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ON UNIFORMLY DISCRETE SUBSETS IN UNIFORM SPACES AND TOPOLOGICAL GROUPS

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We study uniform spaces making use of their uniformly discrete subsets. For a uniform space $(X,\mathcal{U})$, we define the uniformly discrete number $\mathrm{ud}(X)$ of X as the supremum of cardinalities of its uniformly discrete subsets. It is shown that $\mathrm{ud}(X)$ coincides with the index of narrowness $\mathrm{ib}(X)$ of X. Using uniformly discrete subsets of uniform spaces we characterize regular cardinals. For a complete metrizable biuniform space $(X,\mathcal{L},\mathcal{R})$ it is proved the equivalence of the following assertions: (a) the spaces of all bounded uniformly continuous real valued functions on $(X,\mathcal{L})$ and $(X,\mathcal{R})$ coincide; (b) $(X,\mathcal{L})$ and $(X,\mathcal{R})$ have the same families of uniformly discrete subsets; (c) $\mathcal{L}=\mathcal{R}$. This result generalizes the result obtained by the third-named author for Polish groups. Applying the obtained results we extend the Hart-van Mill theorem ([9]) to all locally compact Abelian groups.

1. Introduction. Let $(X,\mathcal{U})$ be a uniform space (we consider only Hausdorff spaces). If $U\in\mathcal{U}$, a subset A of X is called *U-separated* if $(a,b)\notin U$ for every distinct points $a,b\in A$. A subset D of X is called *uniformly discrete* if there is an entourage $U\in\mathcal{U}$ such that D is U-separated. Denote by $\mathfrak{UD}(X,\mathcal{U})$ the family of all uniformly discrete subsets of a uniform space $(X,\mathcal{U})$.

The idea to employ uniformly discrete subsets of uniform spaces for the description of their topological properties is quite natural. For every subset A of a uniform space $(X,\mathcal{U})$ we associate the following cardinal number which characterizes the maximal size of uniformly discrete subsets of A.

Definition 1. Let A be a subset of a uniform space $(X,\mathcal{U})$. *The uniformly discrete number* $\mathrm{ud}(A)$ of A is defined by

$$\mathrm{ud}(A):=\sup\{|D|\colon D \text{ is a uniformly discrete subset of } A\}.$$

Evidently, $\mathrm{ud}(A)$ is finite if and only if A is finite, and in this case $\mathrm{ud}(A)=|A|$.

It turns out that the uniformly discrete number $\mathrm{ud}(X)$ of a uniform space $(X,\mathcal{U})$ coincides with the *index of narrowness* $\mathrm{ib}(X)$ of X. Let us recall that $\mathrm{ib}(X)$ is defined as the minimal cardinal τ such that X is τ-narrow, where $(X,\mathcal{U})$ is called *τ-narrow* if for every $U\in\mathcal{U}$ there exists a subset C of X with $|C|\le\tau$ such that the U-ball of C is the whole X. Note that τ-narrow uniform spaces X and the index of narrowness $\mathrm{ib}(X)$ of X were introduced and studied by I. Guran in [7, 8].

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Theorem 1. *The equality* $\mathrm{ib}(X) = \mathrm{ud}(X)$ *is valid for every uniform space* $(X,\mathcal{U})$.

This theorem shows that to define the index of narrowness $\mathrm{ib}(X)$ we can use only uniformly discrete subsets of the uniform space $(X,\mathcal{U})$, that better emphasizes the uniformity of X. From this point of view it is important to answer the following question: Could $\mathrm{ud}(X)$ be achieved by a uniformly discrete subsets of X? It is natural to consider this question for every subset A of X: Does A contain a uniformly discrete subset of cardinality $\mathrm{ud}(A)$?

Noting that every uniformly discrete subset of a uniform space is closed (see Lemma 2), we attack the last question mainly for closed discrete subsets. Our interest to such subsets is explained also by their importance for the theory of topological groups. For this reason, we select uniform spaces with the following property.

Definition 2. Let κ be a cardinal number. A uniform space $(X,\mathcal{U})$ is said to have the *κ-uniformly discrete property* if every closed discrete subset D of X with $|D| = \kappa$ (if it exists) includes a uniformly discrete subset D_0 of cardinality $|D_0| = |D|$. We say that $(X,\mathcal{U})$ has *the uniformly discrete property* if it has the κ-uniformly discrete property for every κ.

It is natural to ask: Which uniform spaces have the uniform discrete property? In the case of a uniformly locally compact uniform space $(X,\mathcal{U})$ we answer this question in the affirmative. Recall that a uniform space $(X,\mathcal{U})$ is *uniformly locally compact* if there exists an entourage $U \in \mathcal{U}$ such that each U-ball of every element x of X has compact closure.

Theorem 2. *Every uniformly locally compact uniform space has the uniformly discrete property.*

This theorem allows us to extend the Hart-van Mill theorem ([9]) to all locally compact Abelian groups (see Theorem 20 below).

Using uniform spaces with the uniformly discrete property we can characterize regular cardinals as follows (we denote by $\mathrm{uw}(X)$ and $\mathrm{w}(G)$ the weight of a uniform space $(X,\mathcal{U})$ and a topological group G respectively).

Theorem 3. *For a cardinal number κ, the following are equivalent:*

(i) *κ is regular.*

(ii) *Every uniform space $(X,\mathcal{U})$ of uniform weight $\mathrm{uw}(X) < \kappa$ has the κ-uniformly discrete property.*

(iii) *Every complete Abelian topological group G of character $\chi(G) < \kappa$ and $\mathrm{ud}(G) = \kappa$ has the κ-uniformly discrete property.*

Denote by $\tau_{\mathcal{U}}$ the topology on a uniform space $(X,\mathcal{U})$ generated by the uniformity $\mathcal{U}$. We define a natural generalization of topological groups as follows.

Definition 3. Let $\mathcal{L}$ and $\mathcal{R}$ be two uniformities on a set X. If $\tau_{\mathcal{L}} = \tau_{\mathcal{R}}$, the triple $(X,\mathcal{L},\mathcal{R})$ is called a *biuniform space*.

For a uniform space $(X,\mathcal{U})$, we denote by $BUC(X,\mathcal{U})$ the family of all bounded uniformly continuous real-valued functions on $(X,\mathcal{U})$. Given two uniformities $\mathcal{L}$ and $\mathcal{R}$ on a set X, we shall explore the interplay between the following conditions:

(uc) $BUC(X,\mathcal{L}) = BUC(X,\mathcal{R})$;

(ud) $\mathfrak{UD}(X,\mathcal{L}) = \mathfrak{UD}(X,\mathcal{R})$;

(top) $\tau_{\mathcal{L}} = \tau_{\mathcal{R}}$;
(eq) $\mathcal{L} = \mathcal{R}$.

Clearly, (**eq**) implies (**uc**) and (**ud**). In the following important partial case we prove the converse.

Theorem 4. *Let $(X, \mathcal{L}, \mathcal{R})$ be a complete metrizable biuniform space. Then*

$$(\mathbf{eq}) \Leftrightarrow (\mathbf{uc}) \Leftrightarrow (\mathbf{ud}).$$

The article is organized as follows. In Section 2 we characterize precompact subsets in uniform spaces. Theorem 1 is proved in Section 3. Uniformly discrete subsets in uniformly locally compact uniform spaces are studied in Section 4. Theorem 3 is proved in Section 5. In Section 6 we study the size of partitions of uniform spaces into uniformly discrete subsets. Uniformly discrete sets in biuniform spaces and topological groups are considered in Section 7 where we prove Theorem 4. In the last section we prove an extension of the Hart-van Mill theorem ([9]) to all locally compact Abelian groups.

2. A characterization of precompact subsets in uniform spaces. Let $(X,\mathcal{U})$ be a uniform space. For every $x \in X$ and each entourage U, the U-ball of x is the set $B(x,U) = U[x] := \{y \in X : (x,y) \in U\}$. If C is a subset of X and $U \in \mathcal{U}$, we put $B(C,U) = U[C] := \bigcup_{x\in C} B(x,U)$. A subset E of a uniform space $(X,\mathcal{U})$ is *precompact* or *totally bounded* if for every $U \in \mathcal{U}$ there exists a finite subset F of X such that $E \subseteq B(F,U)$. It is known ([5, 8.3.17]) that a uniform space $(X,\mathcal{U})$ is precompact if and only if its completion is compact.

In this section we characterize precompact subsets in uniform spaces (see Theorem 5). To do this we need some propositions which are used also in the sequel. We omit the proof of the following four simple lemmas. Lemma 4 can be proved by a standard application of Zorn's Lemma.

Lemma 1. *Let A be a discrete closed subset of a uniform space $(X,\mathcal{U})$. Then*

(1) *Every subset of A is also discrete and closed.*
(2) *For every compact subset K of X, $A \cap K$ is finite.*
(3) *For every σ-compact subset E of X, $A \cap E$ is at most countable.*

Lemma 2. *Every uniformly discrete subset A of a uniform space $(X,\mathcal{U})$ is closed.*

Recall that, if A is a subset of a uniform space $(X,\mathcal{U})$, then the uniform space $(A,\mathcal{U}|_A)$, where $\mathcal{U}|_A := \{(A \times A) \cap U : U \in \mathcal{U}\}$, is called a *subspace* of $(X,\mathcal{U})$. Note also that the topology induced on A by the uniformity $\mathcal{U}|_A$ coincides with the subspace topology on A, where X has the topology induced by $\mathcal{U}$.

Lemma 3. *If A is a precompact subset of a uniform space $(X,\mathcal{U})$, then the uniform space $(A,\mathcal{U}|_A)$ is precompact.*

Lemma 4. *Let $(X,\mathcal{U})$ be a uniform space. Then, for every $U \in \mathcal{U}$ there exists a maximal (under inclusion) U-separated subset C of X. Moreover, if U is symmetric, then $B(C,U) = X$.*

Proposition 1. *Let E be an infinite subset of a uniform space $(X,\mathcal{U})$. Then E is either precompact or contains an infinite uniformly discrete subset.*

Proof. Assume that E contains an infinite uniformly discrete subset A. Let us show that E is not precompact. Assuming the converse we obtain that $(E,\mathcal{U}|_E)$ is precompact by Lemma 3. Then $(A,\mathcal{U}|_A)$ is precompact as well by [5, 8.3.2]. If A is W-separated, then, for every finite subset F of A, we have $B(F,W_A) = A \cap B(F,W) = F$. Thus $(A,\mathcal{U}|_A)$ is not precompact because A is infinite. This contradiction shows that E is not precompact.

Let now E have no infinite uniformly discrete subsets. We claim that E is precompact. Indeed, let $U \in \mathcal{U}$. Take a symmetric $V \in \mathcal{U}$ such that $V \subseteq U$. By Lemma 4, choose a maximal V-separated subset F in E. By assumption F is finite. Let us show that $E \subseteq B(F,V) \subseteq B(F,U)$. Suppose for a contradiction that there is $g \in E \setminus B(F,V)$. Then, for every $f \in F$, we have $g \notin B(f,V)$, i.e., $(f,g) \notin V$. Since V is symmetric, we have also $(g,f) \notin V$. So the set $F' := \{g\} \cup F$ is V-separated. Since $F \neq F'$ we obtained a contradiction with the maximality of F to be V-separated. Thus $E \subseteq B(F,U)$, and E is precompact. □

For complete uniform spaces we can partially reverse Lemma 2:

Corollary 1. *Every infinite discrete closed subset E of a complete uniform space $(X,\mathcal{U})$ contains an infinite uniformly discrete subset.*

Proof. Choose an arbitrary countably infinite subset A of E. Then, by Lemma 1, A is also discrete and closed. By Proposition 1, we have to show only that A is not precompact.

