УДК 517.55

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SOME PROBLEMS ON PLURISUBHARMONIC SINGULARITIES

A. Rashkovskii. *Some problems on plurisubharmonic singularities*, Mat. Stud. **45** (2016), 104–108.

We present several related problems on residual Monge-Ampère masses of plurisubharmonic functions.

1. Introduction. Here we give a detailed exposition of Questions 7 and 8 from a recent list of open problems in pluripotential theory ([9]). The note is based on the author's talk at the 27th Congress of Nordic Mathematicians, March 19, 2016.

Recall that a function u in a domain $\omega \subset \mathbb{C}^n$ is plurisubharmonic if it is upper semi-continuous and such that the composition $u \circ \gamma$ is subharmonic in the unit disk $\mathbb{D}$ for any holomorphic mapping $\gamma\colon \mathbb{D} \to \omega$. By $\mathrm{PSH}(\omega)$ (resp., PSH_0) we denote the collection of all plurisubharmonic functions in ω (resp., germs of plurisubharmonic functions at $0 \in \mathbb{C}^n$).

A germ $u \in \mathrm{PSH}_0$ is said to be *singular* if $u(0) = -\infty$.

A basic characteristic of singularity of u is its *Lelong number*

$$\nu_u = \nu_u(0) = \liminf_{z\to 0} \frac{u(z)}{\log|z|};$$

this is the largest number $\nu \geq 0$ such that

$$u(z) \leq \nu \log|z| + O(1) \tag{1}$$

near 0. Equivalently,

$$\nu_u = dd^c u \wedge (dd^c \log|z|)^{n-1}(0), \tag{2}$$

where $d = \partial + \bar\partial$, $d^c = (\partial - \bar\partial)/2\pi i$.

If $f \in \mathcal{O}_0$ is a holomorphic function near 0, then $\nu_{\log|f|} = \mathrm{mult}_0 f$, the multiplicity of the f at 0. However, if $f = (f_1, \ldots, f_m)$ is a holomorphic mapping, then $\nu_{\log|f|} = \min_k \mathrm{mult}_0 f_k$ is far from the multiplicity of zero.

Let PSH_0^* be the germs that are locally bounded outside 0 (i.e., with *isolated singularity* at 0). The complex Monge-Ampère operator $(dd^c u)^n$ is well defined on such a function u ([6]), and we denote by $\tau_u = (dd^c u)^n(0)$ its *residual Monge-Ampère mass* at 0.

For the mappings f with an isolated zero, we have $\mathrm{mult}_0 f = \tau_{\log|f|}$ ([6]).

2. Relations between the characteristics of singularity. For a holomorphic mapping f to $\mathbb{C}^n$, by the local Bézout's theorem,

$$\mathrm{mult}_0 f \geq \prod_k \mathrm{mult}_0 f_k \geq (\min_k \mathrm{mult}_0 f_k)^n.$$

2010 *Mathematics Subject Classification:* 32U05, 32U25, 32U35.
Keywords: plurisubharmonic function; Lelong number; Monge-Ampère operator.
doi:10.15330/ms.45.1.104-108

Theorem 1 ([6]). *If PSH_0^*, then $\tau_u \ge \nu_u^n$.*

Proof. This follows from relations (1)–(2) and Demailly's Comparison Theorem ([6])

$$\text{(CT)} \quad u_j \le v_j + O(1) \Rightarrow \bigwedge_j dd^c u_j(0) \ge \bigwedge_j dd^c v_j(0)$$

applied to $u_j = u$, $v_j = \nu_u \log|z|$. □

No reverse bound is possible: take $u = \max\{k\log|z_1|, \log|z_2|\}$, then $\nu_u = 1$ while $\tau_u = k$.

Problem 1 (*Zero Lelong Number Problem*; V. Guedj, A. Rashkovskii, 1999): *Is the implication*

$$(P1) \quad \nu_u = 0 \;\Rightarrow\; \tau_u = 0$$

true whenever $(dd^c u)^n$ is well defined (e.g., for $u \in PSH_0^*$)? (This is Question 7 from [9].)

Note that even for $n = 2$ the condition $\nu_u = 0$ does not imply $dd^c u \wedge dd^c v(0) = 0$ for any v. For example, if $u = \max\{-|\log|z_1||^{1/2}, \log|z_2|\}$ and $v = \log|z_1|$, then $\nu_u = 0$ and $dd^c u \wedge dd^c v = \delta_0$.

3. Finite Łojasiewicz exponent. Denote

$$\gamma_u = \limsup_{z\to 0} \frac{u(z)}{\log|z|},$$

the Łojasiewicz exponent of u at 0.

