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SUM OF ENTIRE FUNCTIONS OF BOUNDED L-INDEX IN DIRECTION

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It is proved that an entire function F has bounded L-index in a direction $\mathbf{b}$ in arbitrary bounded domain G under the assumption that F does not equal identically zero on the slice $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ for all $z^0 \in G$. Also it is obtained sufficient conditions of boundedness of L-index in direction for the sum of entire functions. They are new for entire functions of bounded l-index of one complex variable too. As a corollary, a class of entire functions of strongly bounded L-index in a direction matches with a class of entire functions of bounded L-index in the same direction. Moreover, we gave a negative answer to the question of Prof. S. Yu. Favorov: whether it is possible in theory of bounded L-index in direction to replace the assumption that F is holomorphic in $\mathbb{C}^n$ by the assumption that F is holomorphic on every slice $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ for all $z^0 \in \mathbb{C}^n$.

1. Introduction. To state the problems we need some notation and definitions.

An entire function $F(z)$, $z \in \mathbb{C}^n$, is called (see [1]–[5]) a *function of bounded L-index in a direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\}, \tag{1}$$

where $\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} := F(z)$, $\frac{\partial F(z)}{\partial \mathbf{b}} := \sum\limits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j = \langle \mathbf{grad} F, \overline{\mathbf{b}}\rangle$, $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} := \frac{\partial}{\partial \mathbf{b}}\Big(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\Big)$, $k \ge 2$.

The least such integer $m_0 = m_0(\mathbf{b})$ is called the L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of the entire function $F(z)$ and is denoted by $N_{\mathbf{b}}(F, L) = m_0$. If such m_0 does not exist then F is called a *function of unbounded L-index in the direction* $\mathbf{b}$ and we suppose that $N_{\mathbf{b}}(F, L) = \infty$. If $L(z) \equiv 1$ then $F(z)$ is called *a function of bounded index in the direction* $\mathbf{b}$ and $N_{\mathbf{b}}(F) = N_{\mathbf{b}}(F, 1)$.

Let D be an arbitrary bounded domain in $\mathbb{C}^n$. If inequality (1) holds for all $z \in D$ instead of $\mathbb{C}^n$ then F is called *a function of bounded L-index in the direction* $\mathbf{b}$ *in the domain D*. The least such integer m_0 is called the *L-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *in the domain* D and is denoted by $N_{\mathbf{b}}(F, L, D) = m_0$.

In the case $n = 1$ and $\mathbf{b} = 1$ we obtain the definition of entire function of one variable of bounded l-index (see [6]). Then $N(f, l) = N_1(f, l)$. In the case $n = 1$, $\mathbf{b} = 1$ and $L(z) \equiv 1$ it is reduced to the definition of the function of bounded index, supposed by B. Lepson [7].

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Note that results from our paper [1] are included also in the monograph [5].

For $\eta > 0$, $z \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and a positive continuous function $L: \mathbb{C}^n \to \mathbb{R}_+$ we define

$$\lambda_1^{\mathbf{b}}(z, t_0, \eta) = \inf\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})} : |t-t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})}\right\},$$

$$\lambda_1^{\mathbf{b}}(z,\eta) = \inf\{\lambda_1^{\mathbf{b}}(z,t_0,\eta) : t_0 \in \mathbb{C}\}, \quad \lambda_1^{\mathbf{b}}(\eta) = \inf\{\lambda_1^{\mathbf{b}}(z,\eta) : z \in \mathbb{C}^n\},$$

$$\lambda_2^{\mathbf{b}}(z, t_0, \eta) = \sup\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})} : |t-t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})}\right\},$$

$$\lambda_2^{\mathbf{b}}(z,\eta) = \sup\{\lambda_2^{\mathbf{b}}(z,t_0,\eta) : t_0 \in \mathbb{C}\}, \quad \lambda_2^{\mathbf{b}}(\eta) = \sup\{\lambda_2^{\mathbf{b}}(z,\eta) : z \in \mathbb{C}^n\}.$$

By $Q_{\mathbf{b}}^n$ we denote the class of functions L which satisfy the condition

$$(\forall \eta \ge 0): \ 0 < \lambda_1^{\mathbf{b}}(\eta) \le \lambda_2^{\mathbf{b}}(\eta) < +\infty. \tag{2}$$

If $n = 1$ and $\mathbf{b} = 1$ then we use the notation $Q = Q_1^1$.

It is known that a product of two entire functions of bounded L-index in direction is a function from the same class (see [5], [8]). But the class of entire functions of bounded index is not closed under addition. The example was constructed by W. Pugh (see [9] and [10]). Recently we generalized Pugh's example for entire functions of bounded L-index in direction ([8]).

