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ON SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF ARBITRARY FAST GROWTH IN THE UNIT DISC

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We investigate fast growing solutions of linear differential equations in the unit disc. For that we introduce a general scale to measure the growth of functions of infinite order including arbitrary fast growth. We describe the growth relations between entire coefficients and solutions of the linear differential equation $f^{(n)} + a_{n-1}(z)f^{(n-1)} + \ldots + a_0(z)f = 0$ in this scale and we investigate the growth of solutions where the coefficient of f dominates the other coefficients near a point on the boundary of the unit disc.

1. Introduction. Let us consider the linear differential equations of the form

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \ldots + a_0(z)f = 0, \tag{1}$$

where $k \geq 2$, $a_0 \not\equiv 0$. There has been an increasing interest in studying the growth of analytic solutions of (1) in the unit disc $\mathbb{D} = \{z : |z| < 1\}$. For example, finite order solutions have been studied in [3], [13], [9], [19], [1], [15], [17], [4] as well as solution of finite iterated order in [10], [2].

For $r > 0 \in \mathbb{D}$ define the iterations $\exp_1 r = e^r$, $\exp_{n+1} r = \exp(\exp_n r)$, $n \in \mathbb{N}$, and $\log^+ = \max\{\log x, 0\}$, $\log_1^+ r = \log^+ r$, $\log_{n+1}^+ r = \log^+ \log_n^+ r$, $n \in \mathbb{N}$.

For $p \in \mathbb{N} \cup \{0\}$ the p-th iterated order of an analytic function f in $\mathbb{D}$ is defined by

$$\sigma_{M,p}(f) = \varlimsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,f)}{-\log(1-r)},$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$.

If f is meromorphic in $\mathbb{D}$, then the p-th iterated order is defined by

$$\sigma_p(f) = \varlimsup_{r \to 1^-} \frac{\log_p^+ T(r,f)}{-\log(1-r)},\ p \in \mathbb{N}.$$

where $T(r,f)$ is the Nevanlinna characteristic of f.

Remark 1. Note that $\sigma_{M,p}(f) = \sigma_p(f)$ if $p > 1$ and $\sigma_1(f) \leq \sigma_{M,1}(f) \leq \sigma_1(f) + 1$.

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In the case of solutions of finite order the following results are known.

Theorem A ([5]). *Let $\sigma_{M,0}[a_j] = p_j$ for $j = 0, \dots, k-1$. If*

$$\max_{0\le j\le k-1}\left\{\frac{p_j}{k-j} - 1\right\} = \frac{p_0}{k} - 1 \ge 1,$$

then all nontrivial solutions f of (1) satisfy $\sigma_{M,1} = \frac{p_0}{k} - 1$.

Theorem B ([13]). *Let $a_0, \dots, a_{k-1}$ be analytic functions in $\mathbb{D}$. If $\max\limits_{1\le j\le k-1}\{\alpha_j\} < \alpha_0$, where*

$$\alpha_j = \overline{\lim_{r\to1^-}} \frac{\log(\frac{1}{2\pi}\int_0^{2\pi}|a_j(re^{i\theta})|^{\frac{1}{k-j}}d\theta)}{\log\frac{1}{1-r}},\ j\in\{0,\dots,k-1\},$$

and $\alpha_0 \ge 1$. Then every nontrivial solution of (1) satisfies $\sigma_1(f) = \alpha_0 - 1$.

The following result of J. Heittokangas and al. classifies the growth of finite n-th iterated order solutions of (1) in terms of the growth of the coefficients.

Theorem C ([10]). *Let $n \in \mathbb{N}$ and $\alpha \ge 0$. All solutions f of (1), where the coefficients $a_0(z), \dots, a_{n-1}(z)$ are analytic in $\mathbb{D}$, satisfy $\sigma_{M,n+1}(f) \le \alpha$ if and only if $\sigma_{M,n}(a_j) \le \alpha$ for all $j = 0, 1, \dots, k-1$. Moreover, if $q \in \{0, \dots, k-1\}$ is the largest index for which $\sigma_{M,n}(a_q) = \max_{0\le j\le k-1}\{\sigma_{M,n}(a_j)\}$, then there are at least $k-q$ linearly independent solutions f of (1) such that $\sigma_{M,n+1}(f) = \sigma_{M,n}(a_q)$.*

If the last coefficient a_0 in (1) dominates, one can state more on the order of solutions.

