1. Introduction. Let $f$ be an analytic function in the disc $\{z: |z| < R\}$, $0 < R \leq +\infty$, represented by the power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n. \quad (1)$$

For $r \in (0, R)$ we denote $M_f(r) = \max\{|f(z)|: |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n: n \geq 0\}$. It is well known ([1, p. 9], [2, p. 28]) that for each nonconstant entire function $f(z)$ and every $\varepsilon > 0$ there exists a set $E(\varepsilon, f) \subset [1, +\infty)$ such that Wiman’s inequality

$$M_f(r) \leq \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon}$$

holds for all $r \in [1, +\infty) \setminus E(\varepsilon, f)$ and this set $E(\varepsilon, f)$ has finite logarithmic measure, i.e. $\int_{E(\varepsilon, f)} \frac{dr}{1-r} < +\infty$.

Let $f(z)$ be an analytic function in the unit disc $\mathbb{D} = \{z: |z| < 1\}$. For such a function $f(z)$ and every $\delta > 0$ there exists a set $E_f(\delta) \subset (0, 1)$ of finite logarithmic measure on $(0, 1)$, i.e.

$$\int_{E_f(\delta)} \frac{dr}{1-r} < +\infty,$$

such that for all $r \in (0, 1) \setminus E_f(\delta)$ the inequality

$$M_f(r) \leq \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}$$

holds ([3]). Similar inequality for analytic function in the unit disc one can find in [4].

Also in [3] it is noted that for the function $g(z) = \sum_{n=1}^{+\infty} \exp\{n^\varepsilon\} z^n$, $\varepsilon \in (0, 1)$ we have

$$\lim_{r \to 1^-} \frac{M_g(r)}{\mu_g(r) \ln^{1/2} \frac{\mu_g(r)}{1-r}} \geq C > 0.$$
2. Wiman’s type inequality for analytic functions in $\mathbb{T}$. We consider

$$f(z) = f(z_1, z_2) = \sum_{n+m=0}^{+\infty} a_{nm} z_1^n z_2^m$$

with the domain of convergence $\mathbb{T} = \{z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < +\infty\}$.

By $\mathcal{A}^2$ we denote the class of analytic functions of the form (2) with the domain of convergence $\mathbb{T}$ and $\frac{\partial}{\partial z_2} f(z_1, z_2) \neq 0$ in $\mathbb{T}$.

For $r = (r_1, r_2) \in T := [0, 1] \times [0, +\infty)$ and a function $f \in \mathcal{A}^2$ we denote

$$\Delta_r = \{(t_1, t_2) \in T : t_1 > r_1, t_2 > r_2\}, \quad M_f(r) = \max\{|f(z)|: |z_1| \leq r_1, |z_2| \leq r_2\},$$

$$\mu_f(r) = \max\{|a_{nm}| r_1^n r_2^m : (n, m) \in \mathbb{Z}_+^2\}, \quad \mathfrak{M}_f(r) = \sum_{n+m=0}^{+\infty} |a_{nm}| r_1^n r_2^m.$$

Let $D_f(r) = (D_{ij})$ be a $2 \times 2$ matrix such that

$$D_{ij} = r_i \frac{\partial}{\partial r_i} \left( r_j \frac{\partial}{\partial r_j} \ln M_f(r) \right) = \partial_i \partial_j \ln M_f(r), \quad \partial_i = r_i \frac{\partial}{\partial r_i}, \quad i, j \in \{1, 2\}.$$

We say that $E \subset T$ is set of *asymptotically finite logarithmic measure on $T$* if there exists $r_0 \in T$ such that

$$\nu_{\ln}(E \cap \Delta_{r_0}) := \iint_{E \cap \Delta_{r_0}} \frac{dr_1 dr_2}{(1 - r_1)r_2} < +\infty, \quad (E \in \Upsilon)$$

i.e. the set $E \cap \Delta_{r_0}$ is a set of *finite logarithmic measure on $T$*.

In [5] one can find analogues of Wiman’s inequality for analytic functions from the class $\mathcal{A}^2$.

