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ON FINITE 2-GROUPS WITH NON-DEDEKIND NORM OF ABELIAN NON-CYCLIC SUBGROUPS

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The authors study finite 2-groups with the cyclic center and non-metacyclic non-Dedekind norm of Abelian non-cyclic subgroups. It is found out that such groups are cyclic or metacyclic extensions of their norms of Abelian non-cyclic subgroups. Their structure is described.

1. Introduction. In the group theory findings related to the study of properties of groups with given restrictions on their subgroups and systems of such subgroups are in the focus. In some cases, the group may have a system of subgroups with certain properties, but the impact of this system of subgroups is not significant. On the other hand, the presence of one (usually a characteristic) subgroup with a certain property can be the determining factor for the structure of the group. Nowadays the list of such subgroups can be significantly broaden by means of different Σ-norms of a group.

Let us regard that the Σ-norm of a group G is the intersection of normalizers of all subgroups of a group, included in the system Σ. Obviously, any subgroup which belongs to Σ (assuming that the system Σ is non-empty) is normal in a group in the case of the coincidence of the Σ-norm with a group. For the first time groups with this property were considered in the second part of the XIX century by R. Dedekind, who gave a complete description of finite groups, all subgroups of which are normal (now they are called Dedekind groups). However, a systematic study of groups with arbitrary systems of normal subgroups was continued only in the second part of the XX century that stopped the study of the Σ-norms in a certain way. Currently the structure of groups, which coincide with their Σ-norms, is known for many systems Σ of subgroups. So it is naturally to raise the question about the study of the properties of groups which have proper Σ-norm.

For the first time this problem was posed by R. Baer in 30-s of the previous century (see, e.g. [1]) for the system Σ of all subgroups of this group. Such Σ-norm was called the norm of a group and denoted by $N(G)$. It is clear that the norm $N(G)$ is contained in all other Σ-norms, and those ones can be considered as its generalizations.

The authors continue the study of groups with non-Dedekind Σ-norm, started in [2]–[6] for systems Σ of all Abelian non-cyclic subgroups of a group G, provided that the system of such subgroups in a group is non-empty. In [2] such Σ-norm was called the norm of Abelian

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non-cyclic subgroups and denoted by N_G^A. If $G = N_G^A$, then all Abelian non-cyclic subgroups are normal in G. Periodic non-Abelian groups with this property were studied in [7] and were called $\overline{HA}$-groups (or $\overline{HA}_p$-groups in the case of p-groups).

In [3]–[4] the authors specified infinite locally finite p-groups with non-Dedekind norm of Abelian non-cyclic subgroups. It was found out that all such groups are finite extensions of a quasicyclic subgroup and such groups are $\overline{HA}_p$-groups in the case of $p\neq 2$ or in the case of the infinite norm N_G^A.

Finite p-groups ($p \neq 2$) with non-Dedekind norm of Abelian non-cyclic subgroups were described in [5]. A complete characterization of finite 2-groups with the non-cyclic center and non-Dedekind norm of Abelian non-cyclic subgroups was given in [6].

The purpose of this paper is to study finite 2-groups with the cyclic center and non-metacyclic non-Dedekind norm N_G^A of Abelian non-cyclic subgroups. It is found out that such groups are cyclic or metacyclic extensions of their norms of Abelian non-cyclic subgroups and their structure is described.

This article uses the following notations:

- E the indentity subgroup;
- $Z(G)$ the center of a group G;
- G' the derived subgroup of a group G;
- $A \leftthreetimes B$ the semidirect product of subgroups A and B;
- $\omega_m(G)$ the subgroup, which is generated by all elements, which order does not exceed 2^m, of a group G. In particular, $\omega_1(G){=}\omega(G)$ is the lower layer of a group G. It is the subgroup which is generated by all elements of order 2 of a group G.

2. Preliminary results. The next statement is actively used in the further research.

Proposition 1 (Theorem 1 [7])). *Every finite non-metacyclic non-Hamiltonian $\overline{HA}_2$-group is a group of one of the following types:*

1. $G = (\langle a\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle$, *where* $|a| = 2^n,\ n \geq 2,\ |b| = |c| = 2,\ [a, b] = [a, c] = 1,\ [b, c] = a^{2^{n-1}}$;
2. $G = (H \times \langle b\rangle) \leftthreetimes \langle c\rangle$, *where* $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4,\ h_1^2 = h_2^2 = [h_1, h_2]$, $|b| = |c| = 2$, $[H, \langle b\rangle] = [H, \langle c\rangle] = E$, $[b, c] = h_1^2$;
3. $G = (\langle a\rangle \times \langle b\rangle)\langle c\rangle$, *where* $|a| = |b| = |c| = 4$, $c^2 = a^2b^2$, $[c, b] = c^2, [c, a] = a^2$;
4. $G = (\langle a\rangle \times \langle b\rangle)\langle c\rangle\langle d\rangle$, *where* $|a| = |b| = |c| = |d| = 4$, $c^2 = d^2 = a^2b^2$, $[a, c] = [d, c] = a^2$, $[b, d] = b^2$, $[c, b] = [d, a] = c^2$;
5. $G = H \times \langle c\rangle$, *where H is a quaternion group,* $|c| = 2^n,\ n \geq 2$;
6. $G = H \times Q$, *where H and Q are quaternion groups;*
7. $G = (H \times \langle b\rangle)\langle c\rangle$, *where* $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = |b| = |c| = 4$, $h_1^2 = h_2^2 = [h_1, h_2]$, $c^2 = b^2h_1^2$, $[b, c] = b^2$, $[H, \langle b\rangle] = [H, \langle c\rangle] = E$;
8. $G = \langle c\rangle \leftthreetimes H$, *where* $H = \langle h_1, h_2\rangle$,$|h_1| = |h_2| = 4,\ h_1^2 = h_2^2 = [h_1, h_2]$, $|c| = 2^n > 2$, $[c, h_1] = c^{2^{n-1}}$, $[c, h_2] = 1$;
9. $G = (H \times \langle a\rangle)\langle b\rangle$, *where* $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4, |a| = 2, |b| = 8,\ h_1^2 = h_2^2 = [h_1, h_2] = [a, b]$, $b^2 = h_1$,$[h_2, b] = a$.

