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LEVY'S PHENOMENON FOR ANALYTIC FUNCTIONS IN $\mathbb{D}\times\mathbb{C}$

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In this paper we prove some analogue of Wiman's type inequality for random analytic functions in the domain $\mathbb{T}=\mathbb{D}\times\mathbb{C}=\{z\in\mathbb{C}^2\colon |z_1|<1, |z_2|<+\infty\}$. The obtained inequality is sharp.

1. Introduction. Let f be an analytic function in the disc $\{z\colon |z|<R\}$, $0<R\le+\infty$, represented by the power series

$$f(z)=\sum_{n=0}^{+\infty}a_nz^n. \tag{1}$$

For $r\in(0,R)$ we denote $M_f(r)=\max\{|f(z)|\colon |z|=r\}$, $\mu_f(r)=\max\{|a_n|r^n\colon n\ge0\}$.

It is well known ([1, p. 9], [2, p. 28]) that for each nonconstant entire function $f(z)$ and every $\varepsilon>0$ there exists a set $E(\varepsilon,f)\subset[1,+\infty)$ such that Wiman's inequality

$$M_f(r)\le\mu_f(r)(\ln\mu_f(r))^{1/2+\varepsilon}$$

holds for all $r\in[1,+\infty)\backslash E(\varepsilon,f)$ and this set $E(\varepsilon,f)$ has finite logarithmic measure, i.e. $\int_{E(\varepsilon,f)}\frac{dr}{r}<+\infty$.

Let $f(z)$ be an analytic function in the unit disc $\mathbb{D}=\{z\colon |z|<1\}$. For such a function $f(z)$ and every $\delta>0$ there exists a set $E_f(\delta)\subset(0,1)$ of finite logarithmic measure on $(0,1)$, i.e.

$$\int_{E_f(\delta)}\frac{dr}{1-r}<+\infty,$$

such that for all $r\in(0,1)\backslash E_f(\delta)$ the inequality

$$M_f(r)\le\frac{\mu_f(r)}{(1-r)^{1+\delta}}\ln^{1/2+\delta}\frac{\mu_f(r)}{1-r}$$

holds ([3]). Similar inequality for analytic function in the unit disc one can find in [4].

Also in [3] it is noted that for the function $g(z)=\sum_{n=1}^{+\infty}\exp\{n^\varepsilon\}z^n$, $\varepsilon\in(0,1)$ we have

$$\varliminf_{r\to1-0}\frac{M_g(r)}{\frac{\mu_g(r)}{1-r}\ln^{1/2}\frac{\mu_g(r)}{1-r}}\ge C>0.$$

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2. Wiman's type inequality for analytic functions in $\mathbb{T}$. We consider

$$f(z) = f(z_1, z_2) = \sum_{n+m=0}^{+\infty} a_{nm} z_1^n z_2^m \tag{2}$$

with the domain of convergence $\mathbb{T} = \{z \in \mathbb{C}^2 \colon |z_1| < 1, |z_2| < +\infty\}$.

By $\mathcal{A}^2$ we denote the class of analytic functions of the form (2) with the domain of convergence $\mathbb{T}$ and $\frac{\partial}{\partial z_2} f(z_1, z_2) \not\equiv 0$ in $\mathbb{T}$.

For $r = (r_1, r_2) \in T{:=}[0, 1) \times [0, +\infty)$ and a function $f \in \mathcal{A}^2$ we denote

$$\triangle_r = \{(t_1, t_2) \in T \colon t_1 > r_1,\ t_2 > r_2\},\ M_f(r) = \max\{|f(z)| \colon |z_1| \le r_1, |z_2| \le r_2\},$$

$$\mu_f(r) = \max\{|a_{nm}| r_1^n r_2^m \colon (n, m) \in \mathbb{Z}_+^2\},\ \mathfrak{M}_f(r) = \sum_{n+m=0}^{+\infty} |a_{nm}| r_1^n r_2^m.$$

Let $D_f(r) = (D_{ij})$ be a 2×2 matrix such that

$$D_{ij} = r_i \frac{\partial}{\partial r_i}\Big(r_j \frac{\partial}{\partial r_j} \ln M_f(r)\Big) = \partial_i \partial_j \ln M_f(r), \quad \partial_i = r_i \frac{\partial}{\partial r_i}, \quad i, j \in \{1, 2\}.$$

We say that $E \subset T$ is set of *asymptotically finite logarithmic measure on* T if there exists $r_0 \in T$ such that

$$\nu_{\ln}(E \cap \triangle_{r_0}) {:=} \iint\limits_{E \cap \triangle_{r_0}} \frac{dr_1 dr_2}{(1 - r_1) r_2} < +\infty,\ (E \in \Upsilon)$$

i.e. the set $E \cap \triangle_{r_0}$ is a set of *finite logarithmic measure on* T.