Theorem D ([10]). *Let $n \in \mathbb{N}$. If the coefficients $a_0(z), \dots, a_{k-1}(z)$ are analytic in $\mathbb{D}$ such that $\sigma_{M,n}(a_j) < \sigma_{M,n}(a_0)$ for all $j = 1, \dots, k-1$, then all solutions $f \not\equiv 0$ of (1) satisfy $\sigma_{M,n+1}(f) = \sigma_{M,n}(a_0)$.*

The latter results were generalized on so called $[p, q]$-orders (see e. g. [19], [1], [15], [17]).

But definition p-th iterated order as well as $[p, q]$-order has the disadvantage that it does not cover arbitrary growth, i. e. there exist functions of infinite p-th iterated order for arbitrary $p \in \mathbb{N}$. In the complex plane this case is described in Example 1 in [6].

As well as in the complex plane we consider a more general scale in the unit disc, which does not have this disadvantage.

Let φ be an increasing unbounded function in the unit disc $\mathbb{D}$. We define the orders of the growth of an analytic in $\mathbb{D}$ function f by

$$\widetilde{\sigma}^0_\varphi[f] = \overline{\lim_{r\to1^-}} \frac{\varphi(M(r,f))}{-\log(1-r)},\ \widetilde{\sigma}^1_\varphi[f] = \overline{\lim_{r\to1^-}} \frac{\varphi(\log M(r,f))}{-\log(1-r)}.$$

If g is meromorphic, then the orders are defined by

$$\sigma^0_\varphi[f] = \overline{\lim_{r\to1^-}} \frac{\varphi(e^{T(r,g)})}{-\log(1-r)},\ \sigma^1_\varphi[f] = \overline{\lim_{r\to1^-}} \frac{\varphi(T(r,g))}{-\log(1-r)}.$$

Let Φ be the class of positive unbounded increasing functions φ such that $\varphi(t)$ satisfies

$$\forall c > 0\colon\ \frac{\varphi(e^{ct})}{\varphi(e^t)} \to 1,\ t \to \infty. \tag{2}$$

Remark 2. The regularity condition 2 implies the growth condition $\varphi(r) = O(\log r)$.

Indeed, if $c > 1$, then $\varphi(e^{ct}) < c\varphi(e^t)$ for all $t \geq R$. If $t \geq R$ is fixed, there exists a natural number N such that $c^N \leq t \leq c^{N+1}$. Using the inequality above inductively, we get $\varphi(e^t) = O(t)$, and a change of a variable $r = \log t$ finishes the proof. On the other hand ([6, Proposition 7], see (6)) implies that $(\forall \varepsilon > 0)\ e^{\varphi(r)} = o(r)$, $r \to \infty$. For example, the function $\varphi(r) = \log_j r$, where $j \in \mathbb{N} \setminus \{1\}$ belongs to the class Φ, and $\log r \notin \Phi$.

Our results do not intersect with that from [4].

The following theorem generalize Theorem D and is a counterpart of a result from [6] proved for entire functions.

Theorem 1. *Let $\varphi \in \Phi$, and $a_0, \dots, a_{k-1}$ be analytic functions in $\mathbb{D}$ such that*

$$\widetilde{\sigma}^0_\varphi[a_0] =: \widetilde{\sigma}_0 > \max\{\widetilde{\sigma}^0_\varphi[a_j],\ j = 1, \dots, k-1\}.$$

Then all solutions $f \not\equiv 0$ of (1) satisfy $\tilde{\sigma}^1_\varphi[f] = \widetilde{\sigma}_0$.

Remark 3. If the coefficient a_0 is such that $\log M(r, a_0) = O(\log \frac{1}{1-r})$, then $\widetilde{\sigma}^0_\varphi[f] = 0$ so conditions of Theorem 1 could not be satisfies. On the other hand, the conclusion of Theorem 1 is not true in this case as well (see Theorems A and B).