The following result directly follows from Proposition 1.

Corollary 1. *Let G be a finite 2-group with the cyclic center and non-metacyclic non-Dedekind norm N_G^A of Abelian non-cyclic subgroups. If $G = N_G^A$, then G is a group of one of the types (1), (2), (9) of Proposition 1.*

Lemma 1 ([5])**.** *Let G be a locally finite 2-group with non-Dedekind norm N_G^A of Abelian non-cyclic subgroup. If the center $Z(N_G^A)$ is cyclic, then the central involution a is in every cyclic subgroup of composite order of a group G.*

Lemma 2. *If the norm N_G^A of Abelian non-cyclic subgroups of a finite 2-group G is non-Dedekind and its lower layer $\omega(N_G^A)$ is an elementary Abelian subgroup of order 4, then N_G^A contains all involutions of a group G and $\omega(N_G^A) = \omega(G)$.*

Proof. Let a group G and its norm N_G^A of Abelian non-cyclic subgroups satisfy the conditions of the lemma. Then N_G^A is a group of one of the types (3)–(9) of Proposition 1. By the condition $\omega(N_G^A) \lhd N_G^A$ and the fact, that the subgroup $\omega(N_G^A)$ is characteristic in N_G^A, we have $\omega(N_G^A) \lhd G$ and $\omega(N_G^A) \cap Z(G) \neq E$. Let $\omega(N_G^A) = \langle a_1\rangle \times \langle a_2\rangle$, where $|a_1| = |a_2| = 2$, $a_1 \in Z(G)$ for the definiteness.

Suppose that G contains an involution $x \notin N_G^A$. Then the subgroup $\langle a_1, x\rangle$ is Abelian and normal in the group $G_1 = \langle x\rangle N_G^A$. Since $[G_1 : C_{G_1}(\langle a_1, x\rangle)] \leq 2$, $[y^2, x] = 1$ for any element $y \in N_G^A$. If N_G^A is a group of one of the types (3)–(8) of the Proposition, then

$$\left[\left(N_G^A\right)^2, \langle x\rangle\right] = \left[\omega\left(N_G^A\right), \langle x\rangle\right] = E.$$

It follows that $\langle x\rangle \lhd G_1$ as the intersection of normal subgroups $\langle a_1, x\rangle$ and $\langle a_2, x\rangle$. Hence $G_1 = \langle x\rangle \times N_G^A$ is a non-Hamiltonian $\overline{HA}_2$-group, which contains an elementary Abelian subgroup of order 8. But it contradicts to Lemma 2 ([7]). Therefore in this case $\omega(N_G^A) = \omega(G)$.

Suppose that N_G^A is a group of the type (9) of the proposition. Then $Z(N_G^A) = \langle h_1^2\rangle$, where $h_1 \in H, |h_1| = 4$ and $h_1^2 = a_1 \in Z(G)$. As it is shown above $\left[\langle x\rangle, N_G^A\right] \subseteq \langle a_1\rangle = \langle h_1^2\rangle$ for the involution $x \notin N_G^A$. Hence $[x, b^2] = [x, h_1] = 1$. If $[x, h_2] = 1$, then $\langle x, h_2\rangle \cap N_G^A = \langle h_2\rangle \lhd N_G^A$, which is impossible. Thus $[x, h_2] = h_1^2$ and $|xh_2| = 2$. Since

$$xh_2 \notin N_G^A,\ [xh_2, b] \in \left\langle h_1^2 \right\rangle,\ \left[xh_2, b^2\right] = [xh_2, h_1] = 1.$$

On the other hand, $[xh_2, h_1] = [h_2, h_1] = h_1^2 \neq 1$, which contradicts to the proved above. Thus $\omega(N_G^A) = \omega(G)$. Lemma is proved. □

Corollary 2. *If the norm N_G^A of Abelian non-cyclic subgroups of a finite 2-group G is non-Dedekind and has the non-cyclic center $Z(N_G^A)$, then $\omega(N_G^A) = \omega(G)$.*

Lemma 3. *If the norm N_G^A of Abelian non-cyclic subgroups of a finite 2-group G is non-Dedekind, has the non-cyclic center and the non-central lower layer $\omega(N_G^A)$ in G, then $G = C\langle y\rangle$, where $C = C_G\left(\omega(N_G^A)\right), C \lhd G, |y| > 4, y^2 \in C$. In this case, every Abelian non-cyclic subgroup of a group G is contained in C and $N_G^A = N_C^A \subseteq C$.*

Proof. By the assymptions of the lemma the norm N_G^A is a group of one of the types (3)–(8) of Proposition. In each of these cases $\omega(N_G^A)$ is an elementary Abelian subgroup of order 4 and $\omega(N_G^A) \not\subset Z(G)$ according to the hypothesis of the lemma.

