(t) \leq c_3(1 - r^2)^p(1 - \rho^2)^p \times$$

$$\times \int_{Q^c_r} (|1 - r\rho t_1|^2 - (1 - r^2)(1 - \rho^2))^{-p(n-1)} d\sigma(t) =: c_3(1 - r^2)^p(1 - \rho^2)^p I_r. \quad (18)$$

Since ([9, p. 488])

$$|1 - r\rho e^{i\theta}|^2 - (1 - r^2)(1 - \rho^2) = (\rho - r)^2 + 2r(1 - s) - r^2\rho^2(1 - s^2) + 4r\rho s\sin^2\frac{\theta}{2} \geq$$

$$\geq (r - \rho)^2 + (1 - s)(1 - r) + \frac{\theta^2}{\pi^2}, \quad \min\{\rho r, s\} \geq \frac{1}{2}, \quad (19)$$

by formula 1.4.5(2) in [8],

$$I_r = c_4(n) \int_{Q^c_r} (1 - s^2)^{-n-2}(|1 - r\rho e^{i\theta}|^2 - (1 - r^2)(1 - \rho^2))^{-p(n-1)} ds d\theta \leq$$

$$\leq c_5 \int_{1-c_2(1-r^2)} \left[ c_2(1-r^2) \int_0^{(1-s)(1-r)} (1 - s)^{n-2} \left[ (r - \rho)^2 + (1 - s)(1 - r) + \frac{\theta^2}{\pi^2} \right]^{-p(n-1)} d\theta \right] ds.$$

So

$$I_r \leq c_5 \int_{1-c_2(1-r^2)} (1 - s)^{n-2} \left[ \pi \sqrt{(1-s)(1-r)} \int_0^{\frac{(1-s)(1-r)}{\pi^2}} \left( (1 - s)(1 - r) + \frac{\theta^2}{\pi^2} \right)^{-p(n-1)} d\theta +$$
Therefore from the latter inequalities, (16) and (18) we get

\[ p \leq c_5 \int_{1-c_2(1-r^2)}^{1} (1-s)^{n-2} \left[ \int_{0}^{\pi \sqrt{(1-s)(1-r)}} \left( (1-s)(1-r) \right)^{-p(n-1)} d\theta \right] ds + \left[ \int_{\pi \sqrt{(1-s)(1-r)}}^{c(1-r^2)} \left( \frac{\theta}{\pi} \right)^{-2p(n-1)} d\theta \right] ds. \]

Direct calculation shows that for \( 0 \leq 1-s \leq c_2(1-r^2) \)

\[ \left| \int_{\pi \sqrt{(1-s)(1-r)}}^{c(1-r^2)} \theta^{-2p(n-1)} d\theta \right| \leq \begin{cases} c_6 (1-r)^{1-2p(n-1)}, & p \in (0, 1) \setminus \left\{ \frac{1}{2(n-1)} \right\}; \\ c_6 \ln \frac{1}{1-r}, & p = \frac{1}{2(n-1)}. \end{cases} \]

Let us consider three cases. Firstly, let \( 0 < p < \frac{1}{2(n-1)} \). Since \( 0 < 1-s < 2c_2(1-r) \), we get

\[ I_r \leq c_7 \int_{1-c_2(1-r^2)}^{1} (1-s)^{n-2} (1-r)^{1-2p(n-1)} ds \leq c_8 (1-r^2)^{n-2p(n-1)}. \]

Now let \( 1 \geq p > \frac{1}{2(n-1)} \). Then

\[ I_r \leq c_9 \int_{1-c_2(1-r^2)}^{1} \left( (1-s)^{n-2} (1-r)^{1-2p(n-1)} + (1-s)^{n-2} (1-r)^{1-2p(n-1)} \right) ds \leq c_{10} (1-r^2)^{n-2p(n-1)}. \]

Finally, if \( p = \frac{1}{2(n-1)} \), \( n > 1 \), then

\[ I_r \leq c_9 \int_{1-c_2(1-r^2)}^{1} (1-s)^{n-2} \left( 1 + \ln \frac{1}{1-r} \right) ds \leq c_{11} (1-r^2)^{n-1} \ln \frac{1}{1-r}. \]