Meanwhile, there are sufficient conditions for index boundedness for the sum of two entire functions ([9]). But similar conditions for entire functions of bounded L-index in direction or even of bounded l-index are not known. Therefore, in the present article the following natural **question** is considered: *what are sufficient conditions for L-index boundedness in direction for the sum of two entire functions?*

We need the following theorem.

Theorem 1 ([1, 5]). *Let $L \in Q_{\mathbf{b}}^n$. An entire function $F(z)$ in $\mathbb{C}^n$ is of bounded L-index in the direction $\mathbf{b}$ if and only if for every r_1 and r_2 such that $0 < r_1 < r_2 < +\infty$, there exists a number $P_1 = P_1(r_1, r_2) \ge 1$ such that for each $z^0 \in \mathbb{C}^n$ and $t_0 \in \mathbb{C}$*

$$\max\left\{|F(z^0+t\mathbf{b})| : |t-t_0| = \frac{r_2}{L(z^0+t_0\mathbf{b})}\right\} \le$$
$$\le P_1 \max\left\{|F(z^0+t\mathbf{b})| : |t-t_0| = \frac{r_1}{L(z_0+t_0\mathbf{b})}\right\}. \tag{3}$$

2. Boundedness of L-index in direction in a bounded domain. By $\overline{D}$ we denote the closure of a domain D.

Theorem 2. *Let D be an arbitrary bounded domain in $\mathbb{C}^n$. If $L\colon \mathbb{C}^n \to \mathbb{R}_+$ is a continuous function and $F(z)$ is an entire function such that $(\forall z^0 \in \overline{D})$: $F(z^0+t\mathbf{b}) \not\equiv 0$ then $N_{\mathbf{b}}(F, L, D) < \infty$ for every $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.*

Proof. For every fixed $z^0 \in \overline{D}$ we expand the entire function $F(z^0+t\mathbf{b})$ in a power series by powers of t

$$F(z^0+t\mathbf{b}) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^p F(z^0)}{\partial \mathbf{b}^m} t^m \tag{4}$$

in the disk $\{t \in \mathbb{C} : |t| \le \frac{1}{L(z^0)}\}$.

The quantity $\frac{1}{m!L^m(z^0)}\left|\frac{\partial^m F(z^0)}{\partial \mathbf{b}^m}\right|$ is the modulus of a coefficient of the power series (4) at a point $t \in \mathbb{C}$ such that $|t| = \frac{1}{L(z^0)}$. Since $F(z)$ is entire, for every $z_0 \in \overline{D}$

$$\frac{1}{m!L^m(z^0)}\left|\frac{\partial^m F(z^0)}{\partial \mathbf{b}^m}\right| \to 0 \quad (m \to \infty),$$

i.e. there exists $m_0 = m(z^0, \mathbf{b})$ such that inequality (1) holds at $z = z^0$ for all $m \in \mathbb{Z}_+$.

We prove that $\sup\{m_0 \colon z^0 \in \overline{D}\} < +\infty$. Assume on the contrary that the set of m_0 is unbounded in z^0, i.e. $\sup\{m_0 \colon z^0 \in \overline{D}\} = +\infty$. Hence, for every $m \in \mathbb{Z}_+$ there exist $z^{(m)} \in \overline{D}$ and $p_m > m$

$$\frac{1}{p_m!L^{p_m}(z^{(m)})}\left|\frac{\partial^{p_m} F(z^{(m)})}{\partial \mathbf{b}^{p_m}}\right| > \max\left\{\frac{1}{k!L^k(z^{(m)})}\left|\frac{\partial^k F(z^{(m)})}{\partial \mathbf{b}^k}\right| \colon\ 0 \le k \le m\right\}. \tag{5}$$

Since $\{z^{(m)}\} \subset \overline{D}$, there exists a subsequence $z'^{(m)} \to z' \in \overline{G}$ as $m \to +\infty$. By Cauchy's integral formula

$$\frac{1}{p!}\frac{\partial^p F(z)}{\partial \mathbf{b}} = \frac{1}{2\pi i}\int_{|t|=r}\frac{F(z+t\mathbf{b})}{t^{p+1}}dt$$

for any $p \in \mathbb{N}$, $z \in D$. We rewrite (5) in the form

$$\max\left\{\frac{1}{k!L^k(z^{(m)})}\left|\frac{\partial^k F(z^{(m)})}{\partial \mathbf{b}^k}\right| \colon\ 0 \le k \le m\right\} <$$
$$< \frac{1}{L^{p_m}(z^{(m)})}\int_{|t|=r/L(z^{(m)})}\frac{|F(z^{(m)}+t\mathbf{b})|}{|t|^{p_m+1}}|dt| \le \frac{1}{r^{p_m}}\max\{|F(z)| : z \in D_r\} \tag{6}$$