**Theorem 1** ([5]). Let $f \in \mathcal{A}^2$. For every $\delta > 0$ there exists a set $E = E(\delta, f) \subset \Upsilon$ such that for $r \in T \setminus E$ we obtain

$$M_f(r) \leq \mathfrak{M}_f(r) \leq \frac{\mu_f(r)}{(1 - r)^{1+\delta}} \ln^{1+\delta} \frac{\mu_f(r)}{1 - r} \ln^{1/2+\delta} r_2.$$

(3)

Also in [5] was proved that inequality (3) is sharp. In particular for some $f(z_1, z_2) \in \mathcal{A}^2$ we have

$$E = \{r \in T : M_f(r) > \frac{\mu_f(r)}{(1 - r)^{1+\delta}} \ln^{1/2+\delta} r_2 \} \not\in \mathbb{T}.$$

The aim of this paper is to prove the sharp Wiman’s inequality for random analytic functions in $\mathbb{T}$. We will prove that almost surely one can replace the exponent $1 + \delta$ in inequality (3) by $\frac{1}{2} + \delta$, and this exponent cannot be placed by a number smaller than $\frac{1}{2}$.

3. Wiman’s type inequality for random analytic functions in the $\mathbb{T}$. Let $\Omega = [0, 1]$ and $P$ be the Lebesgue measure on $\mathbb{R}$. We consider the Steinhaus probability space $(\Omega, \mathcal{A}, P)$, where $\mathcal{A}$ is the $\sigma$-algebra of Lebesgue measurable subsets of $\Omega$. In the sequel, the notion “almost surely” will be used in the sense that the corresponding property holds almost everywhere with respect to Lebesgue measure $P$ on $\Omega$. 


Let \( Z = Z_{nm}(t) \) be some sequence of random variables defined in this space.
Let \( X = (X_{nm}(t)) \) be multiplicative system (MS) uniformly bounded by the number 1. That is for all \( n,m \in \mathbb{N} \) and \( t \in [0,1] \) we have \(|X_{nm}(t)| \leq 1 \) for almost all \( t \in [0;1] \) and

\[
\forall (i_1, i_2, \ldots, i_n) \in \mathbb{N}^k, \ 1 \leq i_1 < i_2 < \cdots < i_k, \ : \ M(X_{i_1}X_{i_2} \cdots X_{i_k}) = 0,
\]

where \( M_\xi \) is the expected value of a random variable \( \xi \).

Let \( Z = (Z_{nm}(t)) \) be a sequence of random complex variables \( Z_{nm}(t) = X_{nm}(t) + iY_{nm}(t) \) such that both \( X = X_{nm}(t) \) and \( Y = Y_{nm}(t) \) are real MS. We consider the class of random analytic functions of the form

\[
f(z,t) = \sum_{n+m=0}^{+\infty} a_{n,m}Z_{n,m}(t)z_1^n z_2^m.
\]

For such a functions we prove following statement.

**Theorem 2.** Let \( f \in A^2 \), \( Z \) be a MS uniformly bounded by the number 1, \( \delta > 0 \). Then almost surely in \( t \) there exists a set \( E = E(f,t,\delta), E \subset \Upsilon \) such that for all \( r \in T \setminus E \) we have

\[
M_f(r,t) := \max\{|f(z,t)| : |z_1| \leq r_1, |z_2| \leq r_2\} \leq \frac{\mu_f(r)}{(1-r_1)^{1/2+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r_1} \ln^{1+4+\delta} r_2. \tag{5}
\]

**Lemma 1** ([6]). Let \( X = (X_{nm}(t)) \) be a MS uniformly bounded by the number 1. Then for each \( \beta > 0 \) there exists a constant \( A_\beta > 0 \), which depends only on \( \beta \) only such that for all \( N \geq 4\pi \) and \( \{c_{nm} : n+m \leq N\} \subset \mathbb{C} \) we have

\[
P\left\{ \max\left\{ \sum_{n+m=0}^{+\infty} c_{nm}X_{nm}(t)e^{im_1\psi_1}e^{im_2\psi_2} : \psi \in [0,2\pi]^2 \right\} \geq A_\beta S_N \ln^{3/2} N \right\} \leq \frac{1}{N^\beta}, \tag{6}
\]

where \( S_N^2 = \sum_{n+m=0}^{+\infty} |c_{nm}|^2 \).