Suppose for a contradiction that A is precompact. Then $(A,\mathcal{U}|_A)$ is precompact by Lemma 3. Since A is closed and X is complete, $(A,\mathcal{U}|_A)$ is complete by [5, 8.3.6]. Thus $(A,\mathcal{U}|_A)$ is compact by [5, 8.3.16]. As A is infinite and compact, it contains an accumulation point. Hence A is not discrete. This contradiction shows that A is not precompact. □

The completeness of the uniform space in Corollary 1 is essential as the following example shows.

Example 1. Let $G = \mathbb{Q}$ be the group of all rational numbers with the induced topology from $\mathbb{R}$, and let D be an arbitrary sequence converging to an irrational number α. Then D is discrete and closed in G. Clearly, D has no infinite uniformly discrete subsets.

In the following theorem we characterize precompact subsets of X making use of uniformly discrete subsets.

Theorem 5. *A subset E of a uniform space $(X,\mathcal{U})$ is precompact if and only if every uniformly discrete subset of E is finite.*

Proof. If E is finite, the assertion is trivial. So we will assume that E is infinite.

Assume that E is precompact. Let D be a U-separated subset of E, where $U \in \mathcal{U}$. Choose a symmetric $V \in \mathcal{U}$ satisfying $V \circ V \subseteq U$. Since E is precompact we can find a finite subset F of X such that $E \subseteq B(F,V)$. For every $d \in D$ choose an element $f_d \in F$ such that $d \in B(f_d,V)$. We claim that the map $i(d) := f_d$ from D into F is injective. Indeed, if there are distinct $d,\widetilde{d} \in D$ such that $i(d) = i(\widetilde{d}) = f_d$, then $(d,\widetilde{d}) \in V \circ V \subseteq U$. This contradicts the choise of U. Since the map i is injective and F is finite, D is finite as well.

Conversely, if every uniformly discrete subset of E is finite, then E is precompact by Proposition 1. □

Theorem 5 immediately implies:

Corollary 2. *A uniform space is precompact if and only if every its uniformly discrete subset is finite.*

3. The uniformly discrete number and another cardinals of uniform spaces. We start this section with the proof of Theorem 1.

Proof of Theorem 1. Let us show that $\mathrm{ib}(X) \le \mathrm{ud}(X)$. We have to prove that, for every $U \in \mathcal{U}$, there is a uniformly discrete subset A of X with $|A| \le \mathrm{ud}(X)$ such that $B(A,U) = X$.

Choose a symmetric $V \in \mathcal{U}$ such that $V \circ V \subseteq U$ and let A be a maximal V-separated subset of X (see Lemma 4). Then $|A| \le \mathrm{ud}(X)$ and $B(A,V) = X$ by Lemma 4. Thus $B(A,U) = X$ as well.

Let us prove that $\mathrm{ib}(X) \ge \mathrm{ud}(X)$. Suppose for a contradiction that there is a U-separated subset E of X such that $|E| > \mathrm{ib}(X)$. Choose a symmetric $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. By the definition of $\mathrm{ib}(X)$, there is a set T of cardinality $|T| \le \mathrm{ib}(X)$ such that $B(T,V) = X$. For every $a \in E$, choose arbitrarily $t_a \in T$ such that $(t_a, a) \in V$. So we obtained a map $i\colon E \to T, i(a) := t_a$.

Let us show that i is injective. Indeed, if there are distinct $a, b \in E$ such that $i(a) = i(b) = t_a$, then $(a,b) \in V \circ V \subseteq U$. This contradicts the choice of E. Thus i is injective, and hence $|T| \ge |E| > \mathrm{ib}(X)$, a contradiction. □

Theorem 1 immediately implies the following characterization of τ-narrow uniform spaces.

Corollary 3. *A uniform space $(X,\mathcal{U})$ is τ-narrow if and only if every uniformly discrete subset of X has size $\le \tau$.*

We note that Corollary 3 is actually proved in [17, Assertion 1.2].

Denote by $F(Y)$ and $F(X,\mathcal{U})$ the free topological groups over a Tychonoff space Y and a uniform space $(X,\mathcal{U})$, respectively. It follows from [7] and [14] that $(X,\mathcal{U})$ is τ-narrow if and only if $F(X,\mathcal{U})$ is τ-narrow. This result and Corollary 3 imply

Corollary 4. *Let Y be a Tychonoff space and $(X,\mathcal{U})$ be a uniform space. Then* $\mathrm{ud}(Y) = \mathrm{ud}\big(F(Y)\big)$ *and* $\mathrm{ud}(X,\mathcal{U}) = \mathrm{ud}\big(F(X,\mathcal{U})\big)$.

Let E be a subset of a topological space X. A family $\mathcal{K}$ of subsets of E is called *compact-covering* if $E = \bigcup_{K\in\mathcal{K}} K$ and every $K \in \mathcal{K}$ is a non-empty compact subset of E. Recall that the *compact-covering* number $\mathrm{cc}(E)$ of E is defined as follows

$$\mathrm{cc}(E) := \min\{|\mathcal{K}| \colon \mathcal{K} \text{ is a compact-covering family of } E\}.$$

It is clear that $\mathrm{cc}(E) \le |E|$. Also, if A and B are closed subsets of X and $A \subseteq B$, then $\mathrm{cc}(A) \le \mathrm{cc}(B)$. Clearly, $\mathrm{cc}(E)$ is finite iff $\mathrm{cc}(E) = 1$ iff E is compact.

Proposition 2. *If E is a closed non-compact subset of a uniform space $(X,\mathcal{U})$, then* $\mathrm{ud}(E) \le \mathrm{cc}(E)$.

Proof. Let D be a uniformly discrete subset of E and $\mathcal{K}$ be a compact-covering family of E. Since E is not compact, $\mathcal{K}$ is infinite. As E is closed, D is closed in X (see Lemma 2). Clearly, for every $K \in \mathcal{K}$, K is compact in X and $D \cap K$ is finite by Lemma 1. Hence $|D| \le |\mathcal{K}|$. Thus $\mathrm{ud}(E) \le \mathrm{cc}(E)$. □

Lemma 5. *If D is a discrete closed infinite subset of a uniform space $(X,\mathcal{U})$, then* $\mathrm{cc}(D) = |D|$.

Proof. Let $\mathcal{K}$ be a compact-covering family of D. Since $\mathcal{K}$ contains compact subsets of X, Lemma 1 implies that $D \cap K$ is finite for every $K \in \mathcal{K}$. As D is infinite, we obtain $|D| \leq |\mathcal{K}|$. Hence $|D| \leq \mathrm{cc}(D)$. The converse inequality is trivial. Thus $|D| = \mathrm{cc}(D)$. □

Recall that the *extent* $\mathrm{e}(X)$ of a topological space X is the supremum of cardinalities of closed discrete subsets of X.

Proposition 3. *Let $(X,\mathcal{U})$ be an infinite uniform space.*

(i) *If X is compact, then* $1 = \mathrm{cc}(X) < \mathrm{ud}(X) = \mathrm{e}(X) = \aleph_0$.

(ii) *If X is not compact, then* $\mathrm{ud}(X) \leq \mathrm{e}(X) \leq \mathrm{cc}(X)$.

Proof. (i) follows from Lemma 1.

(ii) By the definitions of $\mathrm{ud}(X)$ and $\mathrm{e}(X)$ and Lemma 2, we have $\mathrm{ud}(X) \leq \mathrm{e}(X)$. Let us prove that $\mathrm{e}(X) \leq \mathrm{cc}(X)$.

Suppose for a contradiction that $\mathrm{e}(X) > \mathrm{cc}(X)$. This means that there is a discrete closed subset D of X such that $|D| > \mathrm{cc}(X)$. Hence there exists a compact-covering family $\mathcal{K} = \{K_i\}_{i\in I}$ of X such that $|I| < |D|$. By Lemma 1, $D \cap K_i$ is finite for every $i \in I$. As X is not compact, I is infinite. Hence $|I| = |\cup_{i\in I} (D \cap K_i)| = |D|$, a contradiction. Thus $\mathrm{e}(X) \leq \mathrm{cc}(X)$. □

As usual, $\chi(G)$ denotes the *character* of a topological group G. In [8] (see Proposition Proposition 5.2.3 of [1]) Guran proved that $\mathrm{w}(G) = \mathrm{ib}(G)\cdot\chi(G)$ for every topological group G. This result and Theorem 1 imply:

Corollary 5. *For every topological group G,* $\mathrm{w}(G) = \mathrm{ud}(G) \cdot \chi(G)$.

Now we introduce a strong version of the cardinals $\mathrm{ud}(X)$, $\mathrm{ib}(X)$ and $\mathrm{e}(X)$ (they are "strong" because in their definitions we use the inequality "<"). Denote by κ^+ the successor of a cardinal number κ.

Definition 4. Let $(X,\mathcal{U})$ be a uniform space.

- the *total uniformly discrete number* of X is the cardinal
$$\overset{\sharp}{\mathrm{ud}}(X) := \sup\{|D|^+ \colon D \subset X \text{ is uniformly discrete}\};$$
- the *total boundedness number* $\mathrm{ib}^\sharp(X)$ of X is the minimal cardinal number κ such that for each entourage $U \in \mathcal{U}$ there is a subset $C \subset X$ of cardinality $|C| < \kappa$ such that $X = B(C, U)$;
- the *total extent* of X is the cardinal
$$\overset{\sharp}{\mathrm{e}}(X) := \sup\{|D|^+ \colon D \text{ is closed and discrete subset of } X\}.$$

In the following proposition we describe relations between $\mathrm{ud}(X)$, $\mathrm{ib}(X)$ and $\mathrm{e}(X)$ and their total versions.

Proposition 4. *Let $(X,\mathcal{U})$ be a uniform space.*

(1) *If X has a uniformly discrete subset of cardinality* $\mathrm{ud}(X)$, *then* $\mathrm{ud}^\sharp(X) = \mathrm{ud}^+(X)$. *Otherwise,* $\mathrm{ud}^\sharp(X) = \mathrm{ud}(X)$.

(2) *If for every* $U \in \mathcal{U}$ *there exists a subset* C *of cardinality* $|C| < \mathrm{ib}(X)$ *such that that* $B(C, U) = X$*, then* $\mathrm{ib}(X) = \mathrm{ib}^\sharp(X)$*. Otherwise,* $\mathrm{ib}^+(X) = \mathrm{ib}^\sharp(X)$.

(3) *If* X *has a closed discrete subset of cardinality* $\mathrm{e}(X)$*, then* $\mathrm{e}^+(X) = \mathrm{e}^\sharp(X)$*. Otherwise,* $\mathrm{e}(X) = \mathrm{e}^\sharp(X)$.

(4) $\mathrm{ud}^\sharp(X) = id^\sharp(X)$.

(5) X *is precompact if and only if* $\mathrm{ud}^\sharp(X) \le \aleph_0$.