Therefore from the latter inequalities, (16) and (18) we get

\[ m_p(G(\cdot, w), r) \leq c_{11} (1-r^2)^{n-1} \left( 1 - \rho^2 \right)^{p(n-1)} (1-r^2)^{n-2p(n-1)} \left( 1 - \rho^2 \right)^{1/p} = \]

\[ = c_{11} \frac{(1-\rho^2)^{n-1}}{(1-r^2)^{n-1-n/p}} \leq c(n, p) (1-r^2)^{n/p}, \quad p \neq \frac{1}{2(n-1)}, \]

\[ m_p(G(\cdot, w), r) \leq c_{12} \left( (1-r^2)(1-\rho^2) \right)^{\frac{1}{2}} (1-r^2)^{n-1} \ln \frac{1}{1-r} \right)^{1/p} \]

\[ \leq c(n)(1-r^2)^{n/p} \ln^{1/p} \frac{1}{1-r}, \quad p = \frac{1}{2(n-1)}. \]
The upper estimates are proved. Let us prove the lower estimate. By (17) we have

\[
\int_S g(\varphi_{re}(\rho t))^p \, d\sigma(t) \geq \tilde{c}_1 \int_{Q^c_r} \left| \varphi_{re}(\rho t) \right|^{-2p(n-1)} \, d\sigma(t) = \\
= \tilde{c}_1 \int_{Q^c_r} \frac{|1 - r\rho t_1|^{2p(n-1)}}{(1 - r^2)(1 - \rho^2)^{p(n-1)}} \, d\sigma(t) \geq \\
\geq \tilde{c}_1 \int_{Q^c_r} \frac{(1 - r\rho)^{2p(n-1)}}{(1 - r^2)(1 - \rho^2)^{p(n-1)}} \, d\sigma(t).
\]

Equality (19) implies

\[
|1 - r\rho se^{i\theta}|^2 - (1 - r^2)(1 - \rho^2) \leq (r - \rho)^2 + 2(1 - s)(1 - r\rho) + \theta^2 \leq \tilde{c}_2(1 - r)^2, \quad se^{i\theta} \in Q^c_r.
\]

Then

\[
\int_S g(\varphi_{re}(\rho t))^p \, d\sigma(t) \geq \tilde{c}_3 |1 - r\rho|^{2p(n-1)} \times \\
\times \int_{1-c(1-r^2)}^1 \left[ \int_0^1 (1 - s^2)^n \cdot (1 - r\rho se^{i\theta})^2 \, ds \right] d\theta \geq \\
\geq \tilde{c}_4 (1 - r)^{2p(n-1)} \int_{1-c(1-r^2)}^1 \left[ \int_0^1 (1 - s^2)^n (1 - r - 2p(n-1)) \, ds \right] d\theta = \tilde{c}_5 (1 - r^2)^n.
\]

So, \( m_p(G(\cdot, w), r) \geq \tilde{c}_6 (1 - r^2)^{n/p}, \ p \in (0; 1] \setminus \left\{ \frac{1}{2(n-1)} \right\}. \)

\[\square\]

3. Proof of Theorem 2. Since, by the convexity, \( m_p(r, G_\mu) \leq m_1(r, G_\mu), \ 0 < p \leq 1, \) it is enough to prove (11) for \( p = 1. \) We follow the scheme from [5].

Let us estimate the absolute values of

\[
u_1(z) := \int_{B^*(z, \frac{1}{4})} G(z, w) \, d\mu(w) \text{ and } \nu_2(z) := \int_{B \setminus B^*(z, \frac{1}{4})} G(z, w) \, d\mu(w).
\]

We start with \( u_1. \) By definition

\[
0 \leq u_1(z) = \int_{B^*(z, \frac{1}{4})} G(z, w) \, d\mu(w) = \int_{B^*(z, \frac{1}{4})} g(\varphi_w(z)) \, d\mu(w).
\]

By (15) we have \( g(z) \leq c |z|^{-2n+2} \) for \( |z| \leq \frac{1}{4} \) and some positive constant \( c. \) Thus,

\[
|u_1(z)| \leq c \int_{B^*(z, \frac{1}{4})} |\varphi_w(z)|^{-2n+2} \, d\mu(w).
\]