Let \( h : \mathbb{R}_+^2 \to \mathbb{R}_+ \) be an increasing function on each variable such that

\[
\int_1^{+\infty} \int_1^{+\infty} \frac{du_1du_2}{h(u_1,u_2)} < +\infty.
\]

**Lemma 2** ([5]). Let \( \delta > 0 \). Then there exists a set \( E \subset T \) of asymptotically finite logarithmic measure such that for all \( r \in T \setminus E \) we have

\[
\frac{\partial}{\partial r_1} \ln M_f(r) \leq \frac{1}{1-r_1} \cdot h\left( \ln M_f(r), \ln r_2 \right) \tag{7},
\]

\[
\frac{\partial}{\partial r_2} \ln M_f(r) \leq \frac{1}{r_2(1-r_1)^\delta} \cdot \left( \ln M_f(r) \right)^{1+\delta} \tag{8}.
\]

**Proof of Theorem 2.** Without loss of generality we may suppose that \( Z = X = (X_{nm}(t)) \) is a MS (see [7]).

For \( k, m \in \mathbb{Z}_+ \) and \( l \in \mathbb{Z} \) such that \( k > -l \) we denote

\[
G_{klm} = \left\{ r = (r_1, r_2) \in T : k \leq \frac{1}{1-r_1} \leq k + 1, \ l \leq \ln \mu_f(r) \leq l + 1, \ m \leq \ln r_2 \leq m + 1 \right\},
\]
Remark that
\[ E_0 = \{ r \in T : \ln \frac{1}{1 - r_1} + \ln \mu_f(r) < 1 \} = \{ r \in T : \frac{\mu_f(r)}{1 - r_1} < e \} \in \mathcal{Y}, \]
because there exists \( r_0 \) such that \( E_0 \cap \Delta_{r_0} = \emptyset \).

By Lemma 2 with \( h(r) = r_1^{1+\delta}r_2^{1+\delta} \) there exists a set \( E_1 \supseteq E_0, E_1 \in \mathcal{Y} \) such that for all \( r \in T \setminus E_1 \) we have
\[ \sum_{n+m=0}^{+\infty} (n+m)|a_{nm}|r_1^n r_2^m = M_f(r)\left( r_1 \frac{\partial}{\partial r_1} (\ln M_f(r_1, r_2)) + r_2 \frac{\partial}{\partial r_2} (\ln M_f(r_1, r_2)) \right) \leq \]
\[ \leq M_f(r) \left( \frac{r_1}{1 - r_1} \ln^{1+\delta} M_f(r) \ln^{1+\delta} r_2 + \frac{r_2}{r_2 (1 - r_1)^\delta} \ln^{1+\delta} M_f(r) \right) \leq \]
\[ \leq \frac{2M_f(r)}{1 - r_1} \ln^{1+\delta} M_f(r) \ln^{1+\delta} r_2 \leq \]
\[ \leq \frac{\mu_f(r)}{1 - r_1} \ln^{1+\delta} \frac{\mu_f(r)}{1 - r_1} \ln^{1/2+\delta} r_2 \left( 3 \ln \mu_f(r) + 3 \ln \frac{1}{1 - r_1} \right). \]

Therefore
\[ \sum_{n+m \geq d} |a_{nm}|r_1^n r_2^m \leq \sum_{n+m \geq d} \frac{n+m}{d} |a_{nm}|r_1^n r_2^m \leq \frac{1}{d} \sum_{n+m=0}^{+\infty} (n+m)|a_{nm}|r_1^n r_2^m \leq \]
\[ \leq \frac{1}{d} \frac{\mu_f(r)}{1 - r_1} \ln^{2+\delta} \frac{\mu_f(r)}{1 - r_1} \ln r_2 \leq \frac{1}{d} \frac{\mu_f(r)}{1 - r_1} \ln^{3+\delta} \frac{\mu_f(r)}{1 - r_1} \leq \mu_f(r), \quad (9) \]
where
\[ d = d(r) = \frac{1}{(1 - r_1)^{2+\delta}} \cdot \ln^{3+\delta} \frac{\mu_f(r)}{1 - r_1}. \]

Let \( G^*_{kl} = G_{kl} \setminus E_2, I = \{(i,j) : G^*_{ij} \neq \emptyset\}, \]
\[ E_2 = E_1 \cup \left( \bigcup_{(i,j) \not\in I} G_{ij} \right). \]

