УДК 517.537.72

O. M. Mulyava, M. M. Sheremeta

A REMARK TO THE GROWTH OF POSITIVE FUNCTIONS AND ITS APPLICATION TO DIRICHLET SERIES

O. M. Mulyava, M. M. Sheremeta. *A remark to the growth of positive functions and its application to Dirichlet series*, Mat. Stud. **44** (2015), 161–170.

For a function Φ continuos on $(-\infty,+\infty)$ increaing to $+\infty$ the lower and upper estimates for $\frac{\alpha(\Phi(q\sigma))}{\alpha^p(\Phi(\sigma))}$ are found, where $p>1, q>1$ and α is positive function continuous on $[x_0,+\infty)$, increasing to $+\infty$. The above results applied to Dirichlet series with positive exponents.

О. М. Мулява, М. Н. Шеремета. *Замечание к росту положительных функций и его приложение к рядам Дирихле* // Мат. Студії. – 2015. – Т.44, №2. – C.161–170.

Для непрерывной возрастающей к $+\infty$ на $(-\infty,+\infty)$ функции Φ найдены оценки сверху и снизу для $\frac{\alpha(\Phi(q\sigma))}{\alpha^p(\Phi(\sigma))}$, где $p>1, q>1$ и α — положительная непрерывная на $[x_0,+\infty)$ функция, возрастающая к $+\infty$. Полученные результаты применены к рядам Дирихле с положительными показателями.

1. Introduction. For an entire transcendental function

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,\quad z=re^{i\theta}, \tag{1}$$

set $M_f(r)=\max\{|f(z)|\colon |z|=r\}$. It is known ([1]) that $M_f(\alpha r)/M_f(r)\nearrow+\infty$ as $r\to+\infty$ for every $\alpha>1$. S. Singh gave a simplier (on its opinion) proof of this fact, using the relation between $M_f(r)$ and the maximal term of the power development of f.

Dirichlet series

$$F(s)=\sum_{n=0}^{\infty}a_n\exp\{s\lambda_n\},\quad s=\sigma+it, \tag{2}$$

with nonnegative increasing to $+\infty$ exponents λ_n are direct generalization of power developments of analytic functions. We suppose that series (2) has an abscissa of absolute convergence $\sigma_a=A\in(-\infty,+\infty]$ and for $\sigma<A$ we put $M(\sigma,F)=\sup\{|F(\sigma+it)|\colon t\in\mathbb{R}\}$. If $A=+\infty$ and series (2) is not reduced to an exponential polynomial then the function $\ln M(\sigma,F)$ is convex on $(-\infty,+\infty)$ and for each number $h>0$ the equality $\ln M(\sigma+h,F)-\ln M(\sigma,F)=\int_\sigma^{\sigma+h}\omega(t)dt$ holds, where $\omega(t)\nearrow+\infty$ as $t\to+\infty$, that is $M(\sigma+h,F)/M(\sigma,F)\nearrow+\infty$ as $\sigma\to+\infty$.

If we replace z with e^s in (1) then we obtain (2) and $M(\sigma,F)=M_f(e^\sigma)$. Hence the result above mentioned for entire functions follows.

2010 *Mathematics Subject Classification:* 30B50.
Keywords: positive functions; Dirichlet series; maximum modulus.
doi:10.15330/ms.44.2.161-170

Using properties of Mittag-Leffler type functions ([3–4]), it is possible to show that unlike $M(\sigma+h,F)/M(\sigma,F)$ the quotient $\ln M(\sigma+h,F)/\ln M(\sigma,F)$ has not a limit and its inferior and superior limits can be any numbers from $[1,+\infty]$.

Here we will investigate asymptotic behaviours as $\sigma\to A$ of some general quantities $\frac{\alpha(\ln M(\sigma+h,F))}{\alpha^p(\ln M(\sigma,F))}$ and $\frac{\alpha(\ln M(q\sigma,F))}{\alpha^p(\ln M(\sigma,F))}$ if $A=+\infty$ and $\frac{\alpha(\ln M(\sigma/q,F))}{\alpha^p(\ln M(\sigma,F))}$ if $A=0$, where $h>0$, $q>1$, $p>1$ and α is positive continuous function on $[x_0,+\infty)$ increasing to $+\infty$.