Proof. (1) Let D be a uniformly discrete subset of X of cardinality $\mathrm{ud}(X)$. Then $\mathrm{ud}(X) = |D| < |D|^+ \le \mathrm{ud}^\sharp(X)$. Hence $\mathrm{ud}^+(X) \le \mathrm{ud}^\sharp(X)$. Clearly, $\mathrm{ud}^\sharp(X) = |D|^+ \le \mathrm{ud}^+(X)$. Thus $\mathrm{ud}^+(X) = \mathrm{ud}^\sharp(X)$.

Assume now that X does not have a uniformly discrete subset of cardinality $\mathrm{ud}(X)$. Clearly, $\mathrm{ud}(X) \le \mathrm{ud}^\sharp(X)$. On the other hand, for every uniformly discrete subset D of X, we have $|D|^+ \le \mathrm{ud}(X)$. This gives $\mathrm{ud}(X) \ge \mathrm{ud}^\sharp(X)$.

(2) Clearly, $\mathrm{ib}(X) \le \mathrm{ib}^\sharp(X) \le \mathrm{ib}^+(X)$. By the definition of $\mathrm{ib}^\sharp(X)$, the assumption implies $\mathrm{ib}^\sharp(X) \le \mathrm{ib}(X)$. Thus $\mathrm{ib}(X) = \mathrm{ib}^\sharp(X)$.

Assume that there exists $U \in \mathcal{U}$ such that for every C of cardinality $|C| < \mathrm{ib}(X)$ we have $B(C, X) \neq X$. This means that $\mathrm{ib}(X) < \mathrm{ib}^\sharp(X)$. Thus $\mathrm{ib}^\sharp(X) = \mathrm{ib}^+(X)$.

(3) follows from definitions as in item (1).

(4) We distinguish between two cases.

Case 1. X has a U-separated subset D of cardinality $\mathrm{ud}(X)$ for some $U \in \mathcal{U}$. Then, by item (1) and Theorem 1, we have

$$\overset{\sharp}{\mathrm{ud}}(X) = \overset{+}{\mathrm{ud}}(X) = \overset{+}{\mathrm{ib}}(X) \ge \overset{\sharp}{\mathrm{ib}}(X).$$

Let us prove the converse inequality. Take a symmetric $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. We claim that, for every $C \subset X$ of cardinality $|C| < \mathrm{ib}(X)$, we have $B(C, V) \neq X$. Indeed, otherwise, for every $d \in D$, we can find $c_d \in C$ such that $d \in B(c_d, V)$. Moreover, the mapping $i\colon D \to C, i(d) := c_d$, is injective (otherwise, if $i(d) = i(d') = c_d$, then $d' \in B(d, V \circ V) \subset B(d, U)$ that contradicts the choice of D). Thus $|D| = \mathrm{ud}(X) \le |C| < \mathrm{ib}(X)$, a contradiction. By item (2), we have $\mathrm{ib}^+(X) = \mathrm{ib}^\sharp(X)$. Thus $\mathrm{ud}^\sharp(X) = id^\sharp(X)$.

Case 2. X does not have a U-separated subset D of cardinality $\mathrm{ud}(X)$ for every $U \in \mathcal{U}$. By item (1) and Theorem 1, we have

$$\overset{\sharp}{\mathrm{ud}}(X) = \mathrm{ud}(X) = \mathrm{ib}(X).$$

We will show that $\mathrm{ib}^\sharp(X) = \mathrm{ib}(X)$ making use of item (2). Let V be an arbitrary symmetric entourage. Lemma 4 implies that there is a maximal V-separated subset C of X for which $B(C, V) = X$. By the assumption, $|C| < \mathrm{ud}(X) = \mathrm{ib}(X)$. Now item (2) yields $\mathrm{ib}^\sharp(X) = \mathrm{ib}(X)$. Thus $\mathrm{ud}^\sharp(X) = id^\sharp(X)$.

(5) follows from the definition of $\mathrm{ud}^\sharp(X)$ and Corollary 2. □

Note that item (1) of this proposition gives also an answer to the problem posed in Introduction: X has a uniformly discrete subset of cardinality $\mathrm{ud}(X)$ if and only if $\mathrm{ud}^\sharp(X) = \mathrm{ud}^+(X)$.

4. Uniformly discrete subsets in uniformly locally compact uniform spaces. To prove Theorem 2 we need some propositions.

Lemma 6. *Let $(X,\mathcal{U})$ be a uniformly locally compact uniform space and $U \in \mathcal{U}$ be such that $B(x,U)$ has compact closure for every $x \in X$. For every compact subset K of X and every symmetric $V \in \mathcal{U}$ such that $V \circ V \subset U$, the set $B(K,V)$ has compact closure.*

Proof. For every $x \in K$ choose $V_x \in \mathcal{U}$ such that $V_x \subset V$. Since K is compact, there are $x_1,\dots,x_n \in K$ such that $K \subset \bigcup_{i=1}^n B(x_i,V_{x_i})$. Then

$$B(K,V) \subset B\Big(\bigcup_{i=1}^n B(x_i,V_{x_i}),V\Big) = \bigcup_{i=1}^n B(x_i,V_{x_i}\circ V) \subset \bigcup_{i=1}^n B(x_i,U).$$

Thus $\mathrm{cl}_X B(K,V) \subset \bigcup_{i=1}^n \mathrm{cl}_X B(x_i,U)$ is compact. □

Proposition 5. *Let $(X,\mathcal{U})$ be a uniformly locally compact uniform space. Then every closed non-compact subset A of X has a uniformly discrete (and hence, closed) subset D of cardinality $\mathrm{cc}(A)$. In particular, $\mathrm{ud}(A)=\mathrm{cc}(A)$ and $\mathrm{ud}(X)=\mathrm{cc}(X)$.*

Proof. Let $\mathcal{K}=\{K_i\}_{i\in I}$ be an arbitrary compact-covering family of A, where K_i is a compact subset of A (and hence it is compact in X). By Zorn's lemma, we may assume that I is a well-ordered set. Choose a symmetric $U \in \mathcal{U}$ such that $B(x,U)$ has compact closure for every $x \in X$. And take a symmetric $V \in \mathcal{U}$ such that $V\circ V\circ V\circ V \subset U$.

Let us define by induction an increasing chain $\{J_\alpha\}_{\alpha\in I}$ of subsets of I as follows. Set $J_0=\{0\}$. Assume that for a non-zero $\alpha \in I$ we defined J_i for every $i<\alpha$. Set

$$K'_\alpha := K_\alpha \setminus \Big(\bigcup_{i<\alpha}\bigcup_{\psi\in J_i} B(K_\psi, V\circ V)\Big).$$

If $K'_\alpha=\varnothing$, we set $J_\alpha=\bigcup_{i<\alpha} J_i$, and, if $K'_\alpha\neq\varnothing$, we set $J_\alpha=\{\alpha\}\cup\bigcup_{i<\alpha}J_i$.

Put $J=\bigcup_{\alpha\in I}J_\alpha$ and note that, by construction,

$$K'_\alpha = K_\alpha \setminus \Big(\bigcup_{i<\alpha}\bigcup_{\psi\in J_i} B(K_\psi, V\circ V)\Big) \neq \varnothing, \text{ for every } \alpha\in J. \tag{1}$$

We claim that the family $\mathcal{A}:=\{A\cap \mathrm{cl}B(K_j,V\circ V)\}_{j\in J}$ is a compact-covering family of A. Indeed, by Lemma 6, the set $A\cap\mathrm{cl}B(K_j,V\circ V)$ is compact for every $j\in J$.

Let us check that $\mathcal{A}$ covers A. Suppose for a contradiction that there is $a\in A$ such that $a\notin\bigcup_{j\in J}\mathrm{cl}B(K_j,V\circ V)$. Since $\mathcal{K}$ covers A, the family of all indices $i\in I$ such that $a\in K_i$ is not empty. As I is well-ordered, there exists the minimal index $\alpha\in I$ such that $a\in K_\alpha$. So

$$a\in K_\alpha\setminus\bigcup_{j\in J}\mathrm{cl}B(K_j,V\circ V)\subseteq K_\alpha\setminus\bigcup_{i<\alpha}\bigcup_{\psi\in J_i}B(K_\psi,V\circ V).$$

Hence $\alpha\in J_\alpha\subseteq J$ and $a\in\mathrm{cl}B(K_j,V\circ V)$, a contradiction.

By the definition of $\mathrm{cc}(A)$, we have $|J|\geq\mathrm{cc}(A)$.

For every $j\in J$, choose an arbitrary element $a_j\in K'_j\subseteq A$ and set $D=\{a_j\}_{j\in J}$. Let us show that D is V-separated. Indeed, assuming that $(a_i,a_j)\in V$ for some $i,j\in J$ with $i<j$, we obtain $a_j\in K'_j\cap B(a_i,V\circ V)$ that contradicts (1). Hence $|D|=|J|\geq\mathrm{cc}(A)$. Since D is closed by Lemma 2, we obtain that $|D|=\mathrm{cc}(D)$ by Lemma 5, and hence $|D|\leq\mathrm{cc}(A)$. Thus $|D|=\mathrm{cc}(A)$.

In particular, we proved that $\mathrm{ud}(A)\geq\mathrm{cc}(A)$. Now the last assertion of the proposition follows from Proposition 2. □

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. Let D be a closed discrete subset of a uniformly locally compact uniform space X. If X is compact, then D must be finite as it is discrete. Then $D_0 := D$ is uniformly discrete as desired.

Assume that X is not compact. If D is finite then $D_0 := D$ is as desired. In the case where D is infinite, the assertion immediately follows from Proposition 5 and Lemma 5. □

Since every uniformly locally compact uniform space is complete, one can ask whether Proposition 5 remains true for uniformly locally *precompact* uniform spaces. Example 1 shows that, in general, the answer to this question is negative.

Corollary 6. *If* $(X,\mathcal{U})$ *is a uniformly locally compact non-compact uniform space, then* $\mathrm{ud}(X) = \mathrm{e}(X) = \mathrm{cc}(X)$.

Proof. By Proposition 3 we have $\mathrm{ud}(X) \le \mathrm{e}(X) \le \mathrm{cc}(X)$. The equality $\mathrm{ud}(X) = \mathrm{cc}(X)$ follows from Proposition 5. □

5. Uniformly discrete subsets of discrete sets. For the sake of completeness we prove the next standard lemma. Denote by $d(X)$ the density of a topological space X.

Lemma 7. *Let* p *be a uniformly continuous map from a* τ*-narrow uniform space* $(X,\mathcal{U})$ *onto a metrizable uniform space* $(M,\mathcal{V})$*. Then* $d(M) \le \tau$.

Proof. Let ρ be a metric on M generated by the uniformity $\mathcal{V}$. For every $n \in \mathbb{N}$, choose an entourage $U_n \in \mathcal{U}$ such that

$$\rho(p(x), p(y)) < \frac{1}{2^n}, \quad \forall (x,y) \in U_n. \tag{2}$$

For every $n \in \mathbb{N}$, take a subset $C_n \subset X$ of cardinality $\le \tau$ such that $X = B(C_n, U_n)$. Set $C := \cup_{n\in\mathbb{N}} C_n$ and $S := p(C)$. Then $|S| \le \tau$. To prove the lemma it is enough to show that S is a dense subset of M.