Theorem 2. *Let $\varphi \in \Phi$, and $a_0, \dots, a_{k-1}$ be analytic functions in $\mathbb{D}$ such that*

$$\sigma^0_\varphi[a_0] =: \sigma_0 > \max\{\sigma^0_\varphi[a_j], j = 1, \dots, k-1\}.$$

Then all solutions $f \not\equiv 0$ of (1) satisfy $\sigma^1_\varphi[f] \geq \sigma_0$.

In general, the conclusion of Theorem 2 is weaker than that of Theorem 1. Nevertheless, Theorem 2 is sharp as can be seen from the following example.

Example 1. Consider the equation $f^{(k)} + a_0 f = 0$, where $k \in \mathbb{N}$, a_0 is analytic and such that $\sigma_1(a_0) = \sigma_{M,1}(a_0) = \sigma > 0$ (see [16]). It follows from Theorem 1, Remark 1 and Proposition 1 that $\sigma_{M,2}(f) = \widetilde{\sigma}^1_\varphi[f] = \sigma = \sigma^1_\varphi[f] = \sigma_2(f)$ for $\varphi(r) = \log_2 r$ and any nontrivial solution f.

There are many generalizations of Theorem D based on the observation that it is sufficient to require that the coefficient a_0 dominates on a subset of $\mathbb{D}$ which is relatively large (see also [12]). For example, the following statement has appeared recently in [8].

Theorem E ([8], Th. 2)**.** *Let $a_0(z), \dots, a_{k-1}(z)$ be meromorphic functions in the unit disc $\mathbb{D}$. If there exist $\omega_0 \in \partial\mathbb{D}$ and a curve $\gamma \subset \mathbb{D}$ tending to ω_0 such that*

$$\lim_{z \to \omega_0} \frac{\sum_{j=1}^{k-1} |a_j(z)| + 1}{|a_0(z)|} \exp_n \left(\frac{\lambda}{(1 - |z|)^\mu} \right) = 0,$$

with $z \in \gamma$, where $n \geq 1$ is an integer and $\lambda > 0$, $\mu > 0$ are constants, then every solution $f(z) \not\equiv 0$ of the differential equation (1) satisfies $\sigma_n(f) = \infty$, and furthermore $\sigma_{n+1}(f) \geq \mu$.

Remark 4. Hypothesis of Theorem E do not provide that a solution is meromorphic in $\mathbb{D}$, so it is a priori assumed that f is meromorphic.

The generalization of Theorem C is formulated as follows

Theorem 3. *Let $a_0(z), \dots, a_{k-1}(z)$ be analytic functions in the unit disc $\mathbb{D}$. If there exist $\omega_0 \in \partial\mathbb{D}$ and a curve $\gamma \in D$ tending to ω_0 such that*

$$\lim_{z\to\omega_0} \frac{\sum_{j=1}^{k-1}|a_j(z)|+1}{|a_0(z)|}\varphi^{-1}\left(\log\frac{1}{(1-|z|)^{\mu}}\right)=0, \quad z\in\gamma, \tag{3}$$

where $\varphi \in \Phi$, $\mu > 0$ is real constant. Then every solution f of the differential equation (1) such that

$$\log\frac{1}{1-r}=o(\log T(r,f)), \qquad r\uparrow 1, \tag{4}$$

satisfies $\sigma^1_\varphi[f] \geq \mu$.

2. Preliminaries. To prove the main results we need several auxiliary results.

The following lemma is a consequence of Theorem 3.1 ([3]). The set $E \subset [0,1)$ in the lemma and thereafter is not necessarily the same at each occurrence, but it is always of finite logarithmic measure on $[0,1)$, that is $\int_E \frac{dr}{1-r} < \infty$.