2. Growth of positive functions. Let L be the class of continuous increasing functions α such that $\alpha(x)\ge 0$ for $x\ge x_0$, $\alpha(x)=\alpha(x_0)$ for $x\le x_0$ and on $[x_0,+\infty)$ the function α increases to $+\infty$. We say that $\alpha\in L^0$ if $\alpha\in L$ and $\alpha(x(1+o(1)))=(1+o(1))\alpha(x)$ as $x\to+\infty$. Further, $\alpha\in L_{si}$ if $\alpha\in L$ and for any $c>0$ $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$. It is easy to see that $L_{si}\subset L^0$. The functions from L_{si} are called slowly increasing. We remark that for every slowly increasing function α there exists a function $\alpha_1\in L$ such that $\alpha_1(x)=(1+o(1))\alpha(x)$ and $\frac{x\alpha_1'(x)}{\alpha_1(x)}\to 0$ as $x\to+\infty$. Therefore, in the sequel we will assume that the function $\alpha\in L_{si}$ satisfies the condition $\frac{x\alpha'(x)}{\alpha(x)}\to 0$ as $x\to+\infty$.

Theorem 1. *Let $q>1$, $p>1$, $\alpha\in L$ and Φ be a positive function continuous on $(-\infty,+\infty)$ and increasing to $+\infty$. If*

$$\varlimsup_{\sigma\to+\infty}\frac{\alpha(\Phi(q\sigma))}{\alpha^p(\Phi(\sigma))}\le\xi_1<+\infty \tag{3}$$

then

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\ln\alpha(\Phi(\sigma))}{\ln\sigma}\le\frac{\ln p}{\ln q}, \tag{4}$$

and if

$$\varliminf_{\sigma\to+\infty}\frac{\alpha(\Phi(q\sigma))}{\alpha^p(\Phi(\sigma))}\ge\xi_2>0 \tag{5}$$

then

$$\varliminf_{\sigma\to+\infty}\frac{\ln\ln\alpha(\Phi(\sigma))}{\ln\sigma}\ge\frac{\ln p}{\ln q}. \tag{6}$$

Proof. By (3), one has $\alpha(\Phi(q\sigma))\le\xi\alpha^p(\Phi(\sigma))$ for all $\xi>\xi_1$ and $\sigma\ge\sigma_0=\sigma_0(\xi)>0$. We put $\sigma_j=\sigma_0q^j$ for $j\ge 1$ and assume that $\sigma_j\le\sigma\le\sigma_{j+1}$. Then

$$\begin{aligned}\alpha(\Phi(\sigma))\le\alpha(\Phi(\sigma_{j+1}))=\alpha(\Phi(q\sigma_j))\le\xi\alpha^p(\Phi(\sigma_j))=\xi\alpha^p(\Phi(q\sigma_{j-1}))\le\xi(\xi\alpha^p(\Phi(\sigma_{j-1})))^p=\\ =\xi^{1+p}\alpha^{p^2}(\Phi(\sigma_{j-1}))\le\cdots\le\xi^{1+p+p^2+\cdots+p^m}\alpha^{p^{m+1}}(\Phi(\sigma_{j-m}))\le\cdots\le\\ \le\xi^{1+p+p^2+\cdots+p^j}\alpha^{p^{j+1}}(\Phi(\sigma_0))=\xi^{\frac{p^{j+1}-1}{p-1}}\alpha^{p^{j+1}}(\Phi(\sigma_0)),\end{aligned} \tag{7}$$

whence

$$\ln\alpha(\Phi(\sigma))\le\frac{p^{j+1}-1}{p-1}\ln\xi+p^{j+1}\ln\alpha(\Phi(\sigma_0))=p^{j+1}\left(\frac{\ln\xi}{p-1}+\ln\alpha(\Phi(\sigma_0))\right)-\frac{\ln\xi}{p-1},$$

i. e. $\ln\ln\alpha(\Phi(\sigma))\le(j+1)\ln p+O(1),\ j\to\infty$. But

$$j+1=\frac{\ln\sigma_{j+1}-\ln\sigma_0}{\ln q}=\frac{\ln\sigma_j+\ln q-\ln\sigma_0}{\ln q}\le\frac{\ln\sigma+\ln q-\ln\sigma_0}{\ln q}.$$