In [5] one can find analogues of Wiman's inequality for analytic functions from the class $\mathcal{A}^2$.

Theorem 1 ([5]). *Let $f \in \mathcal{A}^2$. For every $\delta > 0$ there exists a set $E = E(\delta, f) \subset \Upsilon$ such that for $r \in T \setminus E$ we obtain*

$$M_f(r) \le \mathfrak{M}_f(r) \le \frac{\mu_f(r)}{(1 - r)^{1+\delta}} \ln^{1+\delta} \frac{\mu_f(r)}{1 - r} \ln^{1/2+\delta} r_2. \tag{3}$$

Also in [5] was proved that inequalite (3) is sharp. In particular for some $f(z_1, z_2) \in \mathcal{A}^2$ we have

$$E = \{r \in T : M_f(r) > \frac{\mu_f(r)}{(1 - r)} \ln \frac{\mu_f(r)}{1 - r}\} \notin \Upsilon.$$

The aim of this paper is to prove the sharp Wiman's inequality for random analytic functions in $\mathbb{T}$. We will prove that almost surely one can replace the exponent $1 + \delta$ in inequality (3) by $\frac{1}{2} + \delta$, and this exponent cannot be placed by a number smaller than $\frac{1}{2}$.

3. Wiman's type inequality for random analytic functions in the $\mathbb{T}$. Let $\Omega = [0, 1]$ and P be the Lebesgue measure on $\mathbb{R}$. We consider the Steinhaus probability space $(\Omega, \mathcal{A}, P)$, where $\mathcal{A}$ is the σ-algebra of Lebesgue measurable subsets of Ω. In the sequel, the notion "almost surely" will be used in the sense that the corresponding property holds almost everywhere with respect to Lebesgue measure P on Ω.

Let $Z = Z_{nm}(t)$ be some sequence of random variables defined in this space.

Let $X = (X_{nm}(t))$ be multiplicative system (MS) uniformly bounded by the number 1. That is for all $n, m \in \mathbb{N}$ and $t \in [0,1]$ we have $|X_{nm}(t)| \le 1$ for almost all $t \in [0;1]$ and

$$\forall (i_1, i_2, \dots, i_n) \in \mathbb{N}^k,\ 1 \le i_1 < i_2 < \dots < i_k,\ :\ \mathbf{M}(X_{i_1}X_{i_2}\cdots X_{i_k}) = 0,$$

where $\mathbf{M}\xi$ is the expected value of a random variable ξ.

Let $Z = (Z_{nm}(t))$ be a sequence of random complex variables $Z_{nm}(t) = X_{nm}(t) + iY_{nm}(t)$ such that both $X = X_{nm}(t)$ and $Y = Y_{nm}(t)$ are real MS. We consider the class of random analytic functions of the form

$$f(z,t) = \sum_{n+m=0}^{+\infty} a_{n,m} Z_{n,m}(t) z_1^n z_2^m. \tag{4}$$

For such a functions we prove following statement.

Theorem 2. *Let $f \in \mathcal{A}^2$, Z be a MS uniformly bounded by the number 1, $\delta > 0$. Then almost surely in t there exists a set $E = E(f,t,\delta)$, $E \subset \Upsilon$ such that for all $r \in T \setminus E$ we have*

$$M_f(r,t) := \max\{|f(z,t)| : |z_1| \le r_1, |z_2| \le r_2\} \le \frac{\mu_f(r)}{(1-r_1)^{1/2+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r_1} \cdot \ln^{1/4+\delta} r_2. \tag{5}$$