Lemma. *Let f be a meromorphic function in the unit disc $\mathbb{D}$ such that $f^{(j)}$ does not vanish identically. Let $\varepsilon > 0$ be a constant; k and j be integers satisfying $k > j \geq 0$ and $d \in (0,1)$. Then we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-|z|}, T(s(|z|),f)\right\}\right)^{k-j}, \quad |z|\notin E,$$

where $s(|z|) = 1-d(1-|z|)$. Moreover, if $\sigma_1(f) < \infty$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{(k-j)(\sigma_1(f)+2+\varepsilon)}, \quad |z|\notin E.$$

Proposition 1. *Let $\varphi \in \Phi$ and f be an analytic function in the unit disc $\mathbb{D}$. Then*

$$\sigma^1_\varphi[f] = \widetilde{\sigma}^1_\varphi[f].$$

Proof. By the monotonicity of the function φ and by the known inequality [7, Chap. 7]

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{R+r}{R-r}T(R,f), \quad 0<r<R,$$

we have $\sigma^1_\varphi[f] \leq \widetilde{\sigma}^1_\varphi[f]$. Now we prove the converse inequality. We choose $R = \frac{1+r}{2}$ and estimate the value

$$\varphi\left(\log M(r,f)\right) \leq \varphi\left(\frac{R+r}{R-r}T(R,f)\right) \leq \varphi\left(\frac{4}{1-r}T\left(\frac{1+r}{2},f\right)\right). \tag{5}$$

Now we estimate the value $\frac{\varphi(\log M(r,f))}{\log\frac{1}{1-r}}$ on the set $F = \{r\in[0,1)\colon \log\frac{4}{1-r} < \log T(r,f)\}$.

In view of (5) and the definition of the class Φ we have for $r\in F$, $r\to+\infty$

$$\frac{\varphi(\log M(r,f))}{\log\frac{1}{1-r}} \leq \frac{\varphi(e^{\log\frac{4}{1-r}+\log T\left(\frac{1+r}{2},f\right)})}{\log\frac{1}{1-r}} \leq \frac{\varphi(e^{2\log T(\frac{1+r}{2},f)})}{\log\frac{1}{1-r}} \leq$$
$$\leq \frac{(1+o(1))\varphi(T(R,f))}{\log\frac{1}{1-R}} \leq \sigma^1_\varphi[f]+o(1).$$

Since, ε is small in this case the required inequality is proved.

We then estimate $\frac{\varphi(\log M(r,f))}{\log\frac{1}{1-r}}$ on the complement to the set F that is, on the set $\{r \in [0,1)\colon \log\frac{1}{1-r} \geq \log T(r,f)\}$. Here we use the fact that $\varphi(e^t) = t^{o(1)}$ when $t \to +\infty$ ([18]).

$$\frac{\varphi(\log M(r,f))}{\log\frac{1}{1-r}} \leq \frac{\varphi\left(e^{\log\frac{4}{1-r}+\log\frac{4}{1-r}}\right)}{\log\frac{1}{1-r}} \leq \frac{\left(2\log\frac{4}{1-r}\right)^{o(1)}}{\log\frac{1}{1-r}} = o(1),$$

as $r \to 1^-$. Hence, $\varlimsup\limits_{r\to 1_-} \frac{\varphi(\log M(r,f))}{-\log(1-r)} = \widetilde{\sigma}^1_\varphi[f] \leq \sigma^1_\varphi[f]$, which completes the proof of Proposition 1. □

We need some properties of functions from the class Φ.

Proposition 2 ([6], Prop.7)**.** *If $\varphi \in \Phi$, then*

$$\forall m>0,\ \forall k\geq 0\colon\ \frac{\varphi^{-1}(\log x^m)}{x^k} \to +\infty,\ x\to+\infty; \tag{6}$$

$$\forall \delta>0\colon\ \frac{\log\varphi^{-1}((1+\delta)x)}{\log\varphi^{-1}(x)} \to +\infty,\ x\to+\infty. \tag{7}$$

Proposition 3. *Let $f(z)$ be an analytic function in the unit disc $\mathbb{D}$ with $0 < \widetilde{\sigma}^0_\varphi[f] =: \widetilde{\sigma}_0 < \infty$. Then, for any $0<\mu<\widetilde{\sigma}_0$, there exists a set $F\subset[0,1)$ of infinite logarithmic measure such that for all $r\in F$ one has $\varphi(M(r,f)) > \mu\log\frac{1}{1-r}$.*