Indeed, let $t \in M$ and $n \in \mathbb{N}$. Take $y \in X$ such that $p(y) = t$. Choose $x \in C_n$ for which $(x,y) \in U_n$, and set $s := p(x)$. Then $s \in S$ and, by (2), $\rho(s,t) = \rho(p(x), p(y)) < \frac{1}{2^n}$. Thus S is dense in M. □

The following theorem first proved in [8] is an immediate corollary of Lemma 7 and the proofs of Theorems 8.2.3 and 8.3.8 of [5].

Theorem 6. *Every (complete)* τ*-narrow uniform space is uniformly homeomorphic to a (closed) subspace of the product of a family of (complete) metrizable uniform spaces of density* $\le \tau$.

By Lemma 2, the class of closed discrete subsets of a uniform space $(X,\mathcal{U})$ contains the class of uniformly discrete subsets of X. Corollary 3 shows that every ω-narrow uniform space contains only countable uniformly discrete subsets. So it is natural to ask: Does $\mathrm{ud}(X)(= \mathrm{ib}(X))$ restrict possible values of cardinalities of closed discrete subspaces? In the following theorem we answer this question in the negative. Recall that a cardinal κ is called *non-measurable* if there is no atomless 2-valued σ-additive measure defined on the σ-algebra of all subsets of κ. Note that the first measurable cardinal (if it exists) is strongly inaccessible.

Theorem 7. (i) *Each closed discrete subspace of an ω-narrow complete uniform space $(X,\mathcal{U})$ has non-measurable cardinality.*

(ii) *For every non-measurable cardinal κ there is an ω-narrow complete Abelian topological group G containing a closed discrete subset D of cardinality $|D|=\kappa$.*

Proof. (i) Assume that D is a closed discrete subspace of an ω-bounded complete uniform space X. Being ω-narrow and complete, by Theorem 6, the uniform space X is uniformly homeomorphic to a closed subspace of the direct product $\widetilde{X} := \prod_{\alpha\in A} X_\alpha$ of a family $\{X_\alpha\}_{\alpha\in A}$ of separable complete metric spaces. Since $\widetilde{X}$ is realcompact ([6, 8.2 and 8.11]), the closed discrete subspace D is also realcompact by [6, 8.10]. Thus D has non-measurable cardinality ([6, 12.2]).

(ii) If a cardinal κ is non-measurable, then the space κ endowed with the discrete topology is realcompact by [6, 12.2]. Hence κ is homeomorphic to a closed discrete subspace of the product $\mathbb{R}^\lambda$ for some cardinal λ ([5, 3.11.3]). It remains to remark that the power $\mathbb{R}^\lambda$ is an ω-narrow complete Abelian topological group. □

To prove Theorem 3 we need the following two theorems.

Theorem 8. *Let κ be an uncountable cardinal and $(X,\mathcal{U})$ be uniform space of weight* $\mathrm{uw}(X) < \mathrm{cf}(\kappa)$. *Then X has the κ-uniformly discrete property.*

Proof. Fix arbitrarily a closed discrete subset D of cardinality κ (if such subsets do not exist the theorem holds vacuously). Let $\mathcal{B}\subset\mathcal{U}$ be a base of the uniformity $\mathcal{U}$ having cardinality $|\mathcal{B}|<\mathrm{cf}(\kappa)$ and containing symmetric entourages. For each entourage $U\in\mathcal{B}$ choose a maximal U-separated subset $D_U\subseteq D$ (see Lemma 4).

We claim that $D=\bigcup_{U\in\mathcal{B}} D_U$. Indeed, assuming the converse we can find a point $x\in D\setminus(\bigcup_{U\in\mathcal{B}} D_U)$. Since D is discrete, there is an entourage $U\in\mathcal{B}$ such that $B(x,U)\cap D=\{x\}$. In particular, $B(x,U)\cap D_U=\varnothing$. Hence the set $D_U\cup\{x\}$ is U-separated, that contradicts the maximality of the set D_U to be U-separated. So, $D=\bigcup_{U\in\mathcal{B}} D_U$. Since $|\mathcal{B}|<\mathrm{cf}(\kappa)=\mathrm{cf}(|D|)$, we conclude that $|D_U|=\kappa$ for some $U\in\mathcal{B}$. Then $A=D_U$ is a required U-separated subset in D of cardinality κ. □

Theorem 9. *Let κ be a non-regular cardinal. Then there is a complete Abelian topological group G of weight* $\mathrm{w}(G)=\mathrm{cf}(\kappa)<\kappa$ *which contains a closed discrete subset D such that*

(i) $|D|=\kappa$,

(ii) $\mathrm{ud}(D)=\mathrm{ud}(G)=\kappa$,

(iii) *D does not have uniformly discrete subsets of cardinality κ.*

Proof. The cardinal κ, being non regular, can be represented as $\kappa=\sup\{\kappa_\alpha\}_{\alpha<\mathrm{cf}(\kappa)}$ of a transfinite sequence of cardinals $\kappa_\alpha<\kappa$ of length $\mathrm{cf}(\kappa)<\kappa$.

Let H be a discrete Abelian topological group of cardinality $|H|=\mathrm{cf}(\kappa)$ and $H=\{h_\alpha\colon \alpha<\mathrm{cf}(\kappa)\}$ be a one-to-one enumeration of H. Next, for every $\alpha<\mathrm{cf}(\kappa)$, fix an Abelian discrete topological group G_α of cardinality $|G_\alpha|=\kappa_\alpha$.

Set $G:=H\times\prod_{\alpha<\mathrm{cf}(\kappa)} G_\alpha$. Clearly, G is a complete Abelian topological group of weight $\mathrm{w}(G)=\mathrm{cf}(\kappa)$. Set

$$D=\bigcup_{\alpha<\mathrm{cf}(\kappa)}(h_\alpha+G_\alpha).$$

Let us show that D is as desired.

Step 1. D is *discrete* in G. Indeed, for every $\beta < \mathrm{cf}(\kappa)$ and $g_\beta \in G_\beta$, the set $U = \{h_\beta\} \times \{g_\beta\} \times \prod_{\alpha\neq\beta} G_\alpha$ is an open neighborhood of $h_\beta + g_\beta$ and $U \cap D = \{h_\beta + g_\beta\}$.

Step 2. D is *closed* in G. Indeed, let $g = (h, (g_\alpha)) \notin D$. We have only two cases.

(a) There are $\alpha_1 < \alpha_2$ such that $g_{\alpha_1} \neq 0$ and $g_{\alpha_2} \neq 0$. Then the open neighborhood $\{h\} \times \{g_{\alpha_1}\} \times \{g_{\alpha_2}\} \times \prod_{\alpha\notin\{\alpha_1,\alpha_2\}} G_\alpha$ of g does not intersect with D.

(b) There is a unique β such that $g_\beta \neq 0$. Then $h \neq h_\beta$. In this case the open neighborhood $\{h\} \times \{g_\beta\} \times \prod_{\alpha\neq\beta} G_\alpha$ of g does not intersect with D.

Thus D is closed in G.

Step 3. Let us show that, *for every uniformly discrete subset A of G there are indices $\alpha_1, \dots, \alpha_n$ such that*

$$|A| \le \mathrm{cf}(\kappa) + |G_{\alpha_1}| + \dots + |G_{\alpha_n}| < \kappa. \tag{3}$$

Indeed, let A be V-separated. Then there are indices $\alpha_1, \dots, \alpha_n$ such that the following open neighborhood of zero in G

$$U = \{0_H\} \times \{0_{\alpha_1}\} \times \dots \times \{0_{\alpha_n}\} \times \prod_{\alpha\notin\{\alpha_1,\dots,\alpha_n\}} G_\alpha$$

is contained in V. Thus A is also U-separated. Since U is a clopen subgroup of G, let $q\colon G \to G/U$ be the quotient map. Note that q is injective on A because A is U-separated. Thus $|A| = |q(A)| \le |H| + |G_{\alpha_1}| + \dots + |G_{\alpha_n}| < \kappa$.

Step 4. Clearly, $|D| = \sup\{|G_\alpha|\}_{\alpha<\mathrm{cf}(\kappa)} = \kappa$ and (i) is proved.

Step 5. In order to prove the equality $\mathrm{ud}(D) = \kappa$, note that every subset $h_\alpha + G_\alpha$ is a uniformly discrete subset of D of cardinality κ_α. Hence $\kappa = \sup\{\kappa_\alpha\}_{\alpha<\mathrm{cf}(\kappa)} \le \mathrm{ud}(D)$. Thus $\kappa \le \mathrm{ud}(D) \le |D| = \kappa$, and $\mathrm{ud}(D) = \kappa$. Now (3) implies that $\kappa = \mathrm{ud}(D) \le \mathrm{ud}(G) \le \kappa$, that proves (ii).

Step 6. (iii) immediately follows from (3). □

Let us remark that $\mathrm{ud}(D) = |D| = \kappa$ for the set D in Theorem 9, and the cardinal κ is not achieved by any uniformly discrete subset of D, as in Theorem 8 (see also Corollary 8).

Proof of Theorem 3. (i)⇒(ii) Since κ is regular we have $\mathrm{cf}(\kappa) = \kappa$. Now, if κ is uncountable, (i) implies (ii) by Theorem 8. If $\kappa = \aleph_0$, then every uniform space $(X, \mathcal{U})$ of weight $< \kappa$ is finite and (ii) holds vacuously.

(iii) follows from (ii) trivially, and (iii) implies (i) by Theorem 9. □

6. Partitions of uniform spaces into uniformly discrete subsets. Recall that a family $\mathcal{P} = \{P_i\}_{i\in I}$ is called a *partition* of a set X if $\bigcup_{i\in I} P_i = X$, each A_i is nonvoid, and the sets A_i are pairwise disjoint. A family $\mathcal{P} = \{P_i\}_{i\in I}$ is a *cover* of X if $\bigcup_{i\in I} P_i = X$.

With a uniform space $(X, \mathcal{U})$ we associate the next cardinal (recall that $\mathfrak{UD}(X, \mathcal{U})$ is the family of all uniformly discrete subsets of $(X, \mathcal{U})$):

Definition 5. Let $(X, \mathcal{U})$ be a uniform space. Define

$$\mathrm{pu}(X, \mathcal{U}) := \min\{|\mathcal{P}| : \ \mathcal{P} \subset \mathfrak{UD}(X, \mathcal{U}) \text{ and } \mathcal{P} \text{ is a partition of } X\}.$$

To evaluate $\mathrm{pu}(X,\mathcal{U})$ we use the following cardinal invariants. For $x \in X$, we put

$$\psi(x,\mathcal{U}) = \min\left\{|\mathcal{V}|\colon \mathcal{V} \subseteq \mathcal{U}, \bigcap_{V\in\mathcal{V}} V[x] = \{x\}\right\}, \quad \psi(X,\mathcal{U}) = \sup\{\psi(x,\mathcal{U})\colon x \in X\},$$
$$\delta(x,\mathcal{U}) = \min\{|U[x]|\colon U \in \mathcal{U}\}, \quad \delta(X,\mathcal{U}) = \min\{\delta(x,\mathcal{U})\colon x \in X\}.$$

If $\tau_{\mathcal{U}}$ is a topology on X defined by $\mathcal{U}$ then $\psi(X,\mathcal{U})$ and $\delta(X,\mathcal{U})$ coincide with pseudocharacter and (global) dispersion character of $(X,\tau_{\mathcal{U}})$. Let us introduce a cardinal which describes the maximality of local uniform dispersion of $(X,\tau_{\mathcal{U}})$

$$\Delta(X,\mathcal{U}) := \min\{\sup\{|U[x]|\colon x \in X\}, U \in \mathcal{U}\}.$$

The following proposition express some properties of these cardinals.