where $D_r = \bigcup_{z^* \in \overline{D}}\{z \in \mathbb{C}^n : |z - z^*| \le \frac{|b|r}{L(z^*)}\}$. We can choose $r > 1$ because F is entire. Evaluating the limit for every fixed directional derivative in (6) as $m \to \infty$ we obtain

$$(\forall k \in \mathbb{Z}_+)\colon \quad \frac{1}{k!L^k(z')}\left|\frac{\partial^k F(z')}{\partial \mathbf{b}^k}\right| \le \overline{\lim_{m\to\infty}}\frac{1}{r^{p_m}}\max\{|F(z)| : z \in D_r\} \le 0$$

as $m \to +\infty$. Thus, all derivatives in the direction $\mathbf{b}$ of the function F at the point z' equals 0 and $F(z') = 0$. In view of (4) $F(z' + t\mathbf{b}) \equiv 0$. It is a contradiction.

□

The proof of Theorem 2 is published also in [5, Th.3.2, p.62–64].

Remark 1. Perhaps, the assumption $(\forall z \in \overline{D})\colon\ F(z+t\mathbf{b}) \not\equiv 0$ in Theorem 2 not necessary. But nowadays we do not know a rigorous proof of the theorem without this assumption. There was published proof of Theorem 2 in [2] with gaps. Let $M = \max\{|F(z)| \colon\ z \in D\}$ and $\varepsilon > 0$. If $F(z^0 + t\mathbf{b}) \equiv 0$ for some $z^0 \in \overline{D}$ then by Theorem 2 the function $G(z) = F(z) + M + \varepsilon$ has bounded L-index in domain D in any direction $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$.

3. Sufficient conditions of boundedness of L-index in direction for sum of entire functions. We consider an arbitrary hyperplane $A = \{z \in \mathbb{C}^n \colon \langle z, c\rangle = 1\}$, where $\langle c, \mathbf{b}\rangle \ne 0$. Obviously that $\bigcup_{z^0 \in A}\{z^0 + t\mathbf{b} \colon t \in \mathbb{C}\} = \mathbb{C}^n$.

Let $z^0 \in A$ be a given point. If $F(z^0 + t\mathbf{b}) \not\equiv 0$ as a function of variable $t \in \mathbb{C}$ then there exists $t_0 \in \mathbb{C}$ such that $F(z^0 + t_0\mathbf{b}) \ne 0$. Thus, for every line $\{z^0 + t\mathbf{b} \colon t \in \mathbb{C}\}$

$F(z^0+t\mathbf{b})\not\equiv 0$ we fixe point t_0. By B we denote the union of those points $z^0+t_0\mathbf{b}$ i. e.

$$B=\bigcup_{\substack{z^0\in A\\ F(z^0+t\mathbf{b})\not\equiv 0}}\{z^0+t_0\mathbf{b}\}.$$

Clearly that for every $z\in\mathbb{C}^n$ there exist $z^0\in A$ and $t\in\mathbb{C}$ with the property $z=z^0+t\mathbf{b}$. Indeed,

$$z^0=z+\frac{1-\langle z,c\rangle}{\langle\mathbf{b},c\rangle}\mathbf{b},\ t=\frac{\langle z,c\rangle-1}{\langle\mathbf{b},c\rangle}.$$

Theorem 3. *Let L be the positive continuous function, F, G be entire in $\mathbb{C}^n$ functions satisfying the following conditions:*

1) $G(z)$ has bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$ with $N_b(G,L)=N<+\infty$;

2) there exists $\alpha\in(0,1)$ such that for all $z\in\mathbb{C}^n$ and $p\geq N+1$ $(p\in\mathbb{N})$

$$\frac{1}{p!L^p(z)}\left|\frac{\partial^p G(z)}{\partial\mathbf{b}^p}\right|\leq\alpha\max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k G(z)}{\partial\mathbf{b}^k}\right|:0\leq k\leq N\right\};\tag{7}$$

3) for every $z=z^0+t\mathbf{b}\in\mathbb{C}^n$, where $z^0\in A$, $z^0+t_0\mathbf{b}\in B$ and $r=|t-t_0|L(z^0+t\mathbf{b})$ the inequality

$$\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\leq$$
$$\leq\max\left\{\frac{1}{k!L^k(z^0+t\mathbf{b})}\left|\frac{\partial^k G(z^0+t\mathbf{b})}{\partial\mathbf{b}^k}\right|:\ 0\leq k\leq N\right\}\tag{8}$$

is valid;

4) $(\exists c>0)(\forall z^0+t_0\mathbf{b}\in B)(\forall t\in\mathbb{C},\ |t-t_0|L(z^0+t\mathbf{b})\leq 1)$:

$$\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}\Big/|F(z^0+t_0\mathbf{b})|\leq c<+\infty,\tag{9}$$

or for $L\in Q^n_{\mathbf{b}}$ $(\exists c>0)(\forall z^0+t_0\mathbf{b}\in B)$:

$$\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2\lambda_2^{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}\Big/|F(z^0+t_0\mathbf{b})|\leq c<+\infty.\tag{10}$$

Then for each $\varepsilon\in\mathbb{C}$, $|\varepsilon|\leq\frac{1-\alpha}{2c}$, the function

$$H(z)=G(z)+\varepsilon F(z)\tag{11}$$

is of bounded L-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(H,L)\leq N$.