Therefore, $\ln\ln\alpha(\Phi(\sigma))\le\frac{\ln\sigma\ln p}{\ln q}+O(1),\quad\sigma\to+\infty$, and, thus, inequality (4) is true.

Under condition (5) for $\xi \in (0, \xi_2)$ and all $\sigma \geq \sigma_0 = \sigma_0(\xi) > 0$ we have the inequality $\alpha(\Phi(q\sigma)) \geq \xi\alpha^p(\Phi(\sigma))$. Choosing s_j as above, for $\sigma_j \leq \sigma \leq \sigma_{j+1}$ instead of (7) we obtain $\alpha(\Phi(\sigma)) \geq \alpha(\Phi(\sigma_j)) \geq \cdots \geq \xi^{\frac{p^j-1}{p-1}}\alpha^{p^j}(\Phi(\sigma_0))$, whence $\ln\ln\alpha(\Phi(\sigma)) \geq \frac{\ln\sigma\ln p}{\ln q} + O(1)$, $\sigma \to +\infty$, that is inequality (6) is true. □

Corollary 1. *Let $q > 1$, $p > 1$, $\alpha \in L$ and Φ be a positive continuous function on $(-\infty, +\infty)$ increasing to $+\infty$. If*

$$0 < \varliminf_{\sigma\to+\infty} \frac{\alpha(\Phi(q\sigma))}{\alpha^p(\Phi(\sigma))} \leq \varlimsup_{\sigma\to+\infty} \frac{\alpha(\Phi(q\sigma))}{\alpha^p(\Phi(\sigma))} < +\infty$$

then

$$\lim_{\sigma\to+\infty} \frac{\ln\ln\alpha(\Phi(\sigma))}{\ln\sigma} = \frac{\ln p}{\ln q}. \tag{8}$$

Corollary 2. *Let $h > 0$, $p > 1$, $\alpha \in L$ and Φ be a positive continuous on $(-\infty, +\infty)$ function increasing to $+\infty$. If*

$$\varlimsup_{\sigma\to+\infty} \frac{\alpha(\Phi(\sigma+h))}{\alpha^p(\Phi(\sigma))} \leq \xi_1 < +\infty \tag{9}$$

then

$$\varlimsup_{\sigma\to+\infty} \frac{\ln\ln\alpha(\Phi(\sigma))}{\sigma} \leq \frac{\ln p}{h}, \tag{10}$$

and if

$$\varliminf_{\sigma\to+\infty} \frac{\alpha(\Phi(\sigma+h))}{\alpha^p(\Phi(\sigma))} \geq \xi_2 > 0 \tag{11}$$

then

$$\varliminf_{\sigma\to+\infty} \frac{\ln\ln\alpha(\Phi(\sigma))}{\ln\sigma} \geq \frac{\ln p}{h}. \tag{12}$$

Indeed, let $x = e^\sigma$, $q = e^h$ and $\Phi_1(x) = \Phi(\ln x)$. Then

$$\frac{\alpha(\Phi(\sigma+h))}{\alpha^p(\Phi(\sigma))} = \frac{\alpha(\Phi(\ln(e^h e^\sigma))}{\alpha^p(\Phi(\ln e^\sigma))} = \frac{\alpha(\Phi_1(qx))}{\alpha^p(\Phi_1(x))}.$$

Therefore, (9) implies (3) with Φ_1 instead of Φ and with x instead of σ. By Theorem 1 inequality (4) with such replacement is true. Thus,

$$\varlimsup_{\sigma\to+\infty} \frac{\ln\ln\alpha(\Phi(\sigma))}{\sigma} = \varlimsup_{\sigma\to+\infty} \frac{\ln\ln\alpha(\Phi_1(e^\sigma))}{\sigma} = \varlimsup_{x\to+\infty} \frac{\ln\ln\alpha(\Phi_1(x))}{\ln x} \leq \frac{\ln p}{\ln q} = \frac{\ln p}{h},$$

i. e. inequality (10) is true. By analogy we prove that (11) implies (12).