Proposition 6. *For a uniform space $(X,\mathcal{U})$, the following statements hold:*

(i) $\mathrm{pu}(X,\mathcal{U}) = \min\{|\mathcal{P}|\colon\ \mathcal{P} \subset \mathfrak{UD}(X,\mathcal{U})$ *and* $\mathcal{P}$ *covers* $X\}$.
(ii) $\psi(X,\mathcal{U}) \leqslant \mathrm{pu}(X,\mathcal{U}) \le |X|$.
(iii) $\delta(X,\mathcal{U}) \leqslant \Delta(X,\mathcal{U})$. *If in addition X is a topological group and $\mathcal{U} = \mathcal{L}_X$, then* $\delta(X,\mathcal{U}) = \Delta(X,\mathcal{U})$.
(iv) $\mathrm{pu}(X,\mathcal{U}) \leqslant \Delta(X,\mathcal{U})$.
(v) $\Delta(X,\mathcal{U}) \cdot \mathrm{ud}(X,\mathcal{U}) = |X|$.

Proof. (i) Set $\mu(X,\mathcal{U}) := \min\{|\mathcal{P}|\colon\ \mathcal{P} \subset \mathfrak{UD}(X,\mathcal{U})$ and $\mathcal{P}$ covers $X\}$. Clearly, $\mu(X,\mathcal{U}) \leqslant \mathrm{pu}(X,\mathcal{U})$. Let us prove the converse inequality.

Let $\mathcal{P} = \{P_i\}_{i\in I} \subset \mathfrak{UD}(X,\mathcal{U})$ be a cover of X such that $\mu(X,\mathcal{U}) = |I|$. We can assume that the index-set I is well-ordered. Set $P'_0 := P_0$. For every $\alpha \in I$, set $P'_\alpha := P_\alpha \setminus (\cup_{i<\alpha} P_i)$. Denote by J the set of all indices i such that $P'_i \neq \varnothing$. Clearly, P'_j is uniformly discrete for every $j \in J$ and the family $\mathcal{P}' = \{P'_j\}_{j\in J}$ is a partition of X. Hence $\mathrm{pu}(X,\mathcal{U}) \leqslant |J| \leqslant |I| = \mu(X,\mathcal{U})$. Thus $\mu(X,\mathcal{U}) = \mathrm{pu}(X,\mathcal{U})$.

(ii) Let $\mathcal{P} = \{P_i\}_{i\in I}$ be a partition of X with uniformly discrete cells such that $\mu(X,\mathcal{U}) = |\mathcal{P}|$. Fix $x \in X$. Denote by J_x the set of all indices $i \in I$ such that $x \in P_i$. For every $i \in J_x$ choose $V_i \in \mathcal{U}$ with $V_i[x] \cap P_i = \{x\}$. Since each uniformly discrete subset is closed (Lemma 2), for every $i \in I \setminus J_x$, we can choose $V_i \in \mathcal{U}$ such that $V_i[x] \cap P_i = \varnothing$. Then, for the family $\mathcal{V}_{\mathcal{P}} := \{V_i\colon i \in I\}$, we obtain

$$\{x\} \in \bigcap_{i\in I} V_i[x] = \Big(\bigcap_{i\in I} V_i[x]\Big) \cap \Big(\bigcup_{i\in I} P_i\Big) \subseteq \Big[\bigcup_{i\in I\setminus J_x} (V_i[x] \cap P_i)\Big] \cup \Big[\bigcup_{i\in J_x} (V_i[x] \cap P_i)\Big] = \{x\}.$$

So, by the definition of $\psi(x,\mathcal{U})$, we have $\psi(x,\mathcal{U}) \le |\mathcal{V}_{\mathcal{P}}| = |\mathcal{P}|$. Hence $\psi(x,\mathcal{U}) \leqslant \mu(X,\mathcal{U})$ for every $x \in X$. Thus $\psi(X,\mathcal{U}) \leqslant \mathrm{pu}(X,\mathcal{U})$.

The inequality $\mathrm{pu}(X,\mathcal{U}) \le |X|$ and item (iii) are trivial.

(iv) We consider two cases.

Case 1. Let $\Delta(X,\mathcal{U})$ be *infinite*. Choose a symmetric $U \in \mathcal{U}$ such that

$$\Delta(X,\mathcal{U}) = \sup\{|U[x]|\colon x \in X\}.$$

For every $x \in X$, we have

$$\big|U \circ U[x]\big| = \Big|\bigcup_{y\in U[x]} U[y]\Big| \leqslant \Delta(X,\mathcal{U}) \cdot \Delta(X,\mathcal{U}) = \Delta(X,\mathcal{U}).$$

For every $n \in \mathbb{N}$, set $U^n := U \circ \overset{n}{\cdots} \circ U$. By induction we obtain

$$|U^n[x]| \leqslant \Delta(X,\mathcal{U}), \quad \forall n \in \mathbb{N}, \forall x \in X. \tag{4}$$

Now we define an equivalence $\sim$ on X by $x \sim y \Leftrightarrow \exists n \in \mathbb{N}$: $(x,y) \in U^n$. If E is the equivalence class of x, then $E = \bigcup_{n\in\mathbb{N}} U^n[x]$ and, by (4), we have

$$|E| \leqslant \Delta(X,\mathcal{U}). \tag{5}$$

Let $X = \bigsqcup_{\alpha\in A} E_\alpha$ be the partition of X onto classes of equivalence, and let $E_\alpha = \{x_i^\alpha\}_{i\in I_\alpha}$ be an enumeration of E_α. We can assume that A and I_α are well-ordered sets. Note that $|I_\alpha| \leqslant \Delta(X,\mathcal{U})$ by (5). If $|I_\alpha| < \Delta(X,\mathcal{U})$, for every j such that $|I_\alpha| \leqslant j < \Delta(X,\mathcal{U})$, we put $x_j^\alpha := x_0^\alpha$. For every $i < \Delta(X,\mathcal{U})$, we set $\mathcal{P} = \{P_i\}_{i<\Delta(X,\mathcal{U})}$, where $P_i := \{x_i^\alpha\}_{\alpha\in A}$. Note that $P_i \cap E_\alpha = \{x_i^\alpha\}$ for every $i < \Delta(X,\mathcal{U})$ and $\alpha \in A$. So P_i is a U-separated subset of X for each $i < \Delta(X,\mathcal{U})$. Since $\mathcal{P}$ covers X, item (i) implies $\mathrm{pu}(X,\mathcal{U}) \leqslant |\mathcal{P}| = \Delta(X,\mathcal{U})$.

Case 2. Suppose that $\Delta(X,\mathcal{U})$ is *finite*. Then there is a symmetric $U \in \mathcal{U}$ such that $\Delta(X,\mathcal{U}) = \sup\{|U[x]|: x \in X\} = m \in \mathbb{N}$. If $m = 1$, then X is U-separated. So $\mathrm{pu}(X,\mathcal{U}) = \Delta(X,\mathcal{U}) = 1$.

Assume that $m > 1$. Take a symmetric $V \in \mathcal{U}$ such that $V^{m^2} \subseteq U$ and set $W := V^m$. Let us show that

$$W \circ W[x] = W[x], \quad \forall x \in X. \tag{6}$$

Indeed, first we note that, if $F \circ F[x] = F[x]$ for $F \in \mathcal{U}$ and $x \in X$, then $F^n[x] = F[x]$ for every $n \in \mathbb{N}$. We know that, for every $x \in X$, the sets $V[x], V \circ V[x], \dots, V^{m+1}[x]$ are increasing and contain at most m elements. Hence there is $k_x \leqslant m$ such that $V \circ V^{k_x}[x] = V^{k_x}[x]$, that proves (6).

Taking into account that W is symmetric and (6), we can define an equivalence $\sim$ on X by $x \sim y \Leftrightarrow (x,y) \in W$, and note that each class of equivalence contains at most m elements. Now, repeating the arguments in Case 1, we get $\mathrm{pu}(X,\mathcal{U}) \leqslant m = \Delta(X,\mathcal{U})$.

(v) Clearly, $\Delta(X,\mathcal{U}) \leqslant |X|$ and $\mathrm{ud}(X,\mathcal{U}) \leqslant |X|$. So $\Delta(X,\mathcal{U}) \cdot \mathrm{ud}(X,\mathcal{U}) \leqslant |X|$.

Conversely, if $\Delta(X,\mathcal{U}) = |X|$, then $\Delta(X,\mathcal{U}) \cdot \mathrm{ud}(X,\mathcal{U}) = |X|$. Assume that $\Delta(X,\mathcal{U}) < |X|$. Take a symmetric $U \in \mathcal{U}$ such that $\Delta(X,\mathcal{U}) = \sup\{|U[x]|: x \in X\} < |X|$, and choose a maximal U-separated subset A of X. By Lemma 4, we have $U[A] = X$. So

$$|X| \leqslant |A| \cdot \sup\{|U[a]|: a \in A\} \leqslant |A| \cdot \Delta(X,\mathcal{U}).$$

Hence $|A| \geqslant |X|$. Thus $|A| = |X|$ and $|X| \leqslant \Delta(X,\mathcal{U}) \cdot \mathrm{ud}(X,\mathcal{U})$. □

If $(X,\mathcal{U})$ is a group, in the following examples we assume that $\mathcal{U} = \mathcal{L}_X$ and write simply X. Now we show that all inequalities in Proposition 6 in general are strict.

Example 2. If $(X,\mathcal{U}) = \{n, n + \frac{1}{n}\}_{n\in\mathbb{N}}$ with the usual topology from $\mathbb{R}$. Clearly, $\mathrm{pu}(X,\mathcal{U}) = \Delta(X,\mathcal{U}) = 2 < |X|$.

Example 3. Let $(X,\mathcal{U}) = \mathbb{R}$. Then each uniformly discrete subset of X is countable. Hence $\aleph_0 = \psi(\mathbb{R}) < \delta(\mathbb{R}) = \Delta(\mathbb{R}) = \mathrm{pu}(\mathbb{R}) = |\mathbb{R}| = \mathfrak{c}$.