By (CT), we get

Theorem 2. *Finite Łojasiewicz exponent*

$$(FLE) \quad \gamma_u < \infty$$

implies (P1).

Examples.

1. Any *analytic singularity* $u = \log|f| + O(1) \in PSH_0^*$ has FLE (the classical Łojasiewicz exponent of the mapping f).
2. *Multicircled singularities* $u(z) = u(|z_1|, \dots, |z_n|) + O(1) \in PSH_0^*$ have FLE ([12], [13]).

4. Greenifications. FLE condition might look too restrictive. But, for $n = 1$, any (pluri)subharmonic germ u at 0 represents as $u(z) = g_u + v(z)$, where $g_u(z) = \nu_u \log|z|$ and v is a (pluri)subharmonic function with zero Lelong number, so $\nu_{g_u} = \nu_u$ and g_u definitely has FLE.

For $n \ge 1$, the 'greenification' ([15]) of u is defined in a neighborhood ω of 0 by

$$g_u(z) := \limsup_{x\to z} \sup\{v(x) : \; v \in PSH(\omega),\; v \le 0,\; v \le u + O(1) \text{ near } 0\}.$$

If $u \in PSH_0^*$, then $(dd^c g_u)^n = 0$ on $\omega \setminus 0$; in other words, g_u is a *maximal singularity.* In addition, $\nu_{g_u} = \nu_u$ and $\tau_{g_u} = \tau_u$.

Theorem 3 ([15], [18]). *For $u \in \mathrm{PSH}_0^*$, $g_u \equiv 0$ if and only if $\tau_u = 0$.*

So: (P1) is equivalent to the following question: *For* $u \in \mathrm{PSH}_0^*$, *does* $\nu_u = 0$ *imply* $g_u \equiv 0$?

Problem 1. *Construct a maximal singularity* $\varphi \in PSH_0^*$ *with* $\gamma_\varphi = \infty$.

Remark 1. There exists $u \in PSH(\omega)$ with maximal singularity at 0, well-defined Monge-Ampère operator $(dd^c u)^n$, and such that the set $\{u = -\infty\}$ is dense ([1]).

By removing the condition of isolated singularity, one gets

Problem 2. *Is the implication*

$$(P1') \quad \nu_u = 0 \ \Rightarrow \ g_u \equiv 0$$

true for every $u \in \mathrm{PSH}_0$?

More classes with FLE property to come when considering Problem 2 below.

5. Intermediate Lelong numbers. For $u \in PSH_0^*$, denote

$$e_k = e_k(u) = (dd^c u)^k \wedge (dd^c \log |z|)^{n-k}(0), \quad k = 0, \dots, n,$$

so $e_0 = 1$, $e_1 = \nu_u$, $e_n = \tau_u$. If $u = \log |f|$ for a holomorphic mapping f with the zero set of codimension k, then e_k equals its multiplicity at 0.

As follows from the results of [4], these *intermediate Lelong numbers* satisfy $e_k^2 \le e_{k-1} \cdot e_{k+1}$ which was noticed in [8] (in the analytic setting $u = \log |f|$, it was established in [17]).

Theorem 4 ([8])**.** $e_1 = 0$ *implies* $e_k = 0$ *for all positive* $k < n$.

6. Demailly's approximations. Problem 1 can be approached by approximating u by functions for which the (affirmative) answer to Problem 1 is known: e.g., by those with analytic singularities.

We recall a procedure for analytic approximations due to J.-P. Demailly ([5]). Let $\{f_{k,m}\}_m$ be an orthonormal basis of the weighted Hilbert space

$$H_k(u) = \left\{ f \in \mathcal{O}(\omega) : \int_\omega |f|^2 e^{-2ku} dV < \infty \right\}, \quad u \in PSH(\omega).$$

Then the functions $\mathcal{D}_k u = \frac{1}{2k} \log \sum_m |f_{k,m}|^2 \in PSH(\omega)$ satisfy $u \le \mathcal{D}_k u + \frac{C}{k}$ and converge to u as $k \to \infty$ (in L^1_{loc} and pointwise). Moreover, $\nu_{\mathcal{D}_k u} \to \nu_u$.

Assume $u \in \mathrm{PSH}_0^*$, then $\mathcal{D}_k u \in \mathrm{PSH}_0^*$ have analytic singularities. The condition $\nu_u = 0$ implies $\nu_{\mathcal{D}_k u} = 0$, which in turn, since $\mathcal{D}_k u$ have analytic singularities, gives us $\tau_{\mathcal{D}_k u} = 0$, and it remains to relate these to τ_u.

Problem 2 (Demailly): *Is the convergence*

$$(P2) \quad \tau_{\mathcal{D}_k u} \to \tau_u$$

true? (Question 8 from [9].)