Lemma 1 ([6])**.** *Let $X = (X_{nm}(t))$ be a MS uniformly bounded by the number 1. Then for each $\beta > 0$ there exists a constant $A_\beta > 0$, which depends anly on β only such that for all $N \ge 4\pi$ and $\{c_{nm} \colon n+m \le N\} \subset \mathbb{C}$ we have*

$$P\left\{t \colon \max\left\{\left|\sum_{n+m=0}^{+\infty} c_{nm} X_{nm}(t) e^{in_1\psi_1} e^{im_2\psi_2}\right| \colon \psi \in [0,2\pi]^2\right\} \ge A_\beta S_N \ln^{\frac12} N\right\} \le \frac{1}{N^\beta}, \tag{6}$$

where $S_N^2 = \sum_{n+m=0}^{+\infty} |c_{nm}|^2$.

Let $h \colon \mathbb{R}_+^2 \to \mathbb{R}_+$ be an increasing function on each variable such that

$$\int_1^{+\infty}\int_1^{+\infty} \frac{du_1 du_2}{h(u_1,u_2)} < +\infty.$$

Lemma 2 ([5])**.** *Let $\delta > 0$. Then there exists a set $E \subset T$ of asymptotically finite logarithmic measure such that for all $r \in T \backslash E$ we have*

$$\frac{\partial}{\partial r_1} \ln M_f(r) \le \frac{1}{1-r_1} \cdot h\Big(\ln M_f(r), \ln r_2\Big), \tag{7}$$

$$\frac{\partial}{\partial r_2} \ln M_f(r) \le \frac{1}{r_2(1-r_1)^\delta} \cdot \Big(\ln M_f(r)\Big)^{1+\delta}. \tag{8}$$

Proof of Theorem 2. Without loss of generality we may suppose that $Z = X = (X_{nm}(t))$ is a MS (see [7]).

For $k, m \in \mathbb{Z}_+$ and $l \in \mathbb{Z}$ such that $k > -l$ we denote

$$G_{klm} = \left\{r = (r_1,r_2) \in \mathbb{T} \colon k \le \ln \frac{1}{1-r_1} \le k+1,\ l \le \ln \mu_f(r) \le l+1,\ m \le \ln r_2 \le m+1\right\},$$

$$G_{kl}=\bigcap_{m=1}^{+\infty}G_{klm},\quad G_{kl}^{+}=\bigcup_{i=k}^{+\infty}\bigcup_{j=l}^{+\infty}G_{ij}.$$

Remark that

$$E_0=\Big\{r\in T\colon \ \ln\frac{1}{1-r_1}+\ln\mu_f(r)<1\Big\}=\Big\{r\in T\colon \frac{\mu_f(r)}{1-r_1}<e\Big\}\in\Upsilon,$$

because there exists r_0 such that $E_0\cap\Delta_{r_0}=\varnothing$.

By Lemma 2 with $h(r)=r_1^{1+\delta}r_2^{1+\delta}$ there exists a set $E_1\supset E_0, E_1\in\Upsilon$ such that for all $r\in T\backslash E_1$ we have

$$\sum_{n+m=0}^{+\infty}(n+m)|a_{nm}|r_1^nr_2^m=\mathfrak{M}_f(r)\Big(r_1\frac{\partial}{\partial r_1}(\ln\mathfrak{M}_f(r_1,r_2))+r_2\frac{\partial}{\partial r_2}(\ln\mathfrak{M}_f(r_1,r_2))\Big)\leq$$

$$\leq\mathfrak{M}_f(r)\left(\frac{r_1}{1-r_1}\ln^{1+\delta}\mathfrak{M}_f(r)\ln^{1+\delta}r_2+\frac{r_2}{r_2(1-r_1)^{\delta}}\ln^{1+\delta}\mathfrak{M}_f(r)\right)\leq$$

$$\leq\frac{2\mathfrak{M}_f(r)}{1-r_1}\ln^{1+\delta}\mathfrak{M}_f(r)\ln^{1+\delta}r_2\leq$$

$$\leq\frac{\mu_f(r)}{(1-r_1)^{2+\delta}}\ln^{1+\delta}\frac{\mu_f(r)}{1-r_1}\ln^{1/2+\delta}r_2\Big(3\ln\mu_f(r)+3\ln\frac{1}{1-r_1}\Big).$$