Denote \( z = r\xi, \) where \( r = |z|, \ \frac{1}{2} < r < 1 \) and \( w = |w|\eta, \ \xi, \eta \in S. \) Let

\[
K(z, \sigma_1, \sigma_2) = \{ w \in B : |r - |w|| \leq \sigma_1, d(\xi, \eta) \leq \sigma_2 \}.
\]
In [5] it is proved that
\[ B^* \left( z, \frac{1}{4} \right) \subset K(z, c_{13}(1 - r), c_{14}(1 - r)^{\frac{3}{2}}) \] (20)

where \( c_{13} = \frac{2}{3} \) and \( c_{14} = 4\sqrt{2} \). We denote

\[ K(z) := K \left( z, \frac{2}{3}(1 - r), 4\sqrt{2}(1 - r)^{\frac{3}{2}} \right), \quad \tilde{K}(z) := K \left( z, \frac{2}{3}(1 - r), 8\sqrt{2}(1 - r)^{\frac{3}{2}} \right). \]

The inclusion (20) implies

\[ I_1 := \int_{\nu} |u_1(r\xi)| d\sigma(\xi) \leq c_{15} \int_{\nu} \int_{B^*(r\xi, \frac{1}{4})} |\varphi_w(r\xi)|^{-(2n-2)} d\mu(w) d\sigma(\xi) \leq c_{15} \int_{\nu} \int_{K(r\xi)} \frac{d\mu(w)}{|\varphi_w(r\xi)|^{2n}} d\sigma(\xi) \]

where \( c_{15} = c_{15}(p) \). Then, by Fubini’s theorem we deduce \((z = r\xi, \ w = |w|\eta)\)

\[ I_1 \leq c_{16}(n, p) \int_{|w| - r < \frac{2}{3}(1 - r)} \int_{\eta \in S, \ |\eta - r| < 4\sqrt{2}(1 - r)^{1/2}} \frac{d\sigma(\xi)}{|\varphi_w(r\xi)|^{2n}} d\mu(|w|\eta) \leq c_{16}(p, n) \int_{|w| - r < \frac{2}{3}(1 - r)} \int_{S} \frac{d\sigma(\xi)}{|\varphi_w(r\xi)|^{2n}} d\mu(w). \] (21)

Applying to (21) subsequently (1), (14) and Lemma 1, we obtain that for \( 0 < p \leq 1 \)

\[ \int_{S} \frac{d\sigma(\xi)}{|\varphi_w(r\xi)|^{2n}} = \int_{S} \frac{d\sigma(\xi)}{|\varphi_r(\xi)|^{2n}} \leq \int_{S} g(\varphi_r(\xi)) d\sigma(\xi) \leq c_{17}(1 - r^2)^n, \quad \frac{1}{2} < r < 1. \]

Substituting the estimate of the inner integral into (21) we get

\[ I_1 \leq c_{18}(1 - r)^n \int_{|w| - r < \frac{2}{3}(1 - r)} d\mu(|w|\eta). \] (22)

We need the following lemma that plays a key role in the proof of Theorem C.

**Lemma B** ([5]). Let \( \nu \) be a finite positive Borel measure on \( S \), \( 0 < \delta < \frac{1}{2} \), and \( p \geq 1 \). Then

\[ \int_{S} \nu^{p-1}(D(\xi, \delta)) d\nu(\xi) \leq \frac{N_p}{\delta^{2n}} \int_{S} \nu^p(D(\xi, \delta)) d\sigma(\xi), \]

where \( N \) is a positive constant independent of \( p \) and \( \delta \).

To obtain the final estimate of \( I_1 \), for a fixed \( r \in (\frac{1}{2}, 1) \), we define the measure \( \nu_1 \) on the balls \( \{D(\eta, t): \eta \in S, t > 0\} \) by

\[ \nu_1(D(\eta, t)) = \lambda \left( \left\{ \rho\xi \in B: |\rho - r| < \frac{2}{3}(1 - r), d(\xi, \eta) < t \right\} \right). \]
It can be expanded to the family of all Borel sets on \( B \) in the standard way. It is clear that

\[
\nu_1(D(\eta, t)) \asymp (1 - r)^m \mu \left( \left\{ \rho \zeta \in B : |\rho - r| < \frac{2}{3} (1 - r), d(\zeta, \eta) < t \right\} \right).
\]

By using of (22) and Lemma B we get

\[
I_1 \leq c_{19} \int_{||w|-r|<\frac{2}{3}(1-r)} d\lambda(|w|) = c_{19} \int_S d\nu_1(\eta) \leq \frac{c_{19}N}{(1-r)^n} \int_S \nu_1(D(\eta, 8\sqrt{2}(1-r)^{\frac{1}{4}})) \, d\sigma(\eta) = \frac{c_{20}(n, p)}{(1-r)^n} \int_S \lambda(\tilde{K}(r\eta)) \, d\sigma(\eta).
\]