Proof. The definition of the upper limit implies that there exists an increasing sequence $\{r_m\}$, $r_m\to 1^-$ as $m\to\infty$ satisfying

$$1-\left(1-\frac{1}{m}\right)(1-r_m) < r_{m+1}, \qquad \lim_{m\to+\infty}\frac{\varphi(M(r_m,f))}{\log\frac{1}{1-r_m}} = \widetilde{\sigma}_0.$$

Then, there exists an integer m_0 such that for $m\geq m_0$ and any ε $(0<\varepsilon<\widetilde{\sigma}_0-\mu)$

$$\varphi(M(r,f)) > (\widetilde{\sigma}_0-\varepsilon)\log\frac{1}{1-r_m}. \tag{8}$$

Since $\mu<\widetilde{\sigma}_0-\varepsilon$, there exists an integer m_1 such that for $m\geq m_1$ we have

$$\left(\frac{\widetilde{\sigma}_0-\varepsilon}{\mu}-1\right)\log\frac{1}{1-r_m} > \log\frac{1}{1-\frac{1}{m}},\quad \frac{\widetilde{\sigma}_0-\varepsilon}{\mu}\frac{\log\frac{1}{1-r_m}}{\log\frac{1}{\left(1-\frac{1}{m}\right)(1-r_m)}} > 1. \tag{9}$$

By (8) and (9) for $m\geq m_2=\max\{m_0,m_1\}$ and for any $r\in\left[r_m, 1-\left(1-\frac{1}{m}\right)(1-r_m)\right]$, we obtain

$$\varphi(M(r,f)) \geq \varphi(M(r_m,f)) > (\widetilde{\sigma}_0-\varepsilon)\log\frac{1}{1-r_m} = \frac{\widetilde{\sigma}_0-\varepsilon}{\mu}\mu\frac{\log\frac{1}{1-r_m}}{\log\frac{1}{1-r}}\log\frac{1}{1-r} \geq$$

$$\geq \frac{\widetilde{\sigma}_0-\varepsilon}{\mu}\frac{\log\frac{1}{1-r_m}}{\log\frac{1}{\left(1-\frac{1}{m}\right)(1-r_m)}}\mu\log\frac{1}{1-r} > \mu\log\frac{1}{1-r}.$$

Set $F=\bigcup_{m=m_2}^\infty I_m$, where $I_m=\left[r_m, 1-\left(1-\frac{1}{m}\right)(1-r_m)\right]$. Then

$$m_l(f) = \sum_{m=m_2}^\infty \int_{I_m}\frac{dr}{1-r} = \sum_{m=m_2}^\infty \log\left(\frac{m}{m-1}\right) = \infty.$$

□

Proposition 4. *Let $f(z)$ be an analytic function in the unit disc $\mathbb{D}$ with $0 < \sigma^0_\varphi[f] =: \sigma_0 < \infty$. Then, for any $0 < \beta < \sigma_0$, there exists a set $F_t \subset [0,1)$ of infinite logarithmic measure such that for all $r \in F_t$ one has $\varphi(e^{T(r,f)}) > \beta \log \frac{1}{1-r}$.*

Proposition 4 can be proved similar to Proposition 3 (cf. analogous statement in [6]).

3. Proofs of the main results.

Proof of Theorem 1. First, we prove that $\sigma_1 := \sigma^1_\varphi[f] \geq \widetilde{\sigma}_0$. Suppose the contrary. Let $f \not\equiv 0$ be a solution of the equation (1). In accordance with (1) we have

$$|a_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |a_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \ldots + |a_1(z)| \left|\frac{f'(z)}{f(z)}\right|. \qquad (10)$$