Proof. We write Cauchy's formula for the entire function $F(z^0+t\mathbf{b})$ as a function of one complex variable t

$$\frac{1}{p!}\frac{\partial^p F(z^0+t\mathbf{b})}{\partial\mathbf{b}^p}=\frac{1}{2\pi i}\int_{|t'-t|=\frac{r}{L(z^0+t\mathbf{b})}}\frac{F(z^0+t'\mathbf{b})}{(t'-t)^{p+1}}dt'.\tag{12}$$

For the chosen $r = |t - t_0| L(z^0 + t\mathbf{b})$ the following inequality holds

$$\frac{r}{L(z^0+t\mathbf{b})} = |t'-t| \geq |t'-t_0| - |t-t_0| = |t'-t_0| - \frac{r}{L(z^0+t\mathbf{b})}.$$

Hence,

$$|t'-t_0| \leq \frac{2r}{L(z^0+t\mathbf{b})}. \tag{13}$$

Equality (12) yields

$$\begin{aligned} \frac{1}{p!L^p(z^0+t\mathbf{b})}\left|\frac{\partial^p F(z^0+t\mathbf{b})}{\partial \mathbf{b}^p}\right| &\leq \frac{1}{2\pi L^p(z^0+t\mathbf{b})}\cdot\frac{L^{p+1}(z^0+t\mathbf{b})}{r^{p+1}}\times \\ &\times\frac{2\pi r}{L(z^0+t\mathbf{b})}\cdot\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t| = \frac{r}{L(z^0+t\mathbf{b})}\right\} \leq \\ &\leq \frac{1}{r^p}\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\}. \end{aligned} \tag{14}$$

If $r = |t-t_0|L(z^0+t\mathbf{b}) > 1$ then (14) implies

$$\frac{1}{p!L^p(z^0+t\mathbf{b})}\left|\frac{\partial^p F(z^0+t\mathbf{b})}{\partial \mathbf{b}^p}\right| \leq \max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\}. \tag{15}$$

Let $r = |t-t_0|L(z^0+t\mathbf{b}) \in (0;1]$. Setting $r=1$ in (12) and (13) we similarly deduce

$$\begin{aligned} &\frac{1}{p!L^p(z^0+t\mathbf{b})}\left|\frac{\partial^p F(z^0+t\mathbf{b})}{\partial \mathbf{b}^p}\right| \leq \max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0| = \frac{2}{L(z^0+t\mathbf{b})}\right\} = \\ &= \frac{\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0| = \frac{2}{L(z^0+t\mathbf{b})}\right\}}{\max\left\{|F(z^0+t'\mathbf{b})|\colon |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\}}\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\} \leq \\ &\leq \frac{\max\left\{|F(z^0+t'\mathbf{b})|\colon |t'-t_0| = \frac{2}{L(z^0+t\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\} \leq \\ &\leq c\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\}, \end{aligned} \tag{16}$$

where

$$c = \sup_{z^0+t_0\mathbf{b}\in B}\ \sup_{\substack{t\in\mathbb{C},\\ |t-t_0|L(z^0+t\mathbf{b})\leq 1}} \frac{\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0| = \frac{2}{L(z^0+t\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|} \geq 1.$$

If $L \in Q$ then $\sup\left\{\frac{L(z^0+t_0\mathbf{b})}{L(z^0+t\mathbf{b})}\colon\ |t-t_0| \leq \frac{1}{L(z^0+t\mathbf{b})}\right\} \leq \lambda_2^{\mathbf{b}}(1)$. This means that $L(z^0+t\mathbf{b}) \geq \frac{L(z^0+t_0\mathbf{b})}{\lambda_2^{\mathbf{b}}(1)}$. Using this inequality we choose

$$c := \sup_{z^0+t_0\mathbf{b}\in B} \frac{\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0| = \frac{2\lambda_2^{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|} \geq 1$$

in (16). In view of (15) and (16) we have

$$\frac{1}{p!L^p(z^0+t\mathbf{b})}\left|\frac{\partial^p F(z^0+t\mathbf{b})}{\partial\mathbf{b}^p}\right|\le c\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\qquad(17)$$

for all $p\in\mathbb{N}\cup\{0\}$, $r\ge 0$, $z^0\in A$, $t\in\mathbb{C}$.