Example 4. Let $(X,\mathcal{U}) = \{\mathbb{Q} \cap (-1,1)\} \cup (2,4)$ with the usual metrics induced by $\mathbb{R}$. Clearly,

$$\delta(X,\mathcal{U}) = \min\{|U[0]|, U \in \mathcal{U}\} = \aleph_0 < \Delta(X,\mathcal{U}) = \min\{|U[3]|, U \in \mathcal{U}\} = \mathfrak{c}.$$

Evidently, each uniformly discrete subset of X is finite. Since $|X| = \mathfrak{c}$, we obtain

$$\aleph_0 = \psi(X,\mathcal{U}) = \delta(X,\mathcal{U}) < \Delta(X,\mathcal{U}) = \mathrm{pu}(X,\mathcal{U}) = |X| = \mathfrak{c}.$$

Example 5. Fix an arbitrary cardinal $\eta > \aleph_0$ and let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of groups such that $|G_n| = \eta$ for every $n \in \mathbb{N}$. Put

$$G := \Big\{x = (x_n) \in \prod_{n\in\mathbb{N}} G_n \colon |\mathrm{supp}(x)| < \aleph_0\Big\},$$

where $\mathrm{supp}(x) = \{\alpha \colon x_n \neq e_n\}$ and e_n is the identity of G_n. For every $k \in \mathbb{N}$, set

$$U_k := \{x = (x_n) \in G \colon x_n = e_n \text{ for every } n \leqslant k\},$$
$$D_k := \{x = (x_n) \in G \colon x_n = e_n \text{ for every } k < n\}.$$

Then the sequence $\{U_k \colon k \in \mathbb{N}\}$ forms a base at the identity for some group topology τ on G. Clearly, $\psi(G) = \aleph_0$ and $\Delta(G) = \delta(G) = \min\{|U_k| \colon k \in \mathbb{N}\} = \eta$.

Note that, for every $k \in \mathbb{N}$, D_k is a U_k-separated (discrete) subgroup of G and $G = \bigcup_{k\in\mathbb{N}} D_k$. Define $P_1 := D_1$ and $P_k := D_k \setminus D_{k-1}$ for $k > 1$. Then the family $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$ is a partition of G with uniformly discrete cells. By Proposition 6(i), we get $\psi(G) = \mathrm{pu}(G) = \aleph_0$. Thus $\aleph_0 = \psi(G) = \mathrm{pu}(G) < \delta(G) = \Delta(G) = \eta$.

7. Uniformly discrete sets in biuniform spaces and topological groups. Let G be a topological group. By $\mathcal{N}(G)$ we denote the filter of all open neighborhoods of the unit e. Recall that a subset E of a topological group G is called *left-precompact* (respectively, *right-precompact, precompact*) if, for every $U \in \mathcal{N}(G)$, there exists a finite subset F of G such that $E \subseteq F \cdot U$ (respectively, $E \subseteq U \cdot F$, $E \subseteq F \cdot U$ and $E \subseteq U \cdot F$). If E is symmetric the three different definitions coincide.

For a topological group G the sets of the form

$$V_U^l = \{(x, y) \in G \times G \colon x^{-1}y \in U\} \text{ and } V_U^r = \{(x, y) \in G \times G \colon yx^{-1} \in U\},$$

where $U \in \mathcal{N}(G)$, form respectively a base of the left $\mathcal{U}_l$ and the right $\mathcal{U}_r$ uniform structure on G. Uniformly discrete subsets with respect to $\mathcal{U}_l$ (resp. $\mathcal{U}_r$) will be called *left* (resp. *right*) *uniformly discrete*. Note also that a subset A of G is left (resp. right) uniformly discrete if and only if there is $U \in \mathcal{N}(G)$ such that $aU \cap bU = \varnothing$ (resp. $Ua \cap Ub = \varnothing$) for every distinct elements $a, b \in A$. A subset D of G is called *uniformly discrete* if it is uniformly discrete both in the right and in the left uniformities. For a subset A of a topological group G, *the left* (resp. *right*) *uniformly discrete number* $\mathrm{ud}_l(A)$ (resp. $\mathrm{ud}_r(A)$) of A is defined as $\mathrm{ud}(A)$ with respect to the left (resp. right) uniform structure on G.

Taking into account that a topological group is precompact if and only if it is left precompact ([1, §3.7]) and applying Corollary 2, we get yields.

Corollary 7. *A topological group is precompact if and only if every its left uniformly discrete subset is finite.*

The case of topological groups motivates us to consider two uniformities on a set X and investigate relations between them.

The following theorem generalizes the corresponding result for groups obtained in [16].

Theorem 10. *Let $\mathcal{L}$ and $\mathcal{R}$ be uniformities on a set X. Consider the following statements*

(i) $BUC(X, \mathcal{L}) \subseteq BUC(X, \mathcal{R})$;

(ii) $\forall A \subseteq X \forall L \in \mathcal{L} \exists R \in \mathcal{R} \colon R[A] \subseteq L[A]$;

(iii) $\tau_{\mathcal{L}} \subseteq \tau_{\mathcal{R}}$.

Then (i) *and* (ii) *are equivalent and imply* (iii).

Proof. (i)⇒(ii) Fix arbitrarily $A \subseteq X$ and $L \in \mathcal{L}$. By the Katetov extension theorem [12], there is a uniformly continuous function $f\colon (X,\mathcal{L}) \to [0,1]$ such that $f(X \setminus L[A]) = 0$ and $f(A) = 1$. Since $f \in BUC(X,\mathcal{R})$ as well, we can choose $R \in \mathcal{R}$ such that $(x,y) \in R$ implies $|f(x) - f(y)| < \frac{1}{2}$. Then $R[A] \subseteq L[A]$.

(ii)⇒(i) Suppose for a contradiction that there exists $f \in BUC(X,\mathcal{L}) \setminus BUC(X,\mathcal{R})$. So we can find $\varepsilon > 0$ such that, for every $R \in \mathcal{R}$, there is $(x_R, y_R) \in R$ satisfying $|f(x_R) - f(y_R)| > \varepsilon$.

Since f is bounded, the net $\{f(x_R)\}_{R\in\mathcal{R}}$ has an accumulation point r. So there exists a cofinal family I of $\mathcal{R}$ such that such that $|r - f(x_V)| < \frac{\varepsilon}{4}$ for each $V \in I$. Set $A = \{x_V\colon V \in I\}$. Since $f \in BUC(X,\mathcal{L})$, there is $L \in \mathcal{L}$ such that $(x,y) \in L$ implies $|f(x) - f(y)| < \frac{\varepsilon}{4}$. So, if $y \in L[A]$, then $|f(y) - r| < \frac{\varepsilon}{2}$.

On the other hand, for each $R \in \mathcal{R}$, there is $V \in I$ such that $V \subseteq R$, and hence $(x_V, y_V) \in R$. So

$$|f(y_V) - r| > |f(y_V) - f(x_V)| - |f(x_V) - r| > \frac{3\varepsilon}{4}.$$

Thus $R[A] \setminus L[A] \neq \varnothing$, a contradiction.

(ii)⇒(iii). For each $x \in X$, we put $A = \{x\}$ and apply (ii). □

In the case where $X = G$ is a topological group and $\mathcal{L} = \mathcal{L}_G$, $\mathcal{R} = \mathcal{R}_G$, the following theorems is proved in [15].

Theorem 11. *Let $\mathcal{L}$ and $\mathcal{R}$ be uniformities on a set X. If $\mathcal{L}$ and $\mathcal{R}$ are metrizable, then* (**uc**) ⇒ (**eq**).

Proof. Suppose for a contradiction that $\mathcal{L} \neq \mathcal{R}$, say $\mathcal{L} \setminus \mathcal{R} \neq \varnothing$. Choose a symmetric $L \in \mathcal{L}$ such that $L \circ L \circ L \notin \mathcal{R}$. Let $\{R_n\colon n \in \mathbb{N}\}$ be a base for $\mathcal{R}$. Take a sequence $(x_n, y_n)_{n\in\mathbb{N}}$ in $X \times X$ such that $(x_n, y_n) \in R_n$ but $(x_n, y_n) \notin L \circ L \circ L$ for every $n \in \mathbb{N}$. By the Efremovich lemma ([4]), there is an infinite set $I \subset \mathbb{N}$ such that $(x_n, y_m) \notin L$ for all $m, n \in I$. Set $A = \{y_n\colon n \in I\}$ and note that $R[A] \nsubseteq L[A]$ for each $R \in \mathcal{R}$. This contradicts Theorem 10. □

Theorem 12. *Let $\mathcal{L}$ and $\mathcal{R}$ be uniformities on a set X. If $\mathcal{L}$ and $\mathcal{R}$ are metrizable and complete, then* (**ud**) ∧ (**top**) ⇒ (**eq**).

Proof. We use the following observation which immediately follows from Proposition 1: every infinite subset of a metric space (X, d) contains either infinite uniformly d-discrete subset or an injective Cauchy sequence.

Suppose for a contradiction that $\mathcal{L} \neq \mathcal{R}$, say $\mathcal{L} \setminus \mathcal{R} \neq \varnothing$. Let $\mathcal{L}$ and $\mathcal{R}$ be defined by metrics d and ρ respectively. As in the previous Theorem 11 we can find a sequence $(x_n, y_n)_{n<\omega}$ in $X \times X$ and $\varepsilon > 0$ such that $\rho(x_n, y_n) \to 0$ but $d(x_n, y_n) > \varepsilon$. By the above observation, passing to subsequences, it is enough to consider only two cases.

Case 1. The set $S = \{x_n\colon n \in \mathbb{N}\}$ is uniformly ρ-discrete. Then the set $T = \{y_n\colon n \in \mathbb{N}\}$ is also uniformly ρ-discrete because $\rho(x_n, y_n) \to 0$. By (**ud**), the sets S and T are also uniformly d-discrete. Note that $S \cup T$ is not uniformly ρ-discrete as $\rho(x_n, y_n) \to 0$. Hence, by (**ud**), $S \cup T$ is not uniformly d-discrete as well. Since $d(x_n, y_n) > \varepsilon$, we can choose two injective sequences

$(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $d(x_{n_k}, y_{m_k}) \to 0$, $n_k \neq m_k$, $n_k, m_k \notin \{n_i, m_i \colon i < k\}$. Set $E := \{x_{n_k}, y_{m_l} \colon k, l \in \mathbb{N}\}$. Then E is not uniformly d-discrete as $d(x_{n_k}, y_{m_k}) \to 0$. On the other hand, since S is uniformly ρ-discrete, for some positive number a we obtain

$$0 < a \leqslant \rho\left(x_{n_k}, x_{m_l}\right) \leqslant \rho\left(x_{n_k}, y_{m_l}\right) + \rho\left(y_{m_l}, x_{m_l}\right).$$

As $\rho\left(y_{m_l}, x_{m_l}\right) \to 0$, we obtain that E is uniformly ρ-discrete that contradicts (**ud**).

Case 2. $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, ρ). Since (X, ρ) is complete, the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge in (X, ρ) to some point x. By (**top**), $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ also converge to x in (X, d), and we get a contradiction with $d(x_n, y_n) > \varepsilon$. □

Proof of Theorem 4. Immediately follows from Theorems 11 and 12. □

In the following examples we show that conditions (**uc**) and (**ud**) are not equivalent in general.

Example 6. (**uc**) $\not\Rightarrow$ (**ud**). Let κ and λ be infinite cardinals such that $\lambda \leqslant \kappa$. Set $X := \kappa$. We endow X with the discrete uniformity $\mathcal{L}$. Clearly, each real valued function is uniformly continuous and X is uniformly discrete in $(X, \mathcal{L})$.

We denote by $\mathcal{F}_{\kappa,\lambda}$ the family of all partitions $\mathcal{P}$ of κ of cardinality $|\mathcal{P}| < \lambda$. For each $\mathcal{P}$, we put

$$U_{\mathcal{P}} = \{(x, y) \in X \times X \colon \{x, y\} \subseteq \mathcal{P} \text{ for some } P \in \mathcal{P}\},$$

and denote by $\mathcal{U}_{\kappa,\lambda}$ the uniformity on X with the base $\{U_{\mathcal{P}} \colon \mathcal{P} \in \mathcal{F}_{\kappa,\lambda}\}$. We observe that a subset $A \subseteq X$ is uniformly discrete in $(X, \mathcal{U}_{\kappa,\lambda})$ if and only if $|A| < \lambda$.