Therefore

$$\sum_{n+m\geq d}|a_{nm}|r_1^nr_2^m\leq\sum_{n+m\geq d}\frac{n+m}{d}|a_{nm}|r_1^nr_2^m\leq\frac{1}{d}\sum_{n+m=0}^{+\infty}(n+m)|a_{nm}|r_1^nr_2^m\leq$$

$$\leq\frac{1}{d}\frac{\mu_f(r)}{(1-r_1)^{2+\delta}}\cdot\ln^{2+\delta}\frac{\mu_f(r)}{1-r_1}\ln r_2\leq\frac{1}{d}\frac{\mu_f(r)}{(1-r_1)^{2+\delta}}\cdot\ln^{3+\delta}\frac{\mu_f(r)}{1-r_1}\leq\mu_f(r),\tag{9}$$

where

$$d=d(r)=\frac{1}{(1-r_1)^{2+\delta}}\cdot\ln^{3+\delta}\frac{\mu_f(r)}{1-r_1}.$$

Let $G_{kl}^*=G_{kl}\setminus E_2$, $I=\{(i;j)\colon G_{ij}^*\neq\varnothing\}$,

$$E_2=E_1\cup\left(\bigcup_{(i,j)\notin I}G_{ij}\right).$$

Then $\#I=+\infty$. For $(k,l)\in I$ we choose a sequence $r^{(k,l)}\in G_{kl}^*$ such that $M_f(r^{(k,l)})=\min\limits_{r\in G_{kl}^*}M_f(r)$. So, for all $r\in G_{kl}^*$ we get

$$\mu_f(r^{(k,l)})\leq\mu_f(r)\leq e\mu_f(r^{(k,l)}),\tag{10}$$

$$\frac{1}{e}\frac{1}{1-r_1^{(k,l)}}\leq\frac{1}{1-r_1}\leq e\frac{1}{1-r_1^{(k,l)}},\tag{11}$$

$$\frac{1}{e^2}\frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}}\leq\frac{\mu_f(r)}{1-r_1}\leq\frac{e^2\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}}\tag{12}$$

and also

$$\bigcup_{(k,l)\in I} G^*_{kl} = \bigcup_{(k,l)\in I} G_{kl}\setminus E_2 = \bigcup_{k,l=1}^{+\infty} G_{kl}\setminus E_2 = T\setminus E_2.$$

Denote $N_{kl} = [2d_1(r^{(k,l)})]$, where

$$d_1(r) = \frac{e^{2+\delta}}{(1-r_1)^{2+\delta}}\cdot \ln^{3+\delta}\frac{e^2\mu_f(r)}{1-r_1}.$$

For $r\in G^*_{kl}$ we put

$$W_{N_{kl}}(r,t) = \max\left\{\left|\sum_{n+m\le N_{kl}} a_{nm}r_1^n r_2^m e^{in_1\psi_1+in_2\psi_2}X_{nm}(t)\right| : \psi\in[0,2\pi]^2\right\}.$$

For a Lebesgue measurable set $G\subset G^*_{kl}$ and for $(k,l)\in I$ we denote $\nu_{kl}(G) = \frac{\operatorname{meas}(G)}{\operatorname{meas}(G^*_{kl})}$, where meas denotes the Lebesgue measure on $\mathbb{R}^2$.

Remark that ν_{kl} is a probability measure defined on the family of Lebesgue measurable subsets of G^*_k ([7]). Let $\Omega = \bigcup_{(k,l)\in I} G^*_{kl}$ and

$$k_i, l_{i,j}\colon (k_i,l_{i,j})\in I,\ k_i<k_{i+1},\ l_{i,j}<l_{i,j+1},\ \forall i,j\in\mathbb{Z}_+.$$

For Lebesgue measurable subsets G of Ω we denote

$$\nu(G) = 2^{k_0}\sum_{i=0}^{+\infty}\Bigg(\frac{1}{2^{k_i}}\left(1-\left(\frac12\right)^{k_{i+1}-k_i}\right)\times$$
$$\times\sum_{j=0}^{N_i}\frac{2^{l_{i,0}}}{2^{l_{i,j}}}\frac{\left(1-\left(\frac12\right)^{l_{i,j+1}-l_{i,j}}\right)}{1-\left(\frac12\right)^{l_{i,N_{i+1}}+l_{i,0}}}\nu_{k_{i+1}l_{i+1,j+1}}(G\cap G^*_{k_{j+1}l_{i+1,j+1}})\Bigg), \tag{13}$$

where $N_i = \max\{j\colon (k_i,l_{ij})\in I\}$. Remark that $\nu_{k_{j+1}l_{j+1}}(G^*_{k_{j+1}l_{j+1}}) = \nu(\Omega) = 1$.