We differentiate (11) p times, $p\ge N+1$, and then apply (7), (17) and (8)

$$\begin{gathered}\frac{1}{p!L^p(z^0+t\mathbf{b})}\left|\frac{\partial^p H(z^0+t\mathbf{b})}{\partial\mathbf{b}^p}\right|\le\\ \le\frac{1}{p!L^p(z^0+t\mathbf{b})}\left|\frac{\partial^p G(z^0+t\mathbf{b})}{\partial\mathbf{b}^p}\right|+\frac{|\varepsilon|}{p!L^p(z^0+t\mathbf{b})}\left|\frac{\partial^p F(z^0+t\mathbf{b})}{\partial\mathbf{b}^p}\right|\le\\ \le\alpha\max\left\{\frac{1}{k!L^k(z^0+t\mathbf{b})}\left|\frac{\partial^k G(z^0+t\mathbf{b})}{\partial\mathbf{b}^k}\right|:0\le k\le N\right\}+\\ +c|\varepsilon|\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\le\\ \le(\alpha+c|\varepsilon|)\max\left\{\frac{1}{k!L^k(z^0+t\mathbf{b})}\left|\frac{\partial^k G(z^0+t\mathbf{b})}{\partial\mathbf{b}^k}\right|:0\le k\le N\right\}.\end{gathered}\qquad(18)$$

If $s\le N$, then (17) is true with $p=s$, but (7) does not hold. Therefore, the differentiation of (11) leads to the following estimate

$$\begin{gathered}\frac{1}{s!L^s(z^0+t\mathbf{b})}\left|\frac{\partial^s H(z^0+t\mathbf{b})}{\partial\mathbf{b}^s}\right|\ge\\ \ge\frac{1}{s!L^s(z^0+t\mathbf{b})}\left|\frac{\partial^s G(z^0+t\mathbf{b})}{\partial\mathbf{b}^s}\right|-\frac{|\varepsilon|}{s!L^s(z^0+t\mathbf{b})}\left|\frac{\partial^s F(z^0+t\mathbf{b})}{\partial\mathbf{b}^s}\right|\ge\\ \ge\frac{1}{s!L^s(z^0+t\mathbf{b})}\left|\frac{\partial^s G(z^0+t\mathbf{b})}{\partial\mathbf{b}^s}\right|-c|\varepsilon|\max\left\{|F(z^0+t'\mathbf{b})|\colon\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\},\end{gathered}\qquad(19)$$

where $0\le s\le N$. Hence, (8) and (19) imply that

$$\max_{0\le s\le N}\left\{\frac{1}{s!L^s(z^0+t\mathbf{b})}\left|\frac{\partial^s H(z^0+t\mathbf{b})}{\partial\mathbf{b}^s}\right|\right\}\ge(1-c|\varepsilon|)\max_{0\le s\le N}\left\{\frac{1}{s!L^s(z^0+t\mathbf{b})}\left|\frac{\partial^s G(z^0+t\mathbf{b})}{\partial\mathbf{b}^s}\right|\right\}.\qquad(20)$$

If $c|\varepsilon|<1$, then (18) and (20) yield

$$\frac{1}{p!L^p(z^0+t\mathbf{b})}\left|\frac{\partial^p H(z^0+t\mathbf{b})}{\partial\mathbf{b}^p}\right|\le\frac{\alpha+c|\varepsilon|}{1-c|\varepsilon|}\max_{0\le s\le N}\left\{\frac{1}{s!L^s(z^0+t\mathbf{b})}\left|\frac{\partial^s H(z^0+t\mathbf{b})}{\partial\mathbf{b}^s}\right|\right\}\qquad(21)$$

for $p\ge N+1$. We assume that $\frac{\alpha+c|\varepsilon|}{1-c|\varepsilon|}\le 1$. Hence, $|\varepsilon|\le\frac{1-\alpha}{2c}$.

Let $N_{\mathbf{b}}(z^0+t\mathbf{b},L,F)$ be the L-index in direction of the function F at the point $z^0+t\mathbf{b}$, i. e. $N_{\mathbf{b}}(z^0+t\mathbf{b},L,F)$ is the smallest number m_0 for which inequality (1) holds with $z=z^0+t\mathbf{b}$.

For $|\varepsilon|\le\frac{1-\alpha}{2c}$ validity of inequality (21) means that for any $z^0\in A$ and any $t\in\mathbb{C}$ such that $F(z^0+t\mathbf{b})\ne 0$ the L-index in direction at the point $z^0+t\mathbf{b}$ is not greater than N i. e. $N_{\mathbf{b}}(z^0+t\mathbf{b},F,L)\le N$.