Let's denote $C = C_G\left(\omega(N_G^A)\right)$. Since $\omega(N_G^A) \lhd G, C \lhd G, [G : C] = 2$. Thus $G = C\langle y\rangle$, where $y^2 \in C$.

Since $\omega(N_G^A) \subseteq Z(N_G^A)$, $N_G^A \subseteq C$ and $y \notin N_G^A$. According to Lemma 2 $\omega(N_G^A) = \omega(G)$, so $|y| > 2$. Let $|y| = 4$, then the subgroup $\langle y\rangle \omega(G)$ is a dihedral group of order 8. Since $\langle y\rangle \omega(G) = \langle y, b\rangle$, $|yb| = 2$. But $yb \in \omega(G)$ and $y \in \omega(G)$ by such conditions, which is impossible. Thus $|y| > 4$. Taking into account that every Abelian non-cyclic subgroup contains $\omega(N_G^A)$, we conclude that it is contained in C. Therefore $N_G^A = N_C^A \subseteq C$. □

Lemma 4. *Let G be a finite 2-group with non-Dedekind norm N_G^A of Abelian non-cyclic subgroups, which is a group of one of the types (3)–(7) of the proposition. Then the center $Z(G)$ of a group G is non-cyclic.*

Proof. Let N_G^A be a group of one of the types which noted in the assumptions of the lemma. Then the center $Z(N_G^A)$ of the norm N_G^A is non-cyclic. If the norm N_G^A is a group of the type (5) of the proposition, then $\omega(N_G^A) \subseteq Z(G)$ and a group G has the non-cyclic center.

So we will assume that N_G^A is a group of one of the types (3)–(4) or (6)–(7). In each of these cases $\omega(N_G^A)$ is an elementary Abelian subgroup of order 4. Since $\omega(N_G^A) \subseteq Z(N_G^A)$, $\omega(N_G^A) = \omega(G)$ according to Lemma 2.

Suppose $\omega(N_G^A) \not\subseteq Z(G)$ contrary to the assertion of the lemma. Then $\omega(N_G^A) \cap Z(G) \neq E$ by the condition $\omega(N_G^A) \lhd G$. Let $\omega(N_G^A) = \langle a_1\rangle \times \langle a_2\rangle$, $|a_1| = |a_2| = 2$, where $a_1 \in Z(G)$ and $a_2 \notin Z(G)$.

Let us denote $C = C_G\left(\omega(N_G^A)\right)$, $G = C \cdot \langle y\rangle$, $|y| > 4$, $y^2 \in C$ by Lemma 3, C contains all Abelian non-cyclic subgroups of G, $N_G^A \subseteq N_C^A$ and $N_G^A = N_C^A$. Since the norm N_C^A is non-metacyclic and $Z(C)$ is non-cyclic, C is either non-metacyclic non-Dedekind $\overline{HA}_2$-group by Theorem 12 ([6]) and $C = N_C^A = N_G^A$, or $C = H \cdot Q$ is a product of a quaternion group $H = \langle h_1, h_2\rangle$ of order 8 and a generalized quaternion group $Q = \langle t, q\rangle$, $|t| = 2^k > 8$, $t^{2^{k-1}} = q^2$, $q^{-1}tq = t^{-1}$, $[H, Q] \subseteq \omega(C)$ and $N_C^A = N_G^A = \langle t^{2^{k-2}}\rangle \times H$.

In the previous case N_G^A is a group of the type (5) of Proposition 1 and it is easy to prove that the center $Z(G)$ of a group is non-cyclic.

Thus we will assume that $C = N_G^A$ and $G = N_G^A \cdot \langle y\rangle$, where $y^2 \in N_G^A$. In this case N_G^A is a non-Dedekind $\overline{HA}_2$-group of exponent 4. So $|y| = 8$, $y^4 = a_1 \in Z(G)$ by Lemma 3. It is also easy to prove that the norm N_G^A contains all elements of order 4 of a group G.

Let us consider the quotient-group $\overline{G} = G/\omega(G) \cong \overline{N_G^A} \cdot \langle \overline{y}\rangle$, $\overline{y}^2 \in \overline{N_G^A}$, where $|\overline{y}| = 4$. Since $\omega\left(\overline{G}\right) = \overline{N_G^A} \lhd \overline{G}$, $|\overline{N_G^A}| \geq 8$ and $\overline{y}$ induces an automorphism of order 2 on $\omega(\overline{G})$, there is an involution $\overline{z}$ such that $\langle \overline{y}\rangle \cap \langle \overline{z}\rangle = \overline{E}$ and $[\overline{z}, \overline{y}] = 1$ in $\omega(\overline{G})$. Turning to the preimages, we have $[z, y] = a$, where $a \in \omega(G)$. Since $[z^2, y] = 1$, we conclude that $z^2 = a_1$. Let $a \in \langle a_1\rangle$, then $[z, y^2] = 1$ and $|y^2 z| = 2$. But in this case $y^2 \in \langle z\rangle \omega(G)$ and the intersection $\langle \overline{y}\rangle \cap \langle \overline{z}\rangle$ is non-identity in the quotient-group $\overline{G}$. It is a contradiction. Thus $a \notin \langle a_1\rangle$ and we can assume without loss of generality that $a = a_2$. Then $y^{-1}zy = za_2$, $[z, y^2] = z^2 = a_1$, and $\langle y^2, z\rangle$ is a quaternion group, that is impossible if the norm N_G^A is a group of the type (3) or (4) of Proposition 1.