Since a_j are analytic functions in $\mathbb{D}$ which satisfy $\widetilde{\sigma}^0_\varphi[a_j] < \widetilde{\sigma}_0$, $j = 1, \ldots, k-1$, there exists a constant $\beta_1 > 0$ such that $\widetilde{\sigma}^0_\varphi[a_j] < \beta_1 < \widetilde{\sigma}_0$, $j = 1, \ldots, k-1$. Hence

$$M(r, a_j) < \varphi^{-1}\left(\beta_1 \log \frac{1}{1-r}\right), \quad r \to 1^-. \qquad (11)$$

Without reducing the generality, we can suppose, that

$$\sigma_1 < \beta_1 < \widetilde{\sigma}_0 \qquad (12)$$

holds. We apply Proposition 3 to the coefficient $a_0(z)$ and a constant β_2, where $\beta_1 < \beta_2 < \widetilde{\sigma}_0$. Hence, we have

$$M(r, a_0) > \varphi^{-1}\left(\beta_2 \log \frac{1}{1-r}\right), \quad r \in F, \quad r \to 1^-, \qquad (13)$$

where F is a set of infinite logarithmic measure on $[0,1)$.

The lemma implies the following estimate

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} T(s(|z|), f)\right)^{k-j}, \quad |z| \notin E, \qquad (14)$$

where E is a set of finite logarithmic measure.

Since $F \backslash E$ is a set of infinite logarithmic measure, there exists a sequence of points $|z_n| = r_n \in F \backslash E$ tending to 1. Set $s(|z_n|) = R_n$. We have $1 - |z_n| = \frac{1}{d}(1 - R_n)$, $d \in (0,1)$.

Using (11), (13), (14) and our assumption (12), we obtain from (10)

$$\varphi^{-1}\left(\beta_2 \log \frac{d}{1-R_n}\right) \leq \left(\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon} T(R_n, f)\right)^k +$$

$$+ \left(\left(\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon} T(R_n, f)\right)^{k-1} + \ldots + \left(\frac{d}{1-R_n}\right)^{2+2\varepsilon} T(R_n, f)\right) \times$$

$$\times \varphi^{-1}\left(\beta_1 \log \frac{d}{1-R_n}\right) \leq k\left(\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon} T(R_n, f)\right)^k \varphi^{-1}\left(\beta_1 \log \frac{d}{1-R_n}\right) \leq$$

$$\leq k\left(\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon} \varphi^{-1}\left(\beta_1 \log \frac{1}{1-R_n}\right)\right)^k \varphi^{-1}\left(\beta_1 \log \frac{d}{1-R_n}\right) \leq$$

$$\leq \left(\varphi^{-1}\left((\beta_1 + \varepsilon) \log \frac{d}{1-R_n}\right)\right)^{k+2} \leq \varphi^{-1}\left((\beta_1 + 2\varepsilon) \log \frac{d}{1-R_n}\right), \quad R_n \in F \backslash E, \; R_n \to 1^-.$$

The latter two estimates follow from the properties (6) and (7). By arbitrariness of ε and the monotony of the function φ^{-1} we obtain that $\beta_1 \geq \beta_2$. This contradiction proves the inequality $\widetilde{\sigma}_0 \leq \sigma_1$.

To prove the converse inequality we need the following theorem.

Theorem F ([11]). *Let f be a solution of (1) in $\mathbb{D}_R = \{z\colon |z| < R\}$, where $0 < R \leq \infty$, let $n_c \in \{1, \dots, k\}$ be the number of nonzero coefficients a_j, $j = 0, \dots, k-1$, and let $\theta \in [0, 2\pi)$ and $\varepsilon > 0$. If $z_0 = \nu e^{i\theta} \in \mathbb{D}_R$ is such that $a_j \neq 0$ for some $j = 0, \dots, k-1$, then, for all $\nu < r < R$,*

$$|f(re^{i\theta})| \leq C \exp\left(n_c \int_\nu^r \max_{j=0,\dots,k-1} |a_j(te^{i\theta})|^{\frac{1}{k-j}} dt\right), \tag{15}$$

where $C > 0$ is a constant satisfying

$$C \leq (1+\varepsilon) \max_{j=0,\dots,k-1} \left(\frac{|f^{(j)}(z_0)|}{(n_c)^j \max\limits_{n=0,\dots,k-1} |a_n(z_0)|^{\frac{j}{k-n}}} \right). \tag{16}$$