If for some $z^0\in A$ $F(z^0+t\mathbf{b})\equiv 0$ then $H(z^0+t\mathbf{b})\equiv G(z^0+t\mathbf{b})$ and $N_{\mathbf{b}}(z^0+t\mathbf{b},F,L)=N_{\mathbf{b}}(z^0+t\mathbf{b},G,L)\le N$. Therefore, $H(z)$ is of bounded L-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H,L)\le N$. This completes the proof of Theorem 3. □

Remark 2. Every entire function F with $N_{\mathbf{b}}(F,L)=0$ satisfies inequality (10) (see the proof of necessity of Theorem 2.1 in [5, p. 20–24]).

If $n=1$, $\mathbf{b}=1$, $L=l$, $F=f$ then we obtain the following corollary.

Corollary 1. *Let l be positive continuous function, f, g be entire in $\mathbb{C}$ functions, t_0 be some point such that $f(t_0)\neq 0$, satisfying the following conditions:*

1) *$g(z)$ has bounded l-index with $N(g,l)=N<+\infty$;*
2) *there exists $\alpha\in(0,1)$ such that for all $z\in\mathbb{C}$ and $p\geq N+1$ ($p\in\mathbb{N}$)*
$$\frac{|g^{(p)}(z)|}{p!l^p(z)}\leq\alpha\max\left\{\frac{|g^{(k)}(z)|}{k!l^k(z)}:0\leq k\leq N\right\};$$
3) *for every $t\in\mathbb{C}$, and $r=|t-t_0|l(t)$:*
$$\max\left\{|f(t')|:\ |t'-t_0|=\frac{2r}{l(t)}\right\}\leq\max\left\{\frac{|g^{(k)}(t)|}{k!l^k(t)}:\ 0\leq k\leq N\right\};$$
4) *$(\exists c>0)(\forall t\in\mathbb{C},\ |t-t_0|l(t)\leq 1)$: $\max\left\{|f(t')|:\ |t'-t_0|=\frac{2}{l(t)}\right\}\Big/|f(t_0)|\leq c<+\infty$, or for $l\in Q$ we put $c=\max\left\{|f(t')|:\ |t'-t_0|=\frac{2\lambda_2^{\mathbf{b}}(1)}{l(t_0)}\right\}\Big/|f(t_0)|$.*

Then for each $\varepsilon\in\mathbb{C}$, $|\varepsilon|\leq\frac{1-\alpha}{2c}$, the function $h(z)=g(z)+\varepsilon f(z)$ is of bounded l-index and $N(h,l)\leq N$.

Corollary 1 is a generalization of Pugh's result [9] for l-index.
If $L\in Q^n_{\mathbf{b}}$ then Condition 2) in Theorem 3 always holds. The following theorem is true.

Theorem 4. *Let $L\in Q^n_{\mathbf{b}}$, $\alpha\in(0,1)$ and F, G be entire in $\mathbb{C}^n$ functions satisfying the conditions:*

1) *$G(z)$ has bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$.*
2) *for every $z=z^0+t\mathbf{b}\in\mathbb{C}^n$, where $z^0\in A$, $z^0+t_0\mathbf{b}\in B$ and $r=|t-t_0|L(z^0+t\mathbf{b})$ the following inequality is valid*
$$\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\leq$$
$$\leq\max\left\{\frac{1}{k!L^k(z^0+t\mathbf{b})}\left|\frac{\partial^k G(z^0+t\mathbf{b})}{\partial\mathbf{b}^k}\right|:0\leq k\leq N_{\mathbf{b}}(G_\alpha,L_\alpha)\right\}.$$
3) $$c:=\sup_{z^0+t_0\mathbf{b}\in B}\frac{\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2\lambda_2^{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}<\infty.$$

If $|\varepsilon|\leq\frac{1-\alpha}{2c}$ then the function
$$H(z)=G(z)+\varepsilon F(z)$$
is of bounded L-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H,L)\leq N_{\mathbf{b}}(G_\alpha,L_\alpha)$, where $G_\alpha(z)=G(z/\alpha)$, $L_\alpha(z)=L(z/\alpha)$.