Then \( \#I = +\infty \). For \( (k,l) \in I \) we choose a sequence \( r^{(k,l)} \in G^*_{kl} \) such that \( M_f(r^{(k,l)}) = \min_{r \in G^*_{kl}} M_f(r) \). So, for all \( r \in G^*_{kl} \) we get
\[ \mu_f(r^{(k,l)}) \leq \mu_f(r) \leq e \mu_f(r^{(k,l)}), \quad (10) \]
\[ \frac{1}{e} \frac{1}{1 - r_1^{(k,l)}} \leq \frac{1}{1 - r_1} \leq e \frac{1}{1 - r_1^{(k,l)}}, \quad (11) \]
\[ \frac{1}{e^2} \frac{\mu_f(r^{(k,l)})}{1 - r_1} \leq \frac{\mu_f(r)}{1 - r_1} \leq e^2 \frac{\mu_f(r^{(k,l)})}{1 - r_1^{(k,l)}}. \quad (12) \]
and also
\[ \bigcup_{(k,l) \in I} G_{kl}^* = \bigcup_{(k,l) \in I} G_{kl} \setminus E_2 = \bigcup_{k,l=1}^{+\infty} G_{kl} \setminus E_2 = T \setminus E_2. \]

Denote \( N_{kl} = [2d_1(r_{(k,l)})] \), where
\[ d_1(r) = \frac{e^{2+\delta}}{(1 - r_1)^{2+\delta}} \ln^{3+\delta} \frac{e^2 \mu_f(r)}{1 - r_1}. \]

For \( r \in G_{kl}^* \) we put
\[ W_{N_{kl}}(r, t) = \max \left\{ \left| \sum_{n+m \leq N_{kl}} a_{nm} r_1^n r_2^m e^{in_1 \psi_1 + in_2 \psi_2} X_{nm}(t) \right| : \psi \in [0, 2\pi]^2 \right\}. \]

For a Lebesgue measurable set \( G \subset G_{kl}^* \) and for \((k, l) \in I\) we denote \( \nu_{kl}(G) = \frac{\text{meas}(G)}{\text{meas}(G_{kl}^*)} \), where \( \text{meas} \) denotes the Lebesgue measure on \( \mathbb{R}^2 \).

Remark that \( \nu_{kl} \) is a probability measure defined on the family of Lebesgue measurable subsets of \( G_k^* \) ([7]). Let \( \Omega = \bigcup_{(k,l) \in I} G_{kl}^* \) and
\[ k_i, l_{i,j} : (k_i, l_{i,j}) \in I, \ k_i < k_{i+1}, \ l_{i,j} < l_{i,j+1}, \ \forall i,j \in \mathbb{Z}_+. \]

For Lebesgue measurable subsets \( G \) of \( \Omega \) we denote
\[ \nu(G) = 2^{k_0} \sum_{i=0}^{+\infty} \left( \frac{1}{2^{k_i}} \right) \left( \frac{k_{i+1} - k_i}{2^{k_i}} \right) \times \]
\[ \times \sum_{j=0}^{N_i} 2^{l_{i,j}} \left( 1 - \frac{1}{2^{k_i}} \right) \frac{l_{i,j+1} - l_{i,j}}{1 - \frac{1}{2^{k_i}}} \nu_{k_{i+1}l_{i+1,j+1}}(G \cap G_{k_{i+1}l_{i+1,j+1}}^*), \quad (13) \]

where \( N_i = \max \{ j : (k_i, l_{i,j}) \in I \} \). Remark that \( \nu_{k_{i+1}l_{i+1,j+1}}(G_{k_{i+1}l_{i+1,j+1}}^*) = \nu(\Omega) = 1. \)

Thus \( \nu \) is a probability measure, which is defined on measurable subsets of \( \Omega \). On \([0, 1] \times \Omega\) we define the probability measure \( P_0 = P \times \nu \), which is the direct product of the probability measures \( P \) and \( \nu \). Now for \((k, l) \in I\) we define
\[ F_{kl} = \{(t, r) \in [0, 1] \times \Omega : W_{N_{kl}}(r, t) > A S_{N_{kl}}(r) \ln^{1/2} N_{kl} \}, \]
\[ F_{kl}(r) = \{ t \in [0, 1] : W_{N_{kl}}(r, t) > A S_{N_{kl}}(r) \ln^{1/2} N_{kl} \}, \]

where \( S_{N_{kl}}^2(r) = \sum_{n+m=0}^{N_{kl}} |a_{nm}| r_1^n r_2^m \) and \( A \) is the constant from Lemma 1 with \( \beta = 1. \)