Let N_G^A contains a quaternion group, i.e. N_G^A is a group of the type (4) or (7) of the proposition. Then $G = H \cdot Q$ is a direct or a semidirect product of two quaternion groups H and Q, $[H, Q] \subseteq Q^2$.

Let us consider the group $G = N_G^A \cdot \langle y\rangle$, where $y^2 \in N_G^A$. So $\langle y^2, a_2\rangle$ is Abelian non-cyclic by the inclusion $\omega(N_G^A) \subseteq Z(N_G^A)$ and therefore $\langle y^2, a_2\rangle$ is a normal subgroup in G. The subgroup $\widetilde{N_G^A}$ is an elementary Abelian of order 8 in the quotient-group $\widetilde{G} = G/\langle y^2, a_2\rangle \cong \widetilde{N_G^A} \leftthreetimes \langle \tilde{y}\rangle$. Since $\tilde{y}$ induces an automorphism of order 2 on $\widetilde{N_G^A}$, it is always possible to point involutions

$\widetilde{z_1},\ \widetilde{z_2} \in \widetilde{N_G^A}$, which are permutable with $\tilde{y}$. Turning to preimages we get that $[z_i, y] = y^{2m_i} a^{s_i}$, $i = 1, 2$.

If $s_1 = s_2 = 1$, then $[z_1 z_2, y] = y^{2t}$. If $(t, 2) = 1$, then $|y z_1 z_2| \le 4$ and $y \in N_G^A$ by the proved, which is impossible. Thus $t = 2t_1$ and $[z_1 z_2, y] = y^{4t_1} \in Z(G)$. But $[z_1 z_2, y^2] = \left[(z_1 z_2)^2, y\right] = 1$ by such conditions. From the second part of the equality we have $(z_1 z_2)^2 = a_1 = y^4$ and $|z_1 z_2 y^2| = 2$, which contradicts to the structure of the norm N_G^A.

Thus we can assume that at least one of numbers $s_i = 0$. But then $[z_i, y] = y^{2m_i}$ and we again get the contradiction repeating the above argument. In this case $G = C$ and $\omega\left(N_G^A\right) \subseteq Z(G)$. $\square$

A group, which has the norm N_G^A of one of the types (3)–(7) of the proposition, has the non-cyclic center by Lemma 4. Such groups were studied in [6]. So it remains to study the groups, in which the norm N_G^A is a group of one of the types (1), (2), (8)–(9) of the proposition.

The following example illustrates that in the case, when the norm N_G^A is a group of the type (8) of Proposition 1, the center $Z(G)$ of a group can be cyclic.

Example. $G = (\langle b\rangle \leftthreetimes H)\langle y\rangle$, where $|b| = 4$, $H = \langle h_1, h_2\rangle$, $|h_1| = 4$, $[h_1, h_2] = h_1^2 = h_2^2$, $[b, h_2] = 1$, $y^2 = h_1$, $[y, h_2] = b^2 h_1^2$, $[y, b] = h_2$.

In this group all Abelian non-cyclic subgroups are contained in the group $(\langle b\rangle \leftthreetimes H)$ and are normal in it, so it is easy to verify that $N_G^A = (\langle b\rangle \leftthreetimes H)$. Thus $Z(G) = \langle h_1^2\rangle$ is cyclic.

Lemma 5. *Let G be a finite 2-group, the norm N_G^A of Abelian non-cyclic subgroups of which is a group of the type (9) of Proposition 1. Then all Abelian non-cyclic subgroups are normal in G and $G = N_G^A$.*

Proof. Let N_G^A is a group of the type (9) of Proposition 1, i.e. $N_G^A = (H \times \langle a\rangle)\langle b\rangle$, where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $|a| = 2$, $|b| = 8$, $b^2 = h_1$, $[h_2, b] = a$, $[a, b] = [h_1, h_2] = h_1^2 = h_2^2$. In particular, $\omega\left(N_G^A\right) = \langle h_1^2, a\rangle$ and $Z\left(N_G^A\right) = \langle h_1^2\rangle \subset Z(G)$.