Corollary 3. *Let $h > 0$, $p > 1$, $\alpha \in L$ and Φ be a positive continuous function on $(-\infty, +\infty)$ increasing to $+\infty$. If*

$$0 < \varliminf_{\sigma\to+\infty} \frac{\alpha(\Phi(\sigma+h))}{\alpha^p(\Phi(\sigma))} \leq \varlimsup_{\sigma\to+\infty} \frac{\alpha(\Phi(\sigma+h))}{\alpha^p(\Phi(\sigma))} < +\infty$$

then

$$\lim_{\sigma\to+\infty} \frac{\ln\ln\alpha(\Phi(\sigma))}{\sigma} = \frac{\ln p}{h}.$$

Corollary 4. *Let $q > 1$, $p > 1$, $\alpha \in L$ and Φ be a positive continuous function on $(-\infty, 0)$ increasing to $+\infty$. If*

$$\varlimsup_{\sigma\uparrow 0} \frac{\alpha(\Phi(\sigma/q))}{\alpha^p(\Phi(\sigma))} \le \xi_1 < +\infty \tag{13}$$

then

$$\varlimsup_{\sigma\uparrow 0} \frac{\ln\ln \alpha(\Phi(\sigma))}{\ln(1/|\sigma|} \le \frac{\ln p}{\ln q}, \tag{14}$$

and if

$$\varliminf_{\sigma\uparrow 0} \frac{\alpha(\Phi(\sigma/q))}{\alpha^p(\Phi(\sigma))} \ge \xi_2 > 0 \tag{15}$$

then

$$\varliminf_{\sigma\uparrow 0} \frac{\ln\ln \alpha(\Phi(\sigma))}{\ln(1/|\sigma|} \ge \frac{\ln p}{\ln q}. \tag{16}$$

Indeed, let $\Phi(\sigma) = \Phi_1(1/|\sigma|)$. Then in view of (13)

$$\varlimsup_{\sigma\uparrow 0} \frac{\alpha(\Phi_1(q/|\sigma|))}{\alpha^p(\Phi_1(1/|\sigma|))} = \varlimsup_{\sigma\uparrow 0} \frac{\alpha(\Phi(\sigma/q))}{\alpha^p(\Phi(\sigma))} \le \xi_1 < +\infty$$

and by Theorem 1

$$\varlimsup_{\sigma\uparrow 0} \frac{\ln\ln \alpha(\Phi(\sigma))}{\ln(1/|\sigma|)} = \varlimsup_{\sigma\uparrow 0} \frac{\ln\ln \alpha(\Phi_1(1/|\sigma|))}{\ln(1/|\sigma|)} \le \frac{\ln p}{\ln q},$$

that is we obtain (14). By analogy (15) implies (16).

Corollary 5. *Let $q > 1$, $p > 1$, $\alpha \in L$ and Φ be a positive continuous on $(-\infty, 0)$ function increasing to $+\infty$. If*

$$0 < \varliminf_{\sigma\uparrow 0} \frac{\alpha(\Phi(\sigma/q))}{\alpha^p(\Phi(\sigma))} \le \varlimsup_{\sigma\uparrow 0} \frac{\alpha(\Phi(\sigma/q))}{\alpha^p(\Phi(\sigma))} < +\infty,$$

then

$$\lim_{\sigma\uparrow 0} \frac{\ln\ln \alpha(\Phi(\sigma))}{\ln(1/|\sigma|)} = \frac{\ln p}{\ln q}.$$

3. Asymptotic behaviours of Dirichlet series. We begin with entire Dirichlet series (that is $A = +\infty$) and use the following result from [5].