Take $\lambda = \mathfrak{c}^+$ and set $\mathcal{R} = \mathcal{U}_{\kappa,\mathfrak{c}^+}$. Since $|X| = \kappa \geqslant \lambda$, we obtain that X is not uniformly discrete in $(X, \mathcal{R})$. So (**ud**) does not hold.

Let f be a real-valued function on X. Set $\mathcal{P}_f := \{P_c\}_{c\in \mathrm{Im}(f)}$, where $P_c = f^{-1}(c)$ for every $c \in \mathrm{Im}(f)$. Then $\mathcal{P}_f$ is a partition of X of cardinality $< \lambda$. So f is uniformly continuous in $(X, \mathcal{R})$. Thus $BUC(X, \mathcal{L}) = BUC(X, \mathcal{R})$ and (**uc**) holds.

Example 7. (**ud**) $\wedge$ (**top**) $\not\Rightarrow$ (**uc**). Let $X = \{x_n\}_{n\in\mathbb{N}}$, where $x_{2n-1} = \frac{1}{n+1}$ and $x_{2n} = \frac{n+2}{n+1}$ for every $n \in \mathbb{N}$. Let $\mathcal{L}$ be the uniformity on X induced by the natural uniformity of $\mathbb{R}$. Clearly, $\tau_{\mathcal{L}}$ is discrete. Further, every uniformly discrete subset of $(X, \mathcal{L})$ is finite since X is precompact in $\mathbb{R}$.

For every $n \in \mathbb{N}$, set $R_n = \Delta_X \cup \{(x_i, x_j) \colon i, j \geqslant n\}$, where Δ_X is the diagonal of $X \times X$. Denote by $\mathcal{R}$ the uniformity on X with the base $\{R_n\}_{n\in\mathbb{N}}$. Since $R_{n+1}[x_n] = \{x_n\}$ for every natural number n, we obtain that $\tau_{\mathcal{R}}$ is discrete. Thus (**top**) holds. For every $n \in \mathbb{N}$, each R_n-separated subset intersects with $\{x_n, x_{n+1}, \dots\}$ at most in one point. So every uniformly discrete subset of $(X, \mathcal{R})$ is finite. Therefore (**ud**) is fulfilled as well.

Define $f \colon X \to \{0, 1\}$ by $f(x_{2n-1}) = 0$ and $f(x_{2n}) = 1$ for $n \in \mathbb{N}$. Clearly, $f \in BUC(X, \mathcal{L})$. However, $f \notin BUC(X, \mathcal{R})$ because, for every $n \in \mathbb{N}$, the entourage R_n contains the pair (x_{2n-1}, x_{2n}). Thus (**uc**) does not hold.

Example 8. We note that (**ud**) does not imply even that $(X, \tau_{\mathcal{L}})$ and $(X, \tau_{\mathcal{R}})$ are homeomorphic. Indeed, take an infinite set X which admits two compact topologies τ_0 and τ_1 such that (X, τ_0) and (X, τ_1) are not homeomorphic, and take $\mathcal{L}, \mathcal{R}$ so that $\tau_0 = \tau_{\mathcal{L}}$, $\tau_1 = \tau_{\mathcal{R}}$. Clearly, (**ud**) holds.

The following notions generalize the notions of (left, right) neutral subsets.

Definition 6 ([13])**.** A right uniformly discrete subset A of a topological group (G,τ) is said to be *right strongly neutral* in (G,τ) if for every open neighborhood V of the unit e there is an open neighborhood U of e such that $U \subseteq a^{-1}Va$ for all $a \in A$. In a similar way we define *left strongly neutral* subsets. A subset that is both left and right strongly neutral is said to be *strongly neutral*.

The natural analogue of Definition 6 for uniform spaces is the following:

Definition 7. Let $\mathcal{L}$ and $\mathcal{R}$ be two uniformities on a set X and A be a subset of X. We say that $\mathcal{L}$ is *finer* than $\mathcal{R}$ on A and write $\mathcal{R}(A) \le \mathcal{L}(A)$, if for every entourage $R \in \mathcal{R}$ there is $L \in \mathcal{L}$ such that $L[a] \subseteq R[a]$ for all $a \in A$. This means that the topology $\tau_{\mathcal{L}}$ is *uniformly finer* than $\tau_{\mathcal{R}}$ at all points of A.

In what follows we need the following stronger notion:

Definition 8. Let $\mathcal{L}$ and $\mathcal{R}$ be two uniformities on a set X. A subset A of X is called $(\mathcal{R},\mathcal{L})$-*strongly neutral* if for every entourage $R \in \mathcal{R}$ there is $L \in \mathcal{L}$ such that

(i) $L[a] \subseteq R[a]$, for every $a \in A$, i.e., $\mathcal{R}(A) \le \mathcal{L}(A)$;

(ii) if $x \in R[a]$, then $L[x] \subseteq R \circ R[a]$.

The following proposition shows, in particular, that Definitions 6, 7 and 8 are equivalent in the group case.

Proposition 7. *Let $\mathcal{L}$ and $\mathcal{R}$ be the left and the right uniformities on a topological group (G,τ) and let A be a right uniformly discrete subset of G. Then the following assertions are equivalent:*

(a) *A is right strongly neutral;*

(b) *$\mathcal{R}(A) \le \mathcal{L}(A)$;*

(c) *A is $(\mathcal{R},\mathcal{L})$-strongly neutral.*

Proof. (a) is equivalent to (b) by definition, and (c) trivially implies (b).

$(b) \Rightarrow (c)$ Let $\mathcal{R}(A) \le \mathcal{L}(A)$ and $V \in \mathcal{N}(G)$ be arbitrary. By definition, there exists $U \in \mathcal{N}(G)$ such that $U \subseteq V$ and $aU \subseteq Va$ for all $a \in A$. Hence, for every $x \in Va$ we have $xU \subseteq (Va)U \subseteq V(aU) \subseteq VVa$. Thus A is $(\mathcal{R},\mathcal{L})$-strongly neutral. □

Theorem 13. *Let $\mathcal{L}$ and $\mathcal{R}$ be two uniformities on a space X. Then the following assertions are equivalent:*

(i) *Every $\mathcal{R}$-uniformly discrete subset is $(\mathcal{R},\mathcal{L})$-strongly neutral.*

(ii) $\mathcal{R} \subseteq \mathcal{L}$.

Proof. (i)⇒(ii) Let $R \in \mathcal{R}$. We have to show that $R \in \mathcal{L}$. To this end it is enough to find an $L \in \mathcal{L}$ such that $L \subseteq R$.

Choose a symmetric $R_1 \in \mathcal{R}$ such that

$$R_1 \subseteq R \text{ and } R_1 \circ R_1 \circ R_1 \subseteq R. \tag{7}$$

Choose a maximal R_1-separated subset A of X. Then (see Lemma 4)

$$R_1[A] = X. \tag{8}$$

Choose a symmetric $L \in \mathcal{L}$ satisfying (i)–(ii) of Definition 8 for R_1. Let us show that $L \subseteq R$. For every $(x, y) \in L$, by (8), there is $a \in A$ such that $(x, a) \in R_1$, i.e., $x \in R_1[a]$. Thus $L[x] \subseteq R_1 \circ R_1[a]$. This means that $(a, y) \in R_1 \circ R_1$. Hence, by (7), $(x, y) \in R_1 \circ (R_1 \circ R_1) \subseteq R$ and $L \subseteq R$.

(ii)⇒(i) For every $R \in \mathcal{R}$ it is enough to put $L = R$. □

Definition 9. We say that two uniformities $\mathcal{L}$ and $\mathcal{R}$ on a set X are $(\mathcal{R}, \mathcal{L})$-*connected* if for every $R \in \mathcal{R}$ there is $L \in \mathcal{L}$ such that for every $x \in X$ satisfying $L[x] \subseteq R[x]$ it follows $L[y] \subseteq R \circ R[x]$ for every $y \in L[x]$.

Proposition 8. *Let $\mathcal{L}$ and $\mathcal{R}$ be two $(\mathcal{R}, \mathcal{L})$-connected uniformities on a set X. Then a subset A of X is $(\mathcal{R}, \mathcal{L})$-strongly neutral if and only if $\mathcal{R}(A) \leq \mathcal{L}(A)$.*

Let us recall that a topological group (G, τ) is called *SIN* or a *balanced group* if it has a base consisting of invariant sets (a set A is called *invariant* if $gA = Ag$ for every $g \in G$). It is well known that $\mathcal{U}_l = \mathcal{U}_r$ for SIN groups. All compact, discrete and Abelian topological groups are balanced. Clearly, if D is left U-separated for an invariant U, then D is also right U-separated. Recall also that, a topological group G is called $FSIN$ or $DSIN$ if $BUC(G, \mathcal{L}_G) = BUC(G, \mathcal{R}_G)$ or $\mathfrak{UD}(G, \mathcal{L}_G) = \mathfrak{UD}(G, \mathcal{R}_G)$ respectively. All the following implications

$$FSIN \Rightarrow SIN, \quad DSIN \Rightarrow SIN, \quad FSIN \wedge DSIN \Rightarrow SIN$$

are open problems. For some partial results see [11].

Remark 1. In Theorem 3.4 of [11], G. Itzkowitz asserts that $FSIN \wedge DSIN \Rightarrow SIN$. However, his proof is incomplete. In his notations, in spite of k is left uniformly continuous on aV for every $a \in A$, it is not clear why k is left uniformly continuous on the union $\bigcup_{a \in A} aV$. So the implication $FSIN \wedge DSIN \Rightarrow SIN$ is still open.

Clearly, if G is a $DSIN$-group, then $\mathrm{ud}_l(A) = \mathrm{ud}_r(A)$ for every subset A of G. We do not know an answer to the following question:

Question 14. *Let $\mathrm{ud}_l(A) = \mathrm{ud}_r(A)$ for every subset A of a topological group G. Is G a $DSIN$-group?*

In the following definition we generalize the notions of balanced, F- and D-balanced groups:

Definition 10. A biuniform space $(X, \mathcal{L}, \mathcal{R})$ is said to be

- *balanced* if $\mathcal{L} = \mathcal{R}$;
- *F-balanced* if $BUC(X, \mathcal{L})) = BUC(X, \mathcal{R})$;
- *D-balanced* if $\mathfrak{UD}(X, \mathcal{L}) = \mathfrak{UD}(X, \mathcal{R})$.

Clearly, every balanced biuniform space is F- and D-balanced.

Now we define local connectedness of biuniform spaces.

Definition 11. A uniformity $\mathcal{R}$ on a set X is called *locally connected* if there is a symmetric base $\mathcal{B}_\mathcal{R}$ of $\mathcal{R}$ such that $R[x]$ is connected in the topology $\tau_\mathcal{R}$ defined by $\mathcal{R}$ for all $R \in \mathcal{B}_\mathcal{R}$ and $x \in X$. A biuniform space $(X, \mathcal{L}, \mathcal{R})$ is called *uniformly locally connected* if the uniformities $\mathcal{L}$ and $\mathcal{R}$ are locally connected.