Thus ν is a probability measure, which is defined on measurable subsets of Ω. On $[0,1]\times\Omega$ we define the probability measure $P_0 = P\otimes\nu$, which is the direct product of the probability measures P and ν. Now for $(k;l)\in I$ we define

$$F_{kl} = \{(t,r)\in[0,1]\times\Omega\colon W_{N_{kl}}(r,t) > AS_{N_{kl}}(r)\ln^{1/2}N_{kl}\},$$
$$F_{kl}(r) = \{t\in[0,1]\colon W_{N_{kl}}(r,t) > AS_{N_{kl}}(r)\ln^{1/2}N_{kl}\},$$

where $S^2_{N_{kl}}(r) = \sum_{n+m=0}^{N_{kl}}|a_{nm}|^2 r_1^{2n}r_2^{2m}$ and A is the constant from Lemma 1 with $\beta=1$. Using Fubini's theorem and Lemma 1 with $c_n = a_n r^n$ and $\beta = 1$, we get for $(k,l)\in I$

$$P_0(F_{kl}) = \int_\Omega\left(\int_{F_{kl}(r)} dP\right)d\nu = \int_\Omega P(F_{kl}(r))d\nu \le \frac{1}{N_{kl}}\nu(\Omega) = \frac{1}{N_{kl}}.$$

Note that

$$N_{kl} > \frac{1}{(1-r_1^{(k,l)})^{2+\delta}}\ln^{3+\delta}\frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \ge e^{2k}(l+k)^3.$$

Therefore

$$\sum_{(k,l)\in I} P_0(F_{kl}) \le \sum_{k=1}^{+\infty} \sum_{l=-k+1}^{+\infty} \frac{1}{e^{2k}(l+k)^3} < +\infty.$$

By Borel-Cantelli's lemma the infinite quantity of the events $\{F_{kl}\colon (k,l)\in I\}$ may occur with probability zero. So,

$$P_0(F)=1, \quad F=\bigcup_{s=1}^{+\infty}\bigcup_{m=1}^{+\infty}\bigcap_{\substack{k\ge s,\ l\ge m\\ (k,l)\in I}} \overline{F_{kl}} \subset [0,1]\times\Omega.$$

Then for any point $(t,r)\in F$ there exist $k_0=k_0(t,r)$ and $l_0=l_0(t,r)$ such that for all $k\ge k_0$, $l\ge l_0$, $(k,l)\in I$ we have $W_{N_{kl}}(r,t)\le AS_{N_{kl}}(r)\ln^{1/2}N_{kl}$.

So, $\nu(F^\wedge(t))=1$ (see [7]).

For any $t\in F_1$([7]) and $(k,l)\in I$ we choose a point $r_0^{(k,l)}(t)\in G^*_{kl}$ such that

$$W_{N_{kl}}(r_0^{(k,l)}(t),t)\ge \frac{3}{4}M_{kl}(t),\ \ M_{kl}(t)\overset{\text{def}}{=}\sup\{W_{N_{kl}}(r,t)\colon r\in G^*_{kl}\}.$$

Then from $\nu_{kl}(F^\wedge(t)\cap G^*_{kl})=1$ for all $(k,l)\in I$ it follows that there exists a point $r^{(k,l)}(t)\in G^*_{kl}\cap F^\wedge(t)$ such that

$$|W_{N_{kl}}(r_0^{(k,l)}(t),t)-W_{N_{kl}}(r^{(k,l)}(t),t)|<\frac{1}{4}M_{kl}(t)$$

or

$$\frac{3}{4}M_{kl}(t)\le W_{N_{kl}}(r_0^{(k,l)}(t),t)\le W_{N_{kl}}(r^{(k,l)}(t),t)+\frac{1}{4}M_{kl}(t).$$