Note that if \( \rho \zeta \in \tilde{K}(r\eta) \) then

\[
|1 - \langle \rho \zeta, \eta \rangle| \leq |1 - \langle \zeta, \eta \rangle| + (1 - \rho) |\langle \zeta, \eta \rangle| \leq (4c_{14}^2 + c_{13} + 1)(1 - r) = c_{21}(1 - r).
\]

Hence,

\[
I_1 \leq \frac{c_{20}}{(1-r)^n} \int_S \lambda(C(\eta, c_{21}(1-r))) \, d\sigma(\eta).
\]

By the assumption of the theorem we deduce

\[
I_1 = O((1 - r)^{\gamma - n}), \ r \uparrow 1.
\]

Let us estimate

\[
u_2(z) = \int_B G(z, w)(1 - |w|)^{-n} \, d\tilde{\lambda}(w)
\]

where \( d\tilde{\lambda}(w) = (1 - |w|)^n \chi_{B(1, 2)}(w) \, d\mu(w) \), \( \chi_E \) is the characteristic function of a set \( E \).

We may assume that \( |z| \geq \frac{1}{2} \).

We denote

\[
E_k = E_k(z) = \left\{ w \in B : \left| 1 - \left\langle \frac{z}{|z|}, w \right\rangle \right| < 2^{k+1}(1 - |z|) \right\}, \ k \in \mathbb{Z}_+.
\]

Since \( |1 - \langle z, w \rangle| \geq \frac{1}{2} |1 - \langle \frac{z}{|z|}, w \rangle| \), one has that for \( w \in E_{k+1} \setminus E_k, |1 - \langle z, w \rangle| \geq 2^{k-1}(1 - |z|) \).

Combining Lemma A with the equality in (1) for \( z \in B \) such that \( |z| \geq \frac{1}{2} \) we get that

\[
0 \leq G(z, w) \leq c_{22} \left( \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \langle z, w \rangle|^2} \right)^n
\]

holds. So

\[
|\nu_2(z)| \leq c_{22} \int_B \left( \frac{(1 + |w|)(1 - |z|^2)}{|1 - \langle z, w \rangle|^2} \right)^n \, d\tilde{\lambda}(w) \leq \sum_{k=1}^{\frac{1}{|\log_2 2^{1/\gamma}|}} c_{22} \int_E (2^{2(k-1))(1 - |z|^2)} \, d\tilde{\lambda}(w) + c_{22} \int_{E_1} \frac{(1 + |w|)(1 - |z|^2)}{(1 - |z|^2)} \, d\tilde{\lambda}(w) \leq \sum_{k=1}^{\infty} \int_{E_{k+1} \setminus E_k} \frac{4^n c_{22}}{(2^{2(k-1))(1 - |z|^2)} \, d\tilde{\lambda}(w) + \int_{E_1} \frac{4^n c_{22}}{(1 - |z|^2)} \, d\tilde{\lambda}(w) \leq
\]

\[
\frac{4^n c_{22}}{(1 - |z|^2)} \, d\tilde{\lambda}(w) \leq
\]

\[
\frac{4^n c_{22}}{(1 - |z|^2)} \]
Using Lemma 1 for $p$ and for some $C > 0$ Proposition 1.

4. An example.

Proposition 1. For $n > 1$, $0 < p \leq 1$, $n < \gamma < 2n$, there exists a Borel measure $\mu$ on $B$ satisfying (2) and such that

$$G_{\mu}(z) = O \left( (1 - |z|)^{\gamma-n} \right), \quad |z| \uparrow 1$$

and for some $C > 0$

$$\lambda(C(\xi, \delta)) \geq C\delta^{\gamma}, \quad 0 < \delta < 1.$$  (27)

Proof. We define $d\mu(z) = \frac{dV(z)}{(1-|z|)^{2n+1-\gamma}}$, where $V$ is the Lebesgue measure on $B$.