Since $\widetilde{\sigma}^0_\varphi[a_j] < \widetilde{\sigma}_0$, $j = 1, \dots, k-1$ and from the definition of the $\widetilde{\sigma}^0_\varphi$-order it follows that for arbitrary $j \in \{1, \dots, k-1\}$ $|a_j(z)| < \varphi^{-1}\left((\widetilde{\sigma}_0 + \varepsilon) \log \frac{1}{1-r}\right)$, $|z| = r$, $r \to 1^-$.

Theorem F implies for $\nu < r < R = 1$

$$|f(re^{i\theta})| \leq C \exp\left(n_c \int_\nu^r \max_{j=0,\dots,k-1} |a_j(te^{i\theta})|^{\frac{1}{k-j}} dt\right) \leq$$
$$\leq C \exp\left(n_c \int_\nu^r \max_{j=0,\dots,k-1} \left(\varphi^{-1}\left((\widetilde{\sigma}_0 + \varepsilon) \log \frac{1}{1-t}\right)\right)^{\frac{1}{k-j}} dt\right) \leq$$
$$\leq C \exp\left(n_c \varphi^{-1}\left((\widetilde{\sigma}_0 + \varepsilon) \log \frac{1}{1-r}\right)\right) \leq \exp\left(\varphi^{-1}\left((\widetilde{\sigma}_0 + 2\varepsilon) \log \frac{1}{1-r}\right)\right),$$

where C is a constant which satisfies (16).

From the last inequality in view of arbitrariness of ε we obtain $\sigma_1 \leq \widetilde{\sigma}_0$. □

Proof of Theorem 2. Denote $\sigma_1 := \sigma^1_\varphi[f]$. Suppose the contrary. Let $f \not\equiv 0$ be a solution of the equation (1). Since a_j are analytic functions in $\mathbb{D}$ with satisfy $\sigma^0_\varphi[a_j] < \sigma_0$, $j = 1, \dots, k-1$, then there exists a constant $\beta_1 > 0$ such that $\sigma^0_\varphi[a_j] < \beta_1 < \sigma_0$, $j = 1, \dots, k-1$. Hence

$$T(r, a_j) < \log \varphi^{-1}\left(\beta_1 \log \frac{1}{1-r}\right), \ r \to 1^-. \tag{17}$$

We can suppose that $\sigma_1 < \beta_1 < \sigma_0$ holds. We apply Proposition 4 to the coefficient $a_0(z)$ and a constant β_2, where $\beta_1 < \beta_2 < \sigma_0$. Hence, we have

$$T(r, a_0) > \log \varphi^{-1}\left(\beta_2 \log \frac{1}{1-r}\right), \ r \in F_t, \ r \to 1^-, \tag{18}$$

where F_t is a set of infinite logarithmic measure on $[0, 1)$. Let E be a set of finite logarithmic measure on which the estimate (14) holds. Since $F_t \backslash E$ is a set of infinite logarithmic measure, there exists a sequence of points $|z_n| = r_n \in F_t \backslash E$ tending to 1. Set $s(|z_n|) = R_n$. We have $1 - |z_n| = \frac{1}{d}(1 - R_n)$, $d \in (0, 1)$.

Using (17), (18), (14) and our assumption, we obtain from (10)