It is known ([2]), that the functions $\mathcal{D}_{2^k} u + \frac{C}{2^{k+1}}$ decrease to u and so, by [6], $(dd^c \mathcal{D}_{2^k} u)^n$ converge to $(dd^c u)^n$ as measures (which does not guarantee convergence of their masses at 0).

7. When (P2) is true. For some classes of functions, convergence (P2) is known. Namely:

1. *Analytic singularities* ([3]) $u = c\log|f| + O(1) \in PSH_0^*$, where f are holomorphic mappings with isolated zero.

Such functions constitute a particular case of

2. *Exponentially Hölder continuous* functions ([3]) $\varphi \in \mathrm{PSH}_0^*$, $e^{\varphi(x)} - e^{\varphi(y)} \le A|x-y|^\beta$, $\beta > 0$.

A more general class is:

3. *Tame singularities*: $\varphi \in \mathrm{PSH}_0^*$ with the property that there exists $C > 0$ such that $\forall t > C$ and every $f \in \mathcal{O}_0$ the condition $|f|e^{-t\varphi} \in L^2_{loc}$ implies $\log|f| \le (t-C)\varphi + O(1)$. These functions are characterized by inequalities ([3]) $u + O(1) \le \mathcal{D}_k u \le (1 - C/k)u + O(1)$, so (P2) follows from (CT).

And even more generally, the convergence holds for

4. *Asymptotically analytic singularities* ([16]): $\forall\epsilon > 0\ \exists\varphi_\epsilon$ with analytic singularities such that $(1+\epsilon)\varphi \le \varphi_\epsilon \le (1-\epsilon)\varphi$.

In particular, any isolated multicircled singularity is asymptotically analytic ([16]).

Theorem 5 ([16])**.** *$\varphi \in PSH_0^*$ is asymptotically analytic if and only if the greenifications $g_{\mathcal{D}_k\varphi}$ satisfy $g_{\mathcal{D}_k\varphi}/g_\varphi \to 1$ uniformly on $\omega \setminus 0$.*

Since the greenifications of analytic singularities are continuous ([19]), convergence $g_{\mathcal{D}_k\varphi}/g_\varphi \to 1$ implies $g_\varphi \in C(\omega)$ for an asymptotically analytic φ.

Problem 2.1. *Construct $u \in PSH_0^*$ with discontinuous g_u.*

Problem 2.2. *Construct $u \in PSH_0^*$ whose singularity is not asymptotically analytic.*

The *type of* $u \in \mathrm{PSH}_0$ *relative to a maximal weight* $\varphi \in PSH_0^*$ ([16]) is

$$\sigma(u,\varphi) = \liminf_{z\to 0} \frac{u(z)}{\varphi(z)}.$$

Equivalently, it is the greatest $\sigma \ge 0$ such that $u(z) \le \sigma\varphi(z) + O(1)$.

Example. $\nu_u = \sigma(u, \log|z|)$. More generally, let $\phi_a(z) = \max_i a_i^{-1}\log|z_i|$, $a_i > 0$, then $\sigma(u,\phi_a)$ is Kiselman's directional Lelong number ([11]) in the direction $a = (a_1,\ldots,a_n)$.

It is known that the Lelong numbers of $\mathcal{D}_k u$, both classical and directional, converge to those of u ([5], [13]), and the same do their log canonical thresholds ([7]).

Theorem 6 ([16])**.** *The types $\sigma(\mathcal{D}_k u,\varphi) \to \sigma(u,\varphi)$ for any $u \in PSH_0$ if and only if φ has asymptotically analytic singularity.*

8. Functions with (P2) property.

Theorem 7 ([16])**.** *For $u \in PSH_0^*$, TFAE:*

(i) $\sup_k \tau_{\mathcal{D}_k u} = \tau_u$;

(ii) $\lim_{k\to\infty} \tau_{\mathcal{D}_k u} = \tau_u$;

(iii) $\inf_k g_{\mathcal{D}_k u} = g_u$;

(iv) $\lim_{k\to\infty} g_{\mathcal{D}_k u} = g_u$;

(v) *there exist analytic singularities* $\varphi_j \ge u$ *such that* $\tau_{\varphi_j} \to \tau_u$*;*

(vi) *there exist maximal analytic singularities* φ_j *decreasing to* g_u*;*

(vii) *there exist* $s_k > 0$ *and divisorial valuations* $\mathcal{R}_k$*,* $k = 1, 2, \ldots$*, such that* $\sigma(v, g_u) = \inf_k s_k \mathcal{R}_k(v)\ \forall v \in PSH_0$*.*

Problem 3. *Is (P2) true for every u with FLE?*

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Received 4.04.2016