N_G^A contains all elements of order 2 of a group G by Lemma 2 and $\omega\left(N_G^A\right) = \omega(G)$. Let's denote $C = C_G(\omega(G))$. Then $[G : C] = 2$ and $G = C\langle b\rangle$, $b^2 \in C$. By the proved above, the lower layer $\omega\left(N_G^A\right)$ contains all involutions of the centralizer C, so the quotient-group $\overline{C} = C/\langle a\rangle$ contains only one involution by Lemma 1. Since $\overline{C}$ is non-Abelian, $\overline{C}$ is a quaternion 2-group:

$$\overline{C} \cong \overline{Q} = \langle \overline{x}, \overline{y}\rangle, \ |\overline{x}| = 2^n \ge 4, \ |\overline{y}| = 4, \ \overline{x}^{2^{n-1}} = \overline{y}^2, \ \overline{y}^{-1}\overline{xy} = \overline{x}^{-1}.$$

Turning to the preimages and taking into account Lemma 2, we have that $x^{2^{n-1}} = y^2 = h_1^2, y^{-1}xy = x^{-1}a^m,\ m \in \{0, 1\}$. If $m = 1$, then $y^{-1}xy = x^{-1}a$ and $(xy)^2 = h_1^2 a \notin \langle h_1^2\rangle$, which is impossible. Therefore $m = 0$, $y^{-1}xy = x^{-1}$ and $C = Q \times \langle a\rangle$. We can assume without loss of generality, that $H \subseteq Q$, $h_1 \in \langle x\rangle, \langle h_2\rangle = \langle y\rangle$. If $|Q| > 8$, then $h_2 \notin N_G(\langle a, x h_2\rangle)$, which is impossible, because $h_2 \in N_G^A$. Thus $Q = H$, $C = H \times \langle a\rangle \subset N_G^A$ and $G = C\langle b\rangle = N_G^A$. $\square$

Lemma 6. *If a finite 2-group G has the norm N_G^A of Abelian non-cyclic subgroups, which is a group of the type (2) of Proposition 1, then $G = N_G^A$.*

Proof. Let a group G and its norm N_G^A satisfy the conditions of the lemma, $N_G^A = (H \times \langle b\rangle) \leftthreetimes \langle c\rangle$, where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $[h_1, h_2] = h_1^2 = h_2^2$, $|b| = |c| = 2$, $[H, \langle b\rangle] = [H, \langle c\rangle] = E$, $[b, c] = h_1^2$.

Suppose that $G \neq N_G^A$ and let us prove that N_G^A contains all involutions of a group G. Indeed, otherwise we have $\langle z, h_1^2\rangle \lhd G_1 = \langle z\rangle N_G^A$ for any involution $z \in G\backslash N_G^A$. Therefore $[G_1 : C_{G_1}(\langle z, h_1^2\rangle)] \leq 2$ and $G_1\backslash \langle h_1^2\rangle$ contains an involution $y \neq h_1^2$, which is permutable with z. So it follows $\langle y, z\rangle \cap N_G^A = \langle y\rangle \lhd N_G^A$, which is impossible. Hence all involutions of a group G are contained in N_G^A.

Suppose that an element x of order 4 exists in $G\backslash N_G^A$. By Lemma 1 $x^2 = h_1^2$. Thus any element a of order 4 of the norm N_G^A is not permutable with x, otherwise $|ax|=2$ and $x \in N_G^A$ by the proved above. Let's denote $G_2 = \langle x\rangle N_G^A$ and consider the quotient-group $\overline{G_2} = G_2/\langle h_1^2\rangle$. Since $\overline{N_G^A}$ is an elementary Abelian group of order 16, normal in $\overline{G_2}$ and $\overline{x}$ induces an automorphism of order 2 on $\overline{N_G^A}$, there exist involutions $\overline{y_1},\ \overline{y_2} \in \overline{N_G^A},\quad \langle\overline{y_1}\rangle \cap \langle\overline{y_2}\rangle = \overline{E}$, which are permutable with $\overline{x}$. Turning to the preimages we will have $[x, y_i] \in \langle h_1^2\rangle,\ i = 1, 2$. It is easy to prove that the group $\langle y_1, y_2\rangle$ contains an involution $y \neq h_1^2$, which is permutable with x. Then $\langle x, y\rangle \lhd G_2$ as an Abelian non-cyclic subgroup and $G_2' \subseteq \langle x, y\rangle \cap N_G^A = \langle y, h_1^2\rangle$.

Let t be an arbitrary non-central involution of N_G^A, which differs from y. Let us put $[x, t] = y^m h_1^{2k},\quad m, k \in \{0, 1\}$. Then $[x, t^2] = h_1^{2m}$. On the other hand, $[x, t^2] = 1$, therefore $m = 0$ and $\left[\langle x\rangle, N_G^A\right] \subseteq \langle h_1^2\rangle$. However in this case the group G_2 will contain an involution, which does not belong to N_G^A, that contradicts to the proved above. Therefore N_G^A contains all elements of order 4 of a group G.

According to the assumption $G \neq N_G^A$, we conclude that there is an element $x \in G\backslash N_G^A$, $|x| = 8$. Since $x^2 \in N_G^A,\ |x^2| = 4$ and all cyclic subgroups of order 4 are normal in N_G^A, $\langle x^2\rangle \lhd G_3 = \langle x\rangle N_G^A$. Let us consider the quotient-group $\overline{G_3} = G_2/\langle x^2\rangle$. Since $\overline{N_G^A}$ is a normal elementary Abelian group of order 8 and $\overline{x}$ induces an automorphism of order 2 on it, there exist involutions $\overline{y_1},\ \overline{y_2} \in \overline{N_G^A},\ \langle\overline{y_1}\rangle \cap \langle\overline{y_2}\rangle = \overline{E}$, which are permutable with $\overline{x}$. Turning to the preimages we get $[x, y_i] \in \langle x^2\rangle,\ i = 1, 2$. It is easy to verify that $[x, y_i] \in \langle h_1^2\rangle$ and the group $\langle x^2,\ y_1,\ y_2\rangle$ contains an involution y, which is permutable with x. Then $\langle x, y\rangle \lhd G_3$ as Abelian non-cyclic subgroup and $G_3' \subseteq \langle x, y\rangle \cap N_G^A = \langle y, x^2\rangle$.