Lemma 1. *Let $\alpha_0 \in L_{si}$ and $\beta_0 \in L^0$ be continuously differentiable functions and*

$$\frac{d\beta_0^{-1}(c\alpha_0(x))}{d\ln x} = O(1), \quad x \to +\infty, \tag{17}$$

for each $c \in (0, +\infty)$. Suppose that

$$\varkappa_n = \frac{\ln|a_n| - \ln|a_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow +\infty,$$

$\alpha_0(\lambda_{n+1}) = (1 + o(1))\alpha_0(\lambda_n)$ and $\ln n = o(\lambda_n \beta_0^{-1}(c\alpha_0(\lambda_n))$ as $n \to \infty$ for each $c \in (0, +\infty)$. Then for entire Dirichlet series (1) we have

$$\varliminf_{\sigma\to+\infty} \frac{\alpha_0(\ln M(\sigma, F))}{\beta_0(\sigma)} = \varliminf_{n\to\infty} \frac{\alpha_0(\lambda_n)}{\beta_0\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}.$$

At first using Lemma 1 we prove the following theorem.

Theorem 2. *Let* $A=+\infty$, $p>1$ *and* $\alpha\in L$. *If*

$$\varlimsup_{n\to\infty}\frac{\ln\ln\alpha(\lambda_n)}{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}=\eta^*>0 \tag{18}$$

then

$$\varlimsup_{\sigma\to+\infty}\frac{\alpha(\ln M(q\sigma,F))}{\alpha^p(\ln M(\sigma,F))}=+\infty \tag{19}$$

for each $q>p^{1/\eta^*}$.

If the function α *is continuously differentiable and* $\ln\alpha(e^x)\in L_{si}$, *the coefficients and the exponents satisfy* $\varkappa_n\nearrow+\infty$, $\ln\ln\alpha(\lambda_{n+1})=(1+o(1))\ln\ln\alpha(\lambda_n)$, $\ln n=O(\lambda_n)$ *as* $n\to\infty$ *and*

$$\varliminf_{n\to\infty}\frac{\ln\ln\alpha(\lambda_n)}{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}=\eta_*<+\infty, \tag{20}$$

then

$$\varliminf_{\sigma\to+\infty}\frac{\alpha(\ln M(q\sigma,F))}{\alpha^p(\ln M(\sigma,F))}=0 \tag{21}$$

for each $q<p^{1/\eta_*}$.

Proof. By (18) for every $\eta\in(0,\eta^*)$ there exists an increasing to $+\infty$ sequence (n_k) such that $\ln|a_{n_k}|\ge-\lambda_{n_k}\ln^{1/\eta}\alpha(\lambda_{n_k})$ for all k. We choose $\sigma_k=\ln^{1/\eta}\alpha(\lambda_{n_k})+1$. Then by the Cauchy inequality

$$\ln M(\sigma_k,F)\ge\ln|a_{n_k}|+\sigma_k\lambda_{n_k}\ge-\lambda_{n_k}\ln^{1/\eta}\alpha(\lambda_{n_k})+\lambda_{n_k}(\ln^{1/\eta}\alpha(\lambda_{n_k})+1)=\lambda_{n_k}=$$
$$=\alpha^{-1}(\exp\{(\sigma_k-1)^\eta)\}),$$

whence in view of the arbitrariness of η we obtain the inequality

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\ln\alpha(\ln M(\sigma,F))}{\ln\sigma}\ge\eta^*. \tag{22}$$

Now choose $\Phi(\sigma)=\ln M(\sigma,F)$ in Theorem 1. If

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\ln\alpha(\ln M(\sigma,F))}{\ln\sigma}>\frac{\ln p}{\ln q} \tag{23}$$

then by Theorem 1 equality (19) is true. In view of (22) inequality (23) is true provided $\eta^*>\ln p/\ln q$, that is $q>p^{1/\eta^*}$. First part of Theorem 2 is proved.