Using Fubini’s theorem and Lemma 1 with \( c_n = a_n r^n \) and \( \beta = 1 \), we get for \((k, l) \in I\)
\[ P_0(F_{kl}) = \int_\Omega \left( \int_{F_{kl}(r)} dP \right) d\nu = \int_\Omega P(F_{kl}(r)) d\nu \leq \frac{1}{N_{kl}} \nu(\Omega) = \frac{1}{N_{kl}}. \]

Note that
\[ N_{kl} > \frac{1}{(1 - r_1^{(k,l)})^{2+\delta}} \ln^{3+\delta} \frac{\mu_f(r_{(k,l)})}{1 - r_1^{(k,l)}} \geq e^{2k(l + k)^3}. \]
Therefore

\[
\sum_{(k,l) \in I} P_0(F_{kl}) \leq \sum_{k=1}^{+\infty} \sum_{l=-k+1}^{+\infty} \frac{1}{e^{2k(l+k)^2}} < +\infty.
\]

By Borel-Cantelli’s lemma the infinite quantity of the events \{F_{kl}: (k, l) \in I\} may occur with probability zero. So,

\[
P_0(F) = 1, \quad F = \bigcup_{s=1}^{+\infty} \bigcup_{m=1}^{+\infty} \bigcap_{t \geq s, \ l \geq m} F_{t,l} \subset [0, 1] \times \Omega.
\]

Then for any point \((t, r) \in F\) there exist \(k_0 = k_0(t, r)\) and \(l_0 = l_0(t, r)\) such that for all \(k \geq k_0, l \geq l_0, (k, l) \in I\) we have \(W_{N_{kl}}(r, t) \leq A S_{N_{kl}}(r) \ln^{1/2} N_{kl}\).

So, \(\nu(F \wedge t) = 1\) (see [7]).

For any \((t, r) \in F\) and \((k, l) \in I\) we choose a point \(r_0^{(k, l)}(t) \in G_{kl}^*\) such that

\[
W_{N_{kl}}(r_0^{(k, l)}(t), t) \geq \frac{3}{4} M_{kl}(t), \quad M_{kl}(t) \overset{\text{def}}{=} \sup\{W_{N_{kl}}(r, t) : r \in G_{kl}^*\}.
\]

Then from \(\nu_{kl}(F \wedge t \cap G_{kl}^*) = 1\) for all \((k, l) \in I\) it follows that there exists a point \(r^{(k, l)}(t) \in G_{kl}^* \cap F \wedge t\) such that

\[
|W_{N_{kl}}(r_0^{(k, l)}(t), t) - W_{N_{kl}}(r^{(k, l)}(t), t)| < \frac{1}{4} M_{kl}(t)
\]
or

\[
\frac{3}{4} M_{kl}(t) \leq W_{N_{kl}}(r_0^{(k, l)}(t), t) \leq W_{N_{kl}}(r^{(k, l)}(t), t) + \frac{1}{4} M_{kl}(t).
\]

Since \((t, r^{(k, l)}(t)) \in F\), from inequality (13) we obtain

\[
\frac{1}{2} M_{kl}(t) \leq W_{N_{kl}}(r^{(k, l)}(t), t) \leq A S_{N_{kl}}(r^{(k, l)}(t)) \ln^{1/2} N_{kl}.
\]

Now for \(r^{(k, l)} = r^{(k, l)}(t)\) we get

\[
S_{N_{kl}}^2(r^{(k, l)}) \leq \mu_f(r^{(k, l)}) \mathcal{M}_f(r^{(k, l)}) \leq \frac{\mu_f^2(r^{(k, l)})}{(1 - r_1^{(k, l)})} \ln^{1+\delta} \frac{\mu_f(r^{(k, l)})}{1 - r_1^{(k, l)}} \ln^{1/2 + \delta} r_2^{(k, l)}.
\]

So, for \(t \in F_1\) and all \(k \geq k_0(t), l \geq l_0(t)\), we obtain

\[
S_N(r^{(k, l)}) \leq \mu_f(r^{(k, l)}) \left( \frac{1}{1 - r_1^{(k, l)}} \ln \frac{\mu_f(r^{(k, l)})}{1 - r_1^{(k, l)}} \sqrt{\ln r_2^{(k, l)}} \right)^{1/2 + \delta/2}.
\] (14)