We write

$$G_{\nu}(z) = \int_B G(z, w) d\mu(w) = \int_{B^*(z, \frac{1}{4})} G(z, w) d\mu(w) + \int_{B^c(z, \frac{1}{4})} G(z, w) d\mu(w) =: J_1 + J_2.$$  

Since, by (20) $1 - |w| \asymp 1 - |z|$ holds for $w \in B^*(z, \frac{1}{4})$, we get

$$J_1 \leq c_{27} \int_{B^*(z, \frac{1}{4})} \frac{G(z, w) dV(w)}{(1-|z|)^{2n+1-\gamma}} \leq \frac{c_{27}}{(1-|z|)^{2n+1-\gamma}} \int_{r-1}^{r+c_1(1-r)} \int_S G(z, \rho \eta) d\sigma(\eta) \rho^{2n-1} \rho.$$  

Using Lemma 1 for $p = 1$, we obtain

$$J_1 \leq \frac{c_{28}}{(1-|z|)^{2n+1-\gamma}} \int_{r-1}^{r+c_1(1-r)} (1-\rho)^n \rho^{2n-1} d\rho \leq \frac{c_{29}}{(1-\rho)^{n-\gamma}}.$$  

For $w \in B \setminus B^*(z, \frac{1}{4})$ we have (see (1))

$$0 \leq G(z, w) \leq c \left( \frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z, w \rangle|^2} \right)^n.$$  

Therefore

$$\int_S |u_2(r\xi)| d\sigma(\xi) \leq \frac{c_{23}}{(1-r)^n} \sum_{k=1}^{\infty} \frac{\tilde{\lambda}(E_k(r\xi))}{2^{2n(k-2)}} = \frac{c_{24}}{(1-r)^n} \sum_{k=1}^{\infty} 2^{\gamma(k+1)(1-r)^{\gamma}}$$

$$\leq \frac{c_{25}}{(1-r)^n} \sum_{k=1}^{\infty} 2^{\gamma(k+1)(1-r)^{\gamma}} = \frac{c_{26}(n, \gamma)}{(1-r)^{n-\gamma}}.$$  

Hence

$$m_p(r, G_{\mu}) \leq m_1(r, G_{\mu}) \leq \int_S |u_1(r\xi)| d\sigma(\xi) + \int_S |u_2(r\xi)| d\sigma(\xi) \leq \frac{c(n, \gamma)}{(1-r)^{n-\gamma}}.$$  

4. An example.
Then by the above inequality and [8, Chap.1.4.10] it follows that
\[ J_2 \leq c(1 - |z|)^n \int_B \frac{(1 - |w|^2)^{-n-1+\gamma}}{|1 - \langle z, w \rangle|^{2n}} dV(w) \leq c_30(1 - |z|)^n(1 - |z|)^{-2n+\gamma} = c_30(1 - |z|)^{\gamma - n} . \]

Thus \( m_1(r, G_\mu) = O((1 - r)^{\gamma - n}) \), \( r \uparrow 1 \).

Let us prove (27). We have \( d\lambda(w) = \frac{dV(w)}{(1 - |w|)^{n+1-\gamma}} \). Then
\[
\lambda(C(\xi, \delta)) \geq \int_{C(\xi, \delta) \cap \{ 1 - \frac{\delta}{2} \leq |w| \leq 1 - \frac{\delta}{4} \}} dV(w) \geq c \delta^{\gamma - n - 1} \delta^{n+1} = \delta^{\gamma} .
\]

The latter estimates follow from the inclusion
\[
C(\xi, \delta) \cap \left\{ 1 - \frac{\delta}{2} \leq |w| \leq 1 - \frac{\delta}{4} \right\} \supset \left\{ |w| \eta: \frac{\delta}{4} \leq 1 - |w| \leq \frac{\delta}{2}, d(\xi, \eta) \leq \sqrt{\frac{\delta}{2}} \right\} .
\]

Let us prove this. We denote \( v = (1 - \frac{\delta}{4}) \zeta \in \partial C(\xi, \delta), \zeta \in S \). Since \( \min \{ \delta(\xi, \eta): |w| \eta \in C(\xi, \delta) \cap \{ 1 - \frac{\delta}{2} \leq |w| \leq 1 - \frac{\delta}{4} \} \} \) is attained at \( v \), it is enough to estimate \( d(\xi, \zeta) \) from below.
\[
d(\xi, \zeta) = \sqrt{|1 - \langle \xi, \zeta \rangle|} = \sqrt{|1 - \langle \xi, \zeta \rangle - \langle \xi, |v| \zeta \rangle + \langle \xi, |v| \zeta \rangle|} \geq \\
\geq \sqrt{|1 - \langle \xi, |v| \zeta \rangle - |1 - \langle \xi, v \zeta \rangle - \langle \xi, |v| \zeta \rangle|} = \sqrt{\delta - \frac{\delta}{2} |(\xi, \zeta)|} \geq \sqrt{\delta} .
\]

The estimate (27) is proved.

\[ \square \]

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