For the proof of second part we choose $\alpha_0(x)\equiv\ln\ln\alpha(x)$ and $\beta_0(x)\equiv\ln x$ for $x\ge x_0$. Then $\alpha_0\in L_{si}$, $\beta_0\in L^0$ and $\beta_0^{-1}(c\alpha_0(x))=\ln^c\alpha(x)$. Since $\ln\alpha(e^x)\in L_{si}$ we have $\ln\alpha(x)=\gamma(\ln x)$, where $\gamma\in L_{si}$. Therefore,

$$\frac{d\beta_0^{-1}(c\alpha_0(x))}{d\ln x}=\frac{d\ln^c\alpha(x)}{d\ln x}=\frac{d\gamma^c(\ln x)}{d\ln x}=c\gamma^{c-1}(\ln x)\gamma'(\ln x)=$$
$$=c\frac{\gamma'(\ln x)}{\gamma(\ln x)}\ln x\frac{\gamma^c(\ln x)}{\ln x}\to0,\quad x\to+\infty,$$

because from the slow growth of γ it follows that $\frac{t\gamma'(t)}{\gamma(t)} \to 0$ and $\frac{\gamma^c(t}{t} \to 0$ as $t \to +\infty$ for each $c \in (0, +\infty)$. Thus, condition (17) holds. Finally, the condition $\ln n = o(\lambda_n \beta_0^{-1}(c\alpha_0(\lambda_n))$ as $n \to \infty$ is equivalent to the condition $\ln n = o(\lambda_n \ln^c \alpha(\lambda_n))$ as $n \to \infty$ and holds if $\ln n = O(\lambda_n)$ as $n \to \infty$. Therefore, all the assumptions of Lemma 1 hold and equality (18) by Lemma 1 is equivalent to

$$\varliminf_{\sigma\to+\infty} \frac{\ln\ln\alpha(\ln M(\sigma, F))}{\ln\sigma} = \eta_* < +\infty. \tag{24}$$

On the other hand, if

$$\varliminf_{\sigma\to+\infty} \frac{\ln\ln\alpha(\ln M(\sigma, F))}{\ln\sigma} < \frac{\ln p}{\ln q} \tag{25}$$

then by Theorem 1 equality (21) is true. Since $q < p^{1/\eta_*}$, (24) implies (25). □

For the proof of the following theorem we will use Lemma 1 and Corollary 3.

Theorem 3. *Let $A = +\infty$, $p > 1$ and $\alpha \in L$. If*

$$\varlimsup_{n\to\infty} \frac{\lambda_n \ln\ln\alpha(\lambda_n)}{-\ln|a_n|} = \eta^* > 0, \tag{26}$$

then

$$\varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln M(\sigma + h, F))}{\alpha^p(\ln M(\sigma, F))} = +\infty \tag{27}$$

for each $h > \ln p/\eta^*$.

If the function α is continuously differentiable and $\frac{x\alpha'(x)}{\alpha(x)\ln\alpha(x)} = O(1)$ as $x \to +\infty$, the coefficients and the exponents satisfy the conditions $\varkappa_n \nearrow +\infty$, $\ln\ln\alpha(\lambda_{n+1}) = (1 + o(1))\ln\ln\alpha(\lambda_n)$, $\ln n = o(\lambda_n \ln\ln\alpha(\lambda_n))$ as $n \to \infty$ and

$$\varliminf_{n\to\infty} \frac{\lambda_n \ln\ln\alpha(\lambda_n)}{-\ln|a_n|} = \eta_* < +\infty \tag{28}$$

then

$$\lim_{\sigma\to+\infty} \frac{\alpha(\ln M(\sigma + h, F))}{\alpha^p(\ln M(\sigma, F))} = 0 \tag{29}$$

for each $h < \ln p/\eta_*$.