Let $[x, t] = x^{2m} y^k$, where t is an arbitrary non-central involution of N_G^A, which differs from y. Since N_G^A contains all elements of order 4, $[x, t] \in \langle h_1^2\rangle$ by the condition $[x, t^2] = 1$. But then $[x^2, t] = 1$ and $x^2 \in Z(G_3)$, that is impossible, because the norm N_G^A does not contain non-central elements of order 4. This contradiction proves that $G = N_G^A$. □

3. The main results.

Theorem 1. *Finite 2-groups with non-metacyclic non-Dedekind norm N_G^A of Abelian non-cyclic subgroups and the cyclic center are groups of the following types:*

1. *G is a non-metacyclic non-Hamiltonian $\overline{HA_2}$-group with a cyclic center, $G = N_G^A$;*
2. *$G = (\langle x\rangle \leftthreetimes \langle b\rangle) \leftthreetimes \langle c\rangle,\ |x| = 2^n, n > 3,\ |b| = |c| = 2,\ [x, c] = x^{\pm 2^{n-2}} b,\quad [b, c] = [x, b] = x^{2^{n-1}},\ N_G^A = (\langle x^2\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle$;*
3. *$G = (\langle x\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle \leftthreetimes \langle d\rangle,\ |x| = 2^n, n > 2,\ |b| = |c| = |d| = 2, [x, c] = [x, b] = 1,\ [b, c] = [c, d] = [b, d] = x^{2^{n-1}},\ d^{-1} x d = x^{-1},\quad N_G^A = (\langle x^{2^{n-2}}\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle$;*
4. *$G = (\langle c\rangle \leftthreetimes H)\langle y\rangle, H = \langle h_1, h_2\rangle,\ |h_1| = |h_2| = 4, h_1^2 = h_2^2 = [h_1, h_2], |c| = 4,\ [c, h_1] = c^2,\ [c, h_2] = 1,\ y^2 = h_1, [y, h_2] = c^2 h_1^2,\ [y, c] = h_2^{\pm 1}, N_G^A = \langle c\rangle \leftthreetimes H$.*

Proof. Let a group G and its norm of Abelian non-cyclic subgroups satisfy the conditions of Theorem. If $G = N_G^A$, then according to Corollary 1 G is a group of the type (1) of

Theorem 1. Therefore we will assume that $G \neq N_G^A$. Since G has the cyclic center and non-Dedekind norm N_G^A, the norm N_G^A is a group of one of the types (1) or (8) of Proposition 1 by Lemmas 4–6.

We will continue the proof of Theorem 1 in the following lemmas. □

Lemma 7. *If a finite 2-group G has a non-Dedekind norm $N_G^A \neq G$, which is a group of the type (1) of Proposition, then G is a group of one of the types (2) or (3) of Theorem 1.*

Proof. Let $G \neq N_G^A$ and $N_G^A = (\langle a\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle$, where $|a| = 2^n, n \geq 2$, $|b| = |c| = 2$, $[a, c] = [a, b] = 1, [b, c] = a^{2^{n-1}}$. Since $N_G^A \lhd G$, the intersection $\overline{N_G^A} \cap Z(\overline{G}) \neq \overline{E}$ in the quotient-group $\overline{G} = G/\langle a\rangle$. We can assume without loss of generality that $\overline{b} \in Z(\overline{G})$. Then $\langle a, b\rangle \lhd G, \omega(\langle a, b\rangle) = \langle a^{2^{n-1}}, b\rangle \lhd G$.

Let us denote $C = C_G(\langle a^{2^{n-1}}, b\rangle)$. Then $C \lhd G, [G : C] = 2$ and $G = C \leftthreetimes \langle c\rangle$, where $b \in N_G^A$, $|c| = 2$. By Lemma 1 the quotient-group $\overline{C} = C/\langle a\rangle$ has only one involution and $\overline{C}$ is a cyclic group or a generalized quaternion group.

Let $\overline{C}$ be cyclic, then its full preimage $C = \langle x\rangle \times \langle b\rangle$ is Abelian and $[x, c] \in C \cap N_G^A = \langle a, b\rangle$. Let's put $[x, c] = a^m b^k$. If $|[x, c]| = 2$, then $G' \subset \langle a^2\rangle$ and G is a $\overline{HA}_2$-group contrary to the assumption. Thus $|[x, c]| > 2$. If $|a| = 4$, then $[x, c] = a^{\pm 1} b$ by the condition $[x, c^2] = 1$, so $(xc)^2 \in Z(G)$ and $|x| \leq 8$. So $x^2 = a^{\pm 1} b$. However, $c \notin N_G(\langle a^2\rangle \times \langle xbc\rangle)$ by such conditions, i.e. $c \notin N_G^A$, which is impossible.