$$\log\varphi^{-1}\left(\beta_2\log\frac{d}{1-R_n}\right)\le T\left(R_n,\frac{f^{(k)}}{f}\right)+T\left(R_n,a_{k-1}\right)+T\left(R_n,\frac{f^{k-1}}{f}\right)+\ldots+$$
$$+T(R_n,a_1)+T\left(R_n,\frac{f'}{f}\right)+\log k\le k\log\varphi^{-1}\left(\beta_1\log\frac{d}{1-R_n}\right)+\log M\left(R_n,\frac{f^{(k)}}{f}\right)+$$
$$+\log M\left(R_n,\frac{f^{(k-1)}}{f}\right)+\ldots+\log M\left(R_n,\frac{f'}{f}\right)+\log k\le$$
$$\le k\log\varphi^{-1}\left(\beta_1\log\frac{d}{1-R_n}\right)+k\log\left(\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon}T(R_n,f)\right)^k+\log k\le$$
$$\le k\log\left\{k\varphi^{-1}\left(\beta_1\log\frac{d}{1-R_n}\right)\left(\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon}T(R_n,f)\right)^k\right\}\le$$
$$\le k\log\left\{k\varphi^{-1}\left(\beta_1\log\frac{d}{1-R_n}\right)\left(\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon}\varphi^{-1}\left(\beta_1\log\frac{d}{1-R_n}\right)\right)^k\right\}\le$$
$$\le\log\left(\varphi^{-1}\left((\beta_1+\varepsilon)\log\frac{d}{1-R_n}\right)\right)^{k+2}\le\log\varphi^{-1}\left((\beta_1+2\varepsilon)\log\frac{d}{1-R_n}\right),$$

where $R_n\in F_t\backslash E$, $R_n\to 1^-$.

The latter two estimates follow from the properties of the function φ. By arbitrariness of ε and the monotony of the function φ^{-1} we obtain that $\beta_1\ge\beta_2$. This contradiction proves the inequality $\widetilde{\sigma}_0\le\sigma_1$. □

Proof of Theorem 3. Let $f\not\equiv 0$ be a solution of (1). We rewrite (10) in the form

$$1\le\frac{1}{|a_0(z)|}\left|\frac{f^{(k)}(z)}{f(z)}\right|+\left|\frac{a_{k-1}(z)}{a_0(z)}\right|\left|\frac{f^{(k-1)}}{f(z)}\right|+\ldots+\left|\frac{a_1(z)}{a_0(z)}\right|\left|\frac{f'(z)}{f(z)}\right|. \quad (19)$$

By the assumption (3), we deduce that

$$\lim_{z\to\omega_0}\left|\frac{a_j(z)}{a_0(z)}\right|\varphi^{-1}\left(\log\frac{1}{(1-|z|)^{\mu}}\right)=0, \quad (20)$$

Hence there exist $\varepsilon_1>0$, $\varepsilon_2>0$ such that for $z\in\gamma$ holds

$$\left|\frac{a_j(z)}{a_0(z)}\right|\le\frac{\varepsilon_1}{\varphi^{-1}\left(\log\frac{1}{(1-|z|)^{\mu}}\right)},\quad \frac{1}{|a_0(z)|}\le\frac{\varepsilon_2}{\varphi^{-1}\left(\log\frac{1}{(1-|z|)^{\mu}}\right)}. \quad (21)$$

Substituting (21) and the estimate of the logarithmic derivative of the lemma, where $s(|z|)=1-d(1-|z|)$ and $d\in(0,1)$, E is a set of finite logarithmic measure, we obtain

$$1\le\frac{C}{(1-|z|)^{k(2+2\varepsilon)}\varphi^{-1}\left(\log\frac{1}{(1-|z|)^{\mu}}\right)}(T(s(|z|),f))^k,\quad |z|\notin E,$$

or

$$(1-|z|)^{k(2+2\varepsilon)}\varphi^{-1}\left(\log\frac{1}{(1-|z|)^{\mu}}\right)\le C(T(s(|z|),f))^k,\quad |z|\notin E, \quad (22)$$

where $C > 0$.

Set $s(|z_n|) = R_n$. We have $1 - |z_n| = \frac{1}{d}(1 - R_n)$, $d \in (0, 1)$. In view of (4) we deduce from (22) that $\varphi^{-1}(\log(\frac{d}{1-R_n})^\mu) \le (T(R_n, f))^k(\frac{d}{1-R_n})^{k(2+\varepsilon)} \le (T(R_n, f))^{k+\varepsilon}$. Hence, $\log(\frac{d}{1-R_n})^\mu \le \varphi(C(T(R_n, f))^{k+\varepsilon}) \le \varphi(T(R_n, f))(1+o(1))$. The last inequality implies the required inequality. □

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