Proof. By (26) for every $\eta \in (0, \eta^*)$ there exists an increasing to $+\infty$ sequence (n_k) such that $\ln|a_{n_k}| \geq -\lambda_{n_k} \ln\ln\alpha(\lambda_{n_k})/\eta$. We set $\sigma_k = \ln\ln\alpha(\lambda_{n_k})/\eta + 1$. Then as above we have $\ln M(\sigma_k, F) \geq \lambda_{n_k} = \alpha^{-1}\left(\exp\{e^{\eta(\sigma_k - 1)}\}\right)$, that is

$$\frac{\ln\ln\alpha(\ln M(\sigma_k, F))}{\sigma_k} \geq (1 + o(1))\eta, \quad k \to \infty,$$

whence in view of the arbitrariness of η we obtain

$$\varlimsup_{\sigma\to+\infty} \frac{\ln\ln\alpha(\ln M(\sigma, F))}{\sigma} \geq \eta^*. \tag{30}$$

By Corollary 2 if

$$\varlimsup_{\sigma\to+\infty} \frac{\ln\ln\alpha(\Phi(\sigma))}{\sigma} > \frac{\ln p}{h}, \tag{31}$$

then equality (27) is true. In view of (30) inequality (31) is true provided $\eta^* > \ln p/h$, that is $h > \ln p/\eta^*$. The first part of Theorem 3 is proved.

For the proof of the second part we set $\alpha_0(x) \equiv \ln\ln\alpha(x)$ and $\beta_0(x) \equiv x$ for $x \geq x_0$. Then $\beta_0 \in L^0$ and the condition $\frac{x\alpha'(x)}{\alpha(x)\ln\alpha(x)} = O(1)$ as $x \to +\infty$ implies that $\frac{x\alpha_0'(x)}{\alpha_0(x)} = \frac{x\alpha'(x)}{\alpha(x)\ln\alpha(x)\ln\ln\alpha(x)} \to 0$ as $x \to +\infty$, that is $\alpha_0 \in L_{si}$. In view of the equality $\beta_0^{-1}(c\alpha_0(x)) = c\ln\ln\alpha(x)$ and the condition $\frac{x\alpha'(x)}{\alpha(x)\ln\alpha(x)} = O(1)$ as $x \to +\infty$ condition (17) holds. The condition $\ln n = o(\lambda_n \ln\ln\alpha(\lambda_n))$ as $n \to \infty$ implies that $\ln n = o(\lambda_n\beta_0^{-1}(c\alpha_0(\lambda_n))$ as $n \to \infty$ for each $c \in (0, +\infty)$. Thus, all assumptions of Lemma 1 hold and equality (28) by Lemma 1 implies

$$\varliminf_{\sigma\to+\infty} \frac{\ln\ln\alpha(\ln M(\sigma, F))}{\sigma} = \eta_* < +\infty. \tag{32}$$

On the other hand, if

$$\varliminf_{\sigma\to+\infty} \frac{\ln\ln\alpha(\ln M(\sigma, F))}{\sigma} < \frac{\ln p}{h} \tag{33}$$

then by Corollary 2 equality (29) is true. Since $h < \ln p/\eta_*$, (32) implies (33). □

Now we consider Dirichlet series with null abscissa of absolute convergence. Here we use the following result from [6].

Lemma 2. *Let* $\alpha_0 \in L_{si}$, $\beta_0 \in L_{si}$,

$$\frac{x}{\beta_0^{-1}(c\alpha_0(x))} \uparrow +\infty, \quad x \to +\infty, \tag{34}$$

and

$$\alpha_0\left(\frac{x}{\beta_0^{-1}(c\alpha_0(x))}\right) = (1 + o(1))\alpha_0(x), \quad x \to +\infty, \tag{35}$$

for each $c \in (0, +\infty)$. *Suppose that* $\varkappa_n \nearrow 0$, $\alpha_0(\lambda_{n+1}) = (1 + o(1))\alpha_0(\lambda_n)$ *and* $\alpha_0(\lambda_n) = o\left(\beta_0\left(\frac{\lambda_n}{\ln n}\right)\right)$ *as* $n \to \infty$. *Then for a Dirichlet series (1) with null abscissa of absolute convergence*

$$\varliminf_{\sigma\uparrow 0} \frac{\alpha_0(\ln M(\sigma, F))}{\beta_0\left(\frac{1}{|\sigma|}\right)} = \varliminf_{n\to\infty} \frac{\alpha_0(\lambda_n)}{\beta_0\left(\frac{\lambda_n}{\ln|a_n|}\right)}.$$