Let $|a| > 4$, then $m = 2^{n-2} m_1$, where $(m_1, 2) = 1$, $(k, 2) = 1$. Thus $[x, c] = a^{\pm 2^{n-2}} b$, $(xc)^2 = x^2 a^{\pm 2^{n-2}} b$ and $(xc)^2 \in Z(G)$. Since $Z(G) = \langle a\rangle, |x| > |a|$, we can consider that $(xc)^2 = a$. Let's denote $xc = y$. Then $|y| = 2^{n+1}$, $[y, b] = y^{2^n}$, $[y, c] = y^{\pm 2^{n-1}} b$ and $G = (\langle y\rangle \leftthreetimes \langle b\rangle) \leftthreetimes \langle c\rangle$ is a group of the type (2) of Theorem 1.

Let $\overline{C}$ be a generalized quaternion group $\overline{C} = \langle \overline{h_1}, \overline{h_2}\rangle$, where $|\overline{h_1}| = 2^n$, $n \geq 2, |\overline{h_2}| = 4$, $\overline{h_1}^{2^{n-1}} = \overline{h_2}^2$, $\overline{h_2}^{-1}\overline{h_1 h_2} = \overline{h_1}^{-1}$. Let h_1 and h_2 denote the preimages of elements $\overline{h_1}$ and $\overline{h_2}$ respectively. Since the center $Z(G)$ is cyclic, then $h_1^{2^{n-1}} = h_2^2 = a^{2^{n-1}}$, $h_2^{-1} h_1 h_2 = h_1^{-1} b^m$, $m \in \{0, 1\}$ by Lemma 1. If $m \neq 0$, then $(h_1 h_2)^2 = h_2^2 b = a^{2^{n-1}} b$, which contradicts to Lemma 1. Thus $m = 0$, $C = H \times \langle b\rangle$, $H = \langle h_1, h_2\rangle$ is a generalized quaternion group. We also note that $\langle a\rangle \subseteq \langle h_1\rangle$ by the condition $\langle a\rangle \lhd G$.

Since $[h_2, c] \in \langle h_2, b\rangle \cap \langle b, c\rangle = \langle a^{2^{n-1}}, b\rangle$ and $[h_2, c^2] = 1$, we conclude that $[h_2, c] \in \langle a^{2^{n-1}}\rangle$. Then one of the elements $h_2 c$ or $h_2 bc$ will be of order 2, and hence one of the subgroups $\langle h_2 c,\ a^{2^{n-1}}\rangle$ or $\langle h_2 bc,\ a^{2^{n-1}}\rangle$ is elementary Abelian. Since $\langle a\rangle \subseteq N_G^A$, the element a has to normalize these subgroups, which is possible only if $|a| = 4$.

Based on the fact that $\langle h_1 h_2\rangle \times \langle b\rangle$ is an Abelian non-cyclic subgroup, we have $[h_1 h_2, c] \in (\langle h_1 h_2\rangle \times \langle b\rangle) \cap N_G^A = \langle a^2, b\rangle$. It is easy to prove that $[h_1 h_2, c] \in \langle a^2\rangle$ by Lemma 1. It also follows that $[h_1, c] \in \langle a^2\rangle$. Thus $[H,\ N_G^A] = \langle a^2\rangle$.

Let's denote $B = \langle b, c\rangle$. Since B is a 2-generated non-Abelian subgroup and the commutant $[B, G] \subseteq \langle a^2\rangle$ is of order 2, $G = BC_G(B)$ by [9]. We can assume without loss of generality that $H = C_G(B)$. If $|H| = 8$, then G is a $\overline{HA}_2$-group, which contradicts to the assumption. So $|H| > 8$ and G is a group of the type (3) of Theorem 1. □

Lemma 8. *If a finite 2-group G has the norm $N_G^A \neq G$ which is a group of the type (8) of Proposition, then G is a group of the type (4) of Theorem 1.*

Proof. Let N_G^A be a group of the type (8) of Proposition: $N_G^A = \langle c\rangle \leftthreetimes H$, where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4, h_1^2 = h_2^2 = [h_1, h_2]$, $|c| = 2^n > 2$, $[c, h_1] = c^{2^{n-1}}$, $[c, h_2] = 1$.

Suppose that $N_G^A \neq G$. Since $\omega\left(N_G^A\right) \subset Z\left(N_G^A\right)$ and $\omega\left(N_G^A\right) \not\subset Z\left(G\right)$, $\omega\left(G\right) = \omega\left(N_G^A\right)$, $G = C\left\langle y\right\rangle$, where $C = C_G\left(\omega\left(N_G^A\right)\right) \lhd G,\ y^2 \in C,\ |y| > 4$ by Lemmas 2 and 3. The group C contains all Abelian non-cyclic subgroups of a group G, so $N_G^A \subseteq N_C^A \subseteq C$. Thus C is a 2-group which has a norm of Abelian non-cyclic subgroups of the type (8) of Proposition and a non-cyclic center. We conclude that C is a $\overline{HA_2}$-group and $C = N_G^A = \langle c\rangle \leftthreetimes H$ by the results of [6]. Thus $G = C\left\langle y\right\rangle = \left(\langle c\rangle \leftthreetimes H\right)\langle y\rangle, |y| > 4, y^2 \in C$.