Since $(t,r^{(k,l)}(t))\in F$, from inequality (13) we obtain

$$\frac{1}{2}M_{kl}(t)\le W_{N_{kl}}(r^{(k,l)}(t),t)\le AS_{N_{kl}}(r^{(k,l)}(t))\ln^{1/2}N_{kl}.$$

Now for $r^{(k,l)}=r^{(k,l)}(t)$ we get

$$S^2_{N_{kl}}(r^{(k,l)})\le \mu_f(r^{(k,l)})\mathfrak{M}_f(r^{(k,l)})\le \frac{\mu_f^2(r^{(k,l)})}{(1-r_1^{(k,l)})^{1+\delta}}\ln^{1+\delta}\frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}}\ln^{1/2+\delta} r_2^{(k,l)}.$$

So, for $t\in F_1$ and all $k\ge k_0(t)$, $l\ge l_0(t)$, we obtain

$$S_N(r^{(k,l)})\le \mu_f(r^{(k,l)})\left(\frac{1}{1-r_1^{(k,l)}}\ln\frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}}\sqrt{\ln r_2^{(k,l)}}\right)^{1/2+\delta/2}. \tag{14}$$

It follows from (10)–(12) that $d_1(r^{(k,l)})\ge d(r)$ for $r\in G^*_{kl}$. Then for $t\in F_1$, $r\in F^\wedge(t)\cap G^*_{kl}$, $(k,l)\in I,\ k\ge k_0(t), l\ge l_0(t)$ we get

$$M_f(r,t)\le \sum_{n+m\ge 2d_1(r^{(k,l)})}|a_{nm}|r_1^n r_2^m + W_{N_{kl}}(r,t)\le \sum_{n+m\ge 2d(r)}|a_{nm}|r_1^n r_2^m + M_{kl}(t).$$

Finally for $t\in F_1$, $r\in F^\wedge(t)\cap G^*_{kl}$, $l\ge l_0(t)$ and $k\ge k_0(t)$ we obtain

$$M_f(r^{(k,l)},t)\le \mu_f(r^{(k,l)})+2AS_{N_{kl}}(r^{(k,l)})\ln^{1/2}N_{kl}\le \mu_f(r^{(k,l)})+$$

$$+2A\mu_f(r^{(k,l)})\left(\frac{1}{1-r_1^{(k,l)}}\ln\frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}}\sqrt{\ln r_2^{(k,l)}}\right)^{1/2+\delta/2}\times$$
$$\times\ln^{1/2}\left(\frac{3e^{2+\delta}}{(1-r_j^{(k,l)})^{2+\delta}}\cdot\ln^{3+\delta}\frac{e^2\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}}\right).$$

So, we get for $t\in F_1$, $r\in F^\wedge(t)\cap G^*_{klm}$, $k\geq k_0(t)$ and $l\geq l_0(t)$

$$M_f(r,t)\leq\frac{\mu_f(r)}{(1-r_1)^{1/2+\delta}}\cdot\ln^{1/2+\delta}\frac{\mu_f(r)}{1-r_1}\ln^{1/4+\delta}r_2. \tag{15}$$

Therefore inequality (15) holds almost surely ($t\in F_1$, $P(F_1)=1$) for all

$$r\in\Big(\bigcup_{(k,l)\in I}(G^*_{kl}\cap F^\wedge(t))\cap G^+_{kl}\Big)\backslash E^*=(T\cap G^+_{kl})\setminus(E^*\cup G^*\cup E_1)=T\setminus E_2,$$

where

$$G^+_{kl}=\bigcup_{i=k}^{+\infty}\bigcup_{j=l}^{+\infty}G_{kl},\ \ E_2=E_1\cup G^*\cup E^*,\ \ G^*=\bigcup_{(k,l)\in I}(G^*_{kl}\setminus F^\wedge(t)).$$

It remains to remark that $\nu(G^*)$ satisfies $\nu(G^*)=\sum_{(k,l)\in I}(\nu_{kl}(G^*_{kl})-\nu_{kl}(F^\wedge(t)))=0$. Then for all $(k,l)\in I$ we obtain

$$\nu_{kl}(G^*_{kl}\setminus F^\wedge(t))=\frac{\operatorname{meas}_p(G^*_{kl}\setminus F^\wedge(t))}{\operatorname{meas}_p(G^*_{kl})}=0,$$
$$\operatorname{meas}(G^*_{kl}\setminus F^\wedge(t))=\iint\limits_{G^*_{kl}\setminus F^\wedge(t)}\frac{drdr_2}{(1-r_1)r_2}=0.$$

□

4. Sharpness of Theorem 2. We will prove that no one of powers $1/2+\delta$ in inequality (5) cannot be replaced by smaller number than $1/2$.