Theorem 4. *Let* $A = 0$, $p > 1$, $\alpha(e^x) \in L^0$ *and* $\ln\ln\alpha(x) = o(\ln x)$ *as* $\to +\infty$. *If*

$$\varlimsup_{n\to\infty} \frac{\ln\ln\alpha(\lambda_n)}{\ln(\lambda_n/\ln^+|a_n|)} = \eta^* > 0 \tag{36}$$

then

$$\varlimsup_{\sigma\uparrow 0} \frac{\alpha(\ln M(\sigma/q, F))}{\alpha^p(\ln M(\sigma, F))} = +\infty \tag{37}$$

for each $q > p^{1/\eta^*}$.

If the function α *is continuously differentiable and* $\ln\alpha(x) \in L_{si}$, *the coefficients and the exponents satisfy the conditions* $\varkappa_n \nearrow 0$, $\ln\ln\alpha(\lambda_{n+1}) = (1 + o(1))\ln\ln\alpha(\lambda_n)$ *as* $n \to \infty$,

$$\varlimsup_{n\to\infty} \frac{\ln\ln n}{\ln\lambda_n} = \delta < 1 \tag{38}$$

and

$$\varliminf_{n\to\infty}\frac{\ln\ln\alpha(\lambda_n)}{\ln(\lambda_n/\ln^+|a_n|)}=\eta_*<+\infty, \tag{39}$$

then

$$\varlimsup_{\sigma\uparrow 0}\frac{\alpha(\ln M(\sigma/q,F))}{\alpha^p(\ln M(\sigma,F))}=0 \tag{40}$$

for each $q<p^{1/\eta_*}$.

Proof. By (36) for every $\eta\in(0,\eta^*)$ there exists an increasing to $+\infty$ sequence (n_k) such that $\ln|a_{n_k}|\ge\lambda_{n_k}\ln^{-1/\eta}\alpha(\lambda_{n_k})$. We choose $\sigma_k=-\ln^{-1/\eta}\alpha(\lambda_{n_k})/2$. Then by the Cauchy inequality

$$\ln M(\sigma_k,F)\ge\ln|a_{n_k}|-|\sigma_k|\lambda_{n_k}\ge\lambda_{n_k}(\ln^{-1/\eta}\alpha(\lambda_{n_k})-|\sigma_k|)=|\sigma_k|\lambda_{n_k}=$$
$$=|\sigma_k|\alpha^{-1}\left(\exp\left\{\frac{1}{(2|\sigma_k|)^\eta}\right\}\right).$$

The condition $\ln\ln\alpha(x)=o(\ln x)$ as $\to+\infty$ implies the condition $\ln x=o(\ln\alpha^{-1}(\exp\{e^x\}))$ as $\to+\infty$. Therefore, since $\alpha(e^x)\in L^0$ we have

$$\alpha(\ln M(\sigma_k,F)\ge\alpha\left(\exp\left\{\ln\alpha^{-1}\left(\exp\left\{\frac{1}{(2|\sigma_k|)^\eta}\right\}\right)-\ln\frac{1}{|\sigma_k|}\right\}\right)=$$
$$=\alpha\left(\exp\left\{(1+o(1))\ln\alpha^{-1}\left(\exp\left\{\frac{1}{(2|\sigma_k|)^\eta}\right\}\right)\right\}\right)=(1+o(1))\exp\left\{\frac{1}{(2|\sigma_k|)^\eta}\right\}$$

as $k\to\infty$, whence in view of the arbitrariness of η we obtain the inequality

$$\varlimsup_{\sigma\uparrow 0}\frac{\ln\ln\alpha(\ln M(\sigma,F))}{\ln(1/|\sigma|)}\ge\eta^*>0. \tag{41}$$

By Corollary 4, if

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\ln\alpha(\ln M(\sigma,F))}{\