Let $|y| = 2^k$. Since $y \notin C, \omega\left(G\right)\cap\langle y\rangle \subseteq Z(G)$. Let's denote $\langle a_1\rangle = \omega\left(G\right)\cap\langle y\rangle$ and consider the quotient-group $\overline{G} = G/\omega\left(G\right) \cong \overline{C}\left\langle\overline{y}\right\rangle$. Since the lower layer $\omega\left(\overline{C}\right)$ is an elementary Abelian subgroup of order 8 and $\omega\left(\overline{C}\right) \lhd \overline{G}$, we conclude that $\omega\left(\overline{C}\right)$ contains an involution $\overline{z}$ such that $[\overline{z}, \overline{y}] = \overline{1}$, $\langle\overline{z}\rangle \cap \langle\overline{y}\rangle = \overline{E}$. Turning to the preimages we put $[z, y] = a$, where $|a| = 4,\quad a \in \omega\left(G\right)$. Then $[z^2, y] = 1$ and $z^2 = a_1 \in Z\left(G\right)$. If $a \in Z\left(G\right)$, then $[z, y^2] = 1$, $|y^{2^{k-2}}z| = 2$, which is impossible, because the elements of the order 4 of N_G^A do not have such property. Thus $a \notin Z\left(G\right)$ and $[z, y^2] = a_1$. It follows that $\langle z, y^2\rangle$ is a quaternion group and $|y| = 8$.

If $|c| > 4$, then $a_1 = c^{2^{n-1}} \in Z\left(G\right)$ and $c^{2^{n-1}} \in \langle z, y^2\rangle$. But any quaternion group in N_G^A does not contain $c^{2^{n-1}}$. This means that $|c| = 4,\ c^2 \notin Z\left(G\right)$ and $a_1 = h_1^2 \in \langle z, y^2\rangle$. Taking into account the structure of the quaternion subgroups in N_G^A, we have $\langle z, y^2\rangle = \left\langle h_2c^{2m}, h_1h_2^lc^s\right\rangle$.

Suppose that $\langle y^2\rangle \lhd G$. Then we can assume that $y^2 = h_2c^{2m}$, $z = h_1h_2^lc^s$. Let's consider the quotient-group

$$\check{G} = G/\left\langle y^2\right\rangle \cong \left(\langle\check{c}\rangle \leftthreetimes \left\langle\check{h_1}\right\rangle\right) \leftthreetimes \langle\check{y}\rangle\,.$$

Since $\langle\check{c}\rangle$ is a characteristic subgroup in $\check{N_G^A}$, $\langle\check{c}\rangle \lhd \check{G}$ and $[\check{c}, \check{y}] \in \langle\check{c}^2\rangle$. Turning to the preimages we have $[c, y] = c^{2r}y^{2i}$. So $[c^2, y] = h_2^{2i} \neq 1$ and $i \equiv 1 \pmod 2$. It is easy to verify that in this case $|cy| \leq 4$, which contradicts to the proved.

Thus $\langle y^2\rangle \ntriangleleft G$. Then we can assume that $y^2 = h_1h_2^lc^s$ and $z = h_2c^{2m}$ respectively. Let's consider the quotient-group

$$\overline{G} = G/\omega(G) \cong \left(\langle\overline{c}\rangle \times \left\langle\overline{h_1}\right\rangle \times \left\langle\overline{h_2}\right\rangle\right)\langle\overline{y}\rangle\,.$$

Without loss of generality $\langle\overline{y}\rangle \cap \overline{N}_G^A = \left\langle\overline{h_1}\right\rangle$ and $\overline{z} = \overline{h_2}$. Then $[\overline{y}, \overline{z}] = \left[\overline{y}, \overline{h_2}\right] = \overline{1}$ according to the choice of $\overline{z}$. We get $[\langle\overline{y}\rangle, \overline{N}_G^A] \subseteq \overline{N}_G^A \cap \left\langle\overline{y}, \overline{h_2}\right\rangle = \left\langle\overline{y}^2, \overline{h_2}\right\rangle = \overline{H}$ by the condition $\left\langle\overline{y}, \overline{h_2}\right\rangle \lhd \overline{G}$. Thus $[y, h_2] = c^{2l}h_1^{2s}$ and $[y, c] = c^{2l_1}h_1^mh_2^r$. We have $l \not\equiv 0 \pmod 2$ by the first equality and the condition $[y, c^2] \neq 1$. We have $m \equiv 0 \pmod 2$ and $r \not\equiv 0 \pmod 2$ by the second equality and the condition $[y, c^2] \neq 1$. Thus $[y, h_2] = c^2h_1^{2s}$ and $[y, c] = c^{2l}h_2^{\pm1}$. Further $l_1 \equiv s \pmod 2$, because $[y^2, c] = c^2$.

We can assume without loss of generality that G is a group of the type $G = C\left\langle y\right\rangle = \left(\langle c\rangle \leftthreetimes H\right)\langle y\rangle\,,$ where

$$H = \langle h_1, h_2\rangle\,, |h_1| = |h_2| = 4,\quad h_1^2 = h_2^2 = [h_1, h_2]\,,\ |c| = 4,$$
$$[c, h_1] = c^2,\ [c, h_2] = 1,\ y^2 = h_1,\ [y, h_2] = c^2h_1^2, [y, c] = h_2^{\pm1}.$$

In this group all Abelian non-cyclic subgroups are contained in $\langle c\rangle \leftthreetimes H$ and normalized by this subgroup. At the same time $y \notin N_G^A$, because $y \notin N_G\left(\langle c, h_1^2\rangle\right)$. □

Corollary 3. *Every finite 2-group G with the cyclic center and non-Dedekind non-metacyclic norm N_G^A is a cyclic or metacyclic extension of this norm.*

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