Theorem 3. *Let Z be a sequence of random variables such that $|Z_{nm}|\geq 1$ for almost all $t\in[0;1]$. Then for there exist an analytic function $f\in\mathcal{A}^2$, a constant $C>0$ and $r_0\in T$, such that almost surely in t for all $r\in\Delta_{r_0}$ we get*

$$M_f(r,t)\geq\frac{C\mu_f(r)}{\sqrt{1-r_1}}\cdot\ln^{1/2}\frac{\mu_f(r)}{1-r_1}. \tag{16}$$

Proof. Consider the functions

$$g(z_1,z_2)=\sum_{n+m=0}^{+\infty}\frac{e^{\sqrt{m}}}{n!}z_1^nz_2^m,\quad f(z_1,z_2)=\sum_{n+m=0}^{+\infty}\frac{e^{\sqrt{m}/2}}{\sqrt{n!}}z_1^nz_2^m,$$
$$f(z,t)=\sum_{n+m=0}^{+\infty}Z_{nm}(t)\frac{e^{\sqrt{m}/2}}{\sqrt{n!}}z_1^nz_2^m.$$

Remark, that for all $r \in T$ we have

$$\mu_g(r_1^2, r_2^2) = \max\left\{\frac{e^{\sqrt{m}}}{n!} z_1^n z_2^m : (n,m) \in \mathbb{Z}_2^+\right\} =$$

$$= \max\left\{\left(\frac{e^{\sqrt{m}/2}}{\sqrt{n!}} z_1^n z_2^m\right)^2 : (n,m) \in \mathbb{Z}_2^+\right\} = (\mu_f(r_1, r_2))^2.$$

Using Parseval's equality, we get for almost all t

$$M_g(r_1^2, r_2^2) \leq \sum_{n+m=0}^{+\infty} |Z_{nm}(t)|^2 \frac{e^{\sqrt{m}}}{n!} z_1^n z_2^m =$$

$$= \frac{1}{4\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} |f(r_1 e^{i\theta}, r_2 e^{i\varphi}, t)|^2 d\varphi \leq (M_f(r_1, r_2, t))^2.$$

There exist $r_0 \in T$ ([5]) a set $r_0 \in T$ such that for $r \in \Delta_{r_0}$

$$M_g(r_1, r_2) > \frac{C\mu_g(r_1, r_2)}{1 - r_1} \cdot \ln \frac{\mu_g(r_1, r_2)}{1 - r_1},$$

$$(M_f(r_1, r_2, t))^2 \geq M_g(r_1, r_2) \geq \frac{C\mu_g(r_1^2, r_2^2)}{1 - r_1^2} \cdot \ln \frac{\mu_g(r_1^2, r_2^2)}{1 - r_1^2} =$$

$$= \frac{C\mu_f^2(r_1, r_2)}{(1 - r_1)(1 + r_1)} \ln \frac{\mu_f^2(r_1, r_2)}{(1 - r_1)(1 + r_1)} \geq \frac{C}{2} \frac{\mu_f^2(r_1, r_2)}{1 - r_1} \ln \frac{\mu_f^2(r_1, r_2)}{1 - r_1}$$

and

$$M_f(r, t) \geq \left(\frac{C}{2} \frac{\mu_f^2(r_1, r_2)}{1 - r_1} \ln \frac{\mu_f^2(r_1, r_2)}{1 - r_1}\right)^{1/2} \geq \sqrt{\frac{C}{2}} \frac{\mu_f(r_1, r_2)}{\sqrt{1 - r_1}} \ln^{1/2} \frac{\mu_f(r_1, r_2)}{1 - r_1}.$$

□

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