For locally connected biuniform spaces we prove the following:

Theorem 15. *Let $(X, \mathcal{L}, \mathcal{R})$ be a locally connected biuniform space. If $(X, \mathcal{L}, \mathcal{R})$ is F-balanced and $(\mathcal{R}, \mathcal{L})$-connected, then $\mathcal{R} \le \mathcal{L}$.*

Proof. First we note that $\tau_{\mathcal{L}} = \tau_{\mathcal{R}}$ by Theorem 10.

By Theorem 13 and Proposition 8, it is enough to show that $\mathcal{R}(A) \le \mathcal{L}(A)$ for every $\mathcal{R}$-uniformly discrete subset A. Let $R \in \mathcal{R}$ be such that A is R-separated. Choose a symmetric $R_1 \in \mathcal{B}_{\mathcal{R}}$ such that $R_1 \subseteq R$ and $R_1 \circ R_1 \circ R_1 \subseteq R$.

Let us note that for every $a \in A$, the connected component of a in $R_1[A]$ is $R_1[a]$. Indeed, $R_1[a]$ is connected by definition and $R_1[a] \cap R_1[a'] = \varnothing$ for all distinct $a, a' \in A$ (otherwise, for $x \in R_1[a] \cap R_1[a']$, we have $(x, a) \in R_1$ and $(x, a') \in R_1$, and hence $(a, a') \in R_1 \circ R_1 \subseteq R$, a contradiction).

By Theorem 10, there exists $L \in \mathcal{B}_{\mathcal{L}}$ such that $L[A] \subseteq R_1[A]$. To prove that $\mathcal{R}(A) \le \mathcal{L}(A)$ it is enough to show that $L[a] \subseteq R_1[a]$ for every $a \in A$. The set $L[a]$ is connected and it is contained in $R_1[A]$. Since $L[a]$ has also a non-empty intersection with $R_1[a]$ (since $\{a\} \subseteq L[a] \cap R_1[a]$) and since $R_1[a]$ is the connected component of a in $R_1[A]$, we must get $L[a] \subset R_1[a]$. □

We end this section with the following questions:

Question 16. *Let $(X, \mathcal{L}, \mathcal{R})$ be an F- and D-balanced biuniform space. Is it balanced?*

Question 17. *Does* (**uc**) *imply* (**eq**) *for countable biuniform spaces?*

8. Applications to Abelian topological groups. Now we apply our theory to generalize a result by K. P. Hart and J. van Mill ([9]).

For an Abelian topological group G we denote by $\widehat{G}$ the group of all continuous characters on G. The group G is called *maximally almost periodic* (MAP) if $\widehat{G}$ separates the points of G. If G is a MAP Abelian group we denote by $\sigma(G, \widehat{G})$ *the weak topology* or *the Bohr topology* on G, i.e., the smallest topology in G for which the elements of $\widehat{G}$ are continuous. Set $G^+ := (G, \sigma(G, \widehat{G}))$. Then G^+ is a precompact Abelian group.

Let G be a discrete Abelian group. Eric van Douwen [3, 4.14] posed a question whether G^+ has a closed discrete subset of cardinality $|G|$. K. P. Hart and J. van Mill ([9]) answered this question in the affirmative.

Fact 18. [9, Theorem 0.1] *Let A be a subset of a discrete Abelian group G. Then there is a subset D_0 of A such that*

(1) *D_0 is discrete and closed in G^+, and*

(2) $|D_0| = |A|$.

It is well known that, if G is a discrete Abelian group, then G and G^+ have only finite compact subsets. So, Proposition 3, Corollary 7 and Fact 18 imply:

Corollary 8. *The equalities* $\mathrm{ud}(G^+) = \aleph_0$ *and* $\mathrm{e}(G^+) = \mathrm{cc}(G^+) = |G|$ *hold true for any infinite discrete Abelian group G.*

Since G^+ is not complete, this corollary especially emphasizes the importance of completeness in Corollary 6.

Noting that every subset of a discrete Abelian group is both closed and uniformly discrete, we generalize van Douwen's question as follows.

Question 19. *Let A be a (resp. discrete) closed subset of a MAP non-precompact Abelian group G. Does A contain a subset D_0 such that*

(a) *D_0 is uniformly discrete,*

(b) *D_0 is closed and discrete in G^+,*

(c) *$|D_0| = \mathrm{ud}(A)$ (resp. $|D_0| = |A|$)?*

Note that, in Question 19 the requirement on G to be non-precompact is essential since, for a precompact group G, we have $G = G^+$ and hence every uniformly discrete subset of G is finite by Corollary 7.

In the following theorem, making use of Fact 18, we give a complete answer to Question 19 for locally compact Abelian groups.

Theorem 20. *Let A be a closed non-compact subset of an Abelian locally compact non-compact group G. Then:*

(1) *If a subset D of A is discrete and closed in G^+, then $|D| \le \mathrm{ud}(A)$.*

(2) *There is a subset D_0 of A such that*

  (a) D_0 is uniformly discrete (and hence closed) in G,

  (b) D_0 is discrete and closed in G^+, and

  (c) $|D_0| = \mathrm{ud}(A)$.

Proof. (1) If D is finite, the assertion is trivial.

Assume that D is infinite. Since the identity map $i\colon G \to G^+$ is continuous, D is also discrete and closed in G. By Lemma 5, $\mathrm{cc}(D) = |D|$. Now Proposition 5 implies that

$$|D| = \mathrm{ud}(D) \le \mathrm{ud}(A).$$

(2) The assertion is trivial if D_0 is finite. So we shall assume that D_0 is infinite. Let us note first that A has a uniformly discrete subset D of cardinality $\mathrm{ud}(A)$. Indeed, since A is closed and non-compact, this follows from Proposition 5.

Step 1. Let us prove assertion (2) under the following assumption:

(γ) *there is a closed subgroup H of G such that $|\pi_H(D)| = |D|$ and $\pi_H(D)$ has a discrete and closed subset T in $(G/H)^+$ of cardinality $|\pi_H(D)|$, where $\pi_H\colon G \to G/H$ is the quotient homomorphism.*

Choose an arbitrary subset D_0 of D such that $|D_0 \cap \pi_H^{-1}(t)| = 1$ for every $t \in T$. Let us show that the set D_0 satisfies conditions (a)–(c) that proves (2).

(a) Since D_0 is a subset of D, it is uniformly discrete in G.

(b)–(c) Note that π_H induces the continuous epimorphism $\pi_H^+\colon G^+ \to (G/H)^+$ ([2, Theorem 3.2]). Since $\pi_H^+(D_0) = T$ and π_H^+ is one-to-one on D_0, D_0 is a discrete closed subset of G^+ and

$$|D_0| = |T| = |D| = \mathrm{ud}(A).$$

Step 2. It remains to show that *one can always choose a closed subgroup H such that condition (γ) holds.*

It is well known (see [10, 24.30]) that $G = \mathbb{R}^n \times Y$, where Y has an open compact subgroup K. Let $H_0 = \mathbb{R}^n \times K$ and $\pi\colon G \to G/H_0$ be the natural homomorphism.

We distinguish between two cases.

Case 1. $\mathrm{ud}(A) = \aleph_0$.

Assume that $\pi(D)$ is infinite. Put $H = H_0$. Since G/H is discrete, by Fact 18, there is a subset T of $\pi(D)$ such that $|T| = |\pi(D)|$ and T is discrete and closed in $(G/H)^+$. Lemma 1(3) implies $|\pi(D)| = |D|(= \mathrm{ud}(A))$. So condition (γ) holds.

If $\pi(D)$ is finite, then $n > 0$ (see Corollary 7) and there is an infinite coset D_1 of D. Replacing D with D_1 and taking into account that each shift $g \mapsto g+h$ is a homeomorphism of G^+, without loss of generality we may assume that $D \subset H_0$.

Since D is not compact, without loss of generality we may assume also that the projection of D to the first coordinate is unbounded. Let $\pi_1 \colon G \to \mathbb{R}$ be the projection of G to the first coordinate. Set $H = \ker(\pi_1)$. By our assumption $\pi_1(D)$ is unbounded. Thus there is a subset T of $\pi_1(D)$ such that $|T| = |\pi_1(D)| = \mathrm{ud}(A)$ and T is discrete and closed in $(G/H)^+$ by [3, Theorem 1.1.4]. So, condition (γ) holds also in this case.

Case 2. $\mathrm{ud}(A) > \aleph_0$. Set $H = H_0$. By Lemma 1(3), we have $|\pi(D)| = \mathrm{ud}(A)$. So, by Fact 18, there is a subset T of $\pi(D)$ such that $|T| = |\pi(D)| = \mathrm{ud}(A)$ and T is discrete and closed in $(G/H)^+$. Therefore, condition (γ) holds true. □

It would be interesting to obtain sufficient conditions on a uniformly discrete subset D of G under which it is a closed discrete subset of G^+.

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REFERENCES

1. A.V. Arhangel'skii, M.G. Tkachenko, Topological groups and related strutures, Atlantis Press/World Scientific, Amsterdam-Raris, 2008.
2. W.W. Comfort, S. Hernández, F.J. Trigos-Arrieta, *Relating a locally compact Abelian group to its Bohr compactification*, Adv. Math., **120** (1996), 322–344.
3. E.K. van Douwen, *The maximal totally bounded group topology an G and the biggest minimal G-space, for abelian groups G*, Topology Appl., **34** (1990), 69–91.
4. V. Efremovich, *The geometry of proximity*, Math. Sb., **73** (1952), 189–200.
5. R. Engelking, General Topology, PWN, Polish Scientific Publ, Warszawa, 1977.
6. L. Gillman, M. Jerison, Rings of continuous functions, Van Nostrand, New York, 1960.
7. I.I. Guran, *On topological groups close to being Lindelöf*, Soviet. Math. Dokl., **23** (1981), 173–175.
8. I.I. Guran, *On embeddings of topological groups*, available in VINITI, Moskow, №1483-81, 1981.
9. K.P. Hart, J. van Mill, *Discrete sets and the maximal totally bounded group topology*, J. Pure Appl. Algebra, **70** (1991), 73–80.
10. E. Hewitt, K.A. Ross, Abstract Harmonic Analysis, V.I, 2nd ed. Springer-Verlag, Berlin, 1979.
11. G. Itzkowitz, *Functional balance, discrete balance, and balance in topological groups*, Topology Proc., **28** (2004), 569–577.
12. M. Katetov, *On real valued functions in topological spaces*, Fund. Math., **38** (1951), 85–91.
13. M. Megrelishvili (Levy), P. Nickolas, V. Pestov, *Uniformities and uniformly continuous functions on locally connected groups*, Bull. Austral. Math. Soc., **56** (1997), 279–283.
14. V.G. Pestov, *Some properties of free topological groups*, Moscow Univ. Math. Bull., **37** (1982), 46–49.
15. I. Protasov, *Functionally balanced groups*, Math. Notes, **49** (1991), 614–616.
16. I. Protasov, A. Saryev, *Semigroups of closed subsets of a topological group*, Izv. Akad. Nauk Turkmen SSR, Ser. Fiz.-Tekh. Nauk, **3** (1998), 21–25.

17. M. Tkačenko, *On group uniformities on square of a space and extending pseudometrics II,* Bull. Austral. Math. Soc., **52** (1995), 41–61.

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