It follows from (10)–(12) that \(d_1(r^{(k, l)}) \geq d(r)\) for \(r \in G_{kl}^*\). Then for \(t \in F_1, r \in F \wedge t \cap G_{kl}^*, (k, l) \in I, k \geq k_0(t), l \geq l_0(t)\) we get

\[
M_f(r, t) \leq \sum_{n+m \geq 2d_1(r^{(k, l)})} |a_{nm}| r_1^n r_2^m + W_{N_{kl}}(r, t) \leq \sum_{n+m \geq 2d(r)} |a_{nm}| r_1^n r_2^m + M_{kl}(t).
\]

Finally for \(t \in F_1, r \in F \wedge t \cap G_{kl}^*, l \geq l_0(t)\) and \(k \geq k_0(t)\) we obtain

\[
M_f(r^{(k, l)}, t) \leq \mu_f(r^{(k, l)}) + 2 A S_{N_{kl}}(r^{(k, l)}) \ln^{1/2} N_{kl} \leq \mu_f(r^{(k, l)}) +
\]
Proof. Theorem 3. 4. Sharpness of Theorem 2. Let $Z$ be a sequence of random variables such that $|Z_{nm}| \geq 1$ for almost all $t \in [0; 1]$. Then for there exist an analytic function $f \in \mathcal{A}_2$, a constant $C > 0$ and $r_0 \in T$, such that almost surely in $t$ for all $r \in \Delta_{r_0}$ we get

$$M_f(r, t) \geq \frac{C \mu_f(r)}{\sqrt{1 - r_1}} \cdot \ln^{1/2} \frac{\mu_f(r)}{1 - r_1}. \quad (16)$$

Proof. Consider the functions

$$g(z_1, z_2) = \sum_{n+m=0}^{+\infty} \frac{e^{\sqrt{m}}}{n!} z_1^n z_2^m, \quad f(z_1, z_2) = \sum_{n+m=0}^{+\infty} \frac{e^{\sqrt{m}/2}}{n!} z_1^n z_2^m,$$

$$f(z, t) = \sum_{n+m=0}^{+\infty} Z_{nm}(t) \frac{e^{\sqrt{m}/2}}{\sqrt{n!}} z_1^n z_2^m.$$
Remark, that for all \( r \in T \) we have

\[
\mu_g(r_1^2, r_2^2) = \max \left\{ \frac{e^{\sqrt{m}}}{n!} z_1^n z_2^m : (n, m) \in \mathbb{Z}_+^2 \right\} = \\
= \max \left\{ \left( \frac{e^{\sqrt{m}/2}}{\sqrt{n!}} z_1^n z_2^m \right)^2 : (n, m) \in \mathbb{Z}_+^2 \right\} = (\mu_f(r_1, r_2))^2.
\]

Using Parseval’s equality, we get for almost all \( t \)

\[
M_g(r_1^2, r_2^2) \leq \sum_{n+m=0}^{+\infty} |Z_{nm}(t)|^2 \frac{e^{\sqrt{m}}}{n!} z_1^n z_2^m = \\
= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} |f(r_1 e^{i\theta}, r_2 e^{i\varphi}, t)|^2 dr d\theta \leq (M_f(r_1, r_2, t))^2.
\]

There exist \( r_0 \in T \) ([5]) a set \( r_0 \in \Delta r_0 \)

\[
M_g(r_1, r_2) > \frac{C \mu_g(r_1, r_2)}{1 - r_1} \ln \frac{\mu_g(r_1, r_2)}{1 - r_1},
\]

\[
(M_f(r_1, r_2, t))^2 \geq M_g(r_1, r_2) \geq \frac{C \mu_g(r_1^2, r_2^2)}{1 - r_1^2} \ln \frac{\mu_g(r_1^2, r_2^2)}{1 - r_1^2} = \\
= \frac{C \mu_f^2(r_1, r_2)}{(1 - r_1)(1 + r_1)} \ln \frac{\mu_f^2(r_1, r_2)}{(1 - r_1)(1 + r_1)} \geq \frac{C \mu_f^2(r_1, r_2)}{2} \ln \frac{\mu_f^2(r_1, r_2)}{1 - r_1}
\]

and

\[
M_f(r, t) \geq \left( \frac{C \mu_f^2(r_1, r_2)}{2} \ln \frac{\mu_f^2(r_1, r_2)}{1 - r_1} \right)^{1/2} \geq \sqrt{\frac{C \mu_f(r_1, r_2)}{2}} \ln^{1/2} \frac{\mu_f(r_1, r_2)}{1 - r_1}.
\]

\[\square\]

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