Proof. Condition 2) in Theorem 3 always holds for $N = N_b(G_\alpha, L_\alpha)$ instead of $N = N_{\mathbf{b}}(G, L)$, where $G_\alpha(z) = G(z/\alpha)$, $L_\alpha(z) = L(z/\alpha)$, $\alpha \in (0,1)$. Indeed, by Theorem 1 inequality (3) holds for the function G. Substituting $\frac{z^0}{\alpha}$, $\frac{t}{\alpha}$ and $\frac{t_0}{\alpha}$ instead of z^0, t and t_0 in (3) we obtain

$$\max\left\{|G((z^0+t\mathbf{b})/\alpha)|\colon |t-t_0| = \frac{r_2\alpha}{L((z^0+t_0\mathbf{b})/\alpha)}\right\} \leq$$
$$\leq P_1 \max\left\{|G((z^0+t\mathbf{b})/\alpha)|\colon |t-t_0| = \frac{r_1\alpha}{L((z_0+t_0\mathbf{b})/\alpha)}\right\}. \tag{22}$$

By Theorem 1 inequality (22) implies that $G_\alpha = G(z/\alpha)$ has bounded L_α-index in the direction $\mathbf{b}$ and vice versa. Hence, for $p \geq N_{\mathbf{b}}(G_\alpha, L_\alpha) + 1$ and $\alpha \in (0,1)$

$$\frac{1}{p!L_\alpha^p(z)}\left|\frac{\partial^p G_\alpha(z)}{\partial \mathbf{b}^p}\right| = \frac{1}{p!\alpha^p L^p(z/\alpha)}\left|\frac{\partial^p G(z/\alpha)}{\partial \mathbf{b}^p}\right| \leq$$
$$\leq \max\left\{\frac{1}{s!L_\alpha^s(z)}\left|\frac{\partial^s G_\alpha(z)}{\partial \mathbf{b}^s}\right|:\ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\} =$$
$$= \max\left\{\frac{1}{s!\alpha^s L^s(z/\alpha)}\left|\frac{\partial^s G(z/\alpha)}{\partial \mathbf{b}^s}\right|:\ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\}.$$

Multiplying by α^p we deduce

$$\frac{1}{p!L^p(z/\alpha)}\left|\frac{\partial^p G(z/\alpha)}{\partial \mathbf{b}^p}\right| \leq \max\left\{\frac{\alpha^{p-s}}{s!L^s(z/\alpha)}\left|\frac{\partial^s G(z/\alpha)}{\partial \mathbf{b}^s}\right|:\ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\} \leq$$
$$\leq \alpha \max\left\{\frac{1}{s!L^s(z/\alpha)}\left|\frac{\partial^s G(z/\alpha)}{\partial \mathbf{b}^s}\right|:\ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\}. \tag{23}$$

In view of arbitrariness of z inequality (23) imply (7). □

Remark 3. It is easy to prove that $N_{\mathbf{b}}(G_\alpha, L_\alpha) \leq N_{\mathbf{b}}(G, L)$ for $\alpha \in (0,1)$. Therefore $N_{\mathbf{b}}(G_\alpha, L_\alpha)$ in Theorem 4 can be replaced by $N_{\mathbf{b}}(G, L)$.

Unfortunately, Theorem 4 does not allow to remove a constraint $(\forall z \in \overline{D})\colon\ F(z+t\mathbf{b}) \not\equiv 0$ in Theorem 2.

Corollary 2. *Let $l \in Q$, $\alpha \in (0,1)$ and f, g be entire in $\mathbb{C}$ functions satisfying the conditions:*

1) $g(z)$ has bounded l-index;

2) for every $t \in \mathbb{C}$, and $r = |t - t_0|l(t)$:

$$\max\left\{|f(t')|\colon\ |t'-t_0| = \frac{2r}{l(t)}\right\} \leq \max\left\{\frac{|g^{(k)}(t)|}{k!l^k(t)} : 0 \leq k \leq N(g_\alpha, l_\alpha)\right\}.$$

If $|\varepsilon| \leq \frac{1-\alpha}{2c}$ then the function $h(z) = g(z) + \varepsilon f(z)$ is of bounded l-index with $N(h,l) \leq N(g_\alpha, l_\alpha)$, where $g_\alpha(z) = g(z/\alpha)$, $l_\alpha(z) = l(z/\alpha)$, $c = \max\left\{|f(t')|\colon |t'-t_0| = \frac{2\lambda_2^{\mathbf{b}}(1)}{l(t_0)}\right\}\Big/|f(t_0)|$.

Corollary 2 is new even in the case $n = 1$ and $l \equiv 1$, i.e. for entire functions of bounded index.

5. An example of function of unbounded index in direction in a bounded domain.

In our investigations of boundedness of L-index in direction we often consider the slices $\{z^0 + t\mathbf{b}\colon t \in \mathbb{C}\}$. Then we fix $z^0 \in \mathbb{C}^n$ and apply arguments from the one-dimensional case. Afterwards we deduce uniform estimations in z^0. This is a short description of the method.

Prof. S. Yu. Favorov (2015) posed the following **problem** in conversation with Prof. O. B. Skaskiv.

Problem 1. *Let $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$ be a given direction, $L\colon\mathbb{C}^n\to\mathbb{R}_+$ be a continuous function. Is it possible to replace the assumption that F is holomorphic in $\mathbb{C}^n$ by the assumption that F is holomorphic on all slices of the form $z^0+t\mathbf{b}$ and deduce known properties of entire functions of bounded L-index in direction?*

Our answer to Favorov's question is negative. This relaxation of restriction on the function F does not imply certain theorems.

Theorem 5. *For every direction $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$ there exist a function $F(z)$ and a bounded domain $D\subset\mathbb{C}^n$ with following properties:*

1) F is holomorphic in every slice $\{z^0+t\mathbf{b}\colon t\in\mathbb{C}\}$ for all $z^0\in\mathbb{C}^n$;

2) F is not entire in $\mathbb{C}^n$;

3) F does not satisfy (1) in $\overline{D}$, i.e. for any $p\in\mathbb{Z}_+$ there exist $m\in\mathbb{Z}_+$ and $z_p\in\overline{D}$

$$\frac{1}{m!}\left|\frac{\partial^m F(z_p)}{\partial\mathbf{b}^m}\right|>\max\left\{\frac{1}{k!}\left|\frac{\partial^k F(z_p)}{\partial\mathbf{b}^k}\right|\colon 0\le k\le p\right\}.$$

Proof. Without loss of generality we assume that $n=2$ and $\mathbf{b}=(0,1)$. Let

$$F(z_1,z_2)=\begin{cases}-1+z_1\sin\frac{z_2}{z_1}, & z_1\neq 0,\\ -1, & z_1=0.\end{cases}.$$

For every fixed $z_1^0\in\mathbb{C}$ the function $F(z_1,z_2)$ is holomorphic in variable z_2, i. e. F is holomorphic on every slice $z=z^0+t\mathbf{b}$, where $z^0=(z_1^0,0)$, $t\in\mathbb{C}$. On the other hand, F is not entire in $\mathbb{C}^2$.

If $z_1=0$ then $\frac{\partial^k F}{\partial\mathbf{b}^k}=0$ and if $z_1\neq 0$ then

$$\frac{\partial^k F}{\partial\mathbf{b}^k}=z_1^{1-k}\sin(\frac{z_2}{z_1}+\frac{\pi k}{2})\ (k\in\mathbb{N}). \tag{24}$$

Hence, for every fixed $z_1^0\in\mathbb{C}$ the function $F(z_1^0,z_2)$ has bounded index in variable z_2, because $\frac{(z_1^0)^{1-k}}{k!}\to 0$ as $k\to\infty$.

Nevertheless, the function $F(z_1,z_2)$ is of unbounded index in the direction $\mathbf{b}$. Moreover, F has unbounded index in any closed bounded domain G, that contains a part of plane $z_1=0$ with some neighborhood: $D\supset\{(z_1,z_2)\colon |z_1|\le R,|z_2|\le R\}$.

Denote $g_0(z_2)=F(z_1^0,z_2)$. The function $\sin t$ $(t\in\mathbb{C})$ has index 2. Therefore, in view of (24), index of the function $g_0(z_2)$ can be established from the following inequalities:

$$\frac{|z_1^0|^{1-k}}{k!}\ge\frac{|z_1^0|^{1-(k+2)}}{(k+2)!}\Longleftrightarrow |z_1^0|^2\ge\frac{1}{(k+1)(k+2)}\Longleftrightarrow$$
$$(k+1)(k+2)\ge\frac{1}{|z_1^0|^2}\Longrightarrow(k+2)^2>\frac{1}{|z_1^0|^2}\Longrightarrow k>\frac{1}{|z_1^0|}-2.$$

Thus, index of the function g_0 is greater than $\frac{1}{|z_1^0|}-2$, i. e. $N(g_0)>\frac{1}{|z_1^0|}-2$. If $z_1^0\to 0$ then $N(g_0)\to+\infty$. Hence, the function F has unbounded index in the direction $\mathbf{b}$: $N_{\mathbf{b}}(F)=\sup\limits_{z_1^0\in\mathbb{C}}N(g_0)=+\infty$, i.e. F does not satisfy (1) in $\mathbb{C}^2$.

Similarly, it can be proved that (1) does not hold for the function F in the domain D. It is easy to see that the function F is not continuous in joint variables. It is discontinuous for $z_1=0$. □

Remark 4. If we replace holomorphy in $\mathbb{C}^n$ by holomorphy on the slices $\{z^0 + t\mathbf{b} \colon t \in \mathbb{C}\}$ then the conclusion of Theorem 2 is not valid. Thus, Theorem 5 gives the negative answer to Problem 1. But careful analysis of the proof of Theorem 2 reveals that we implicitly use continuity in joint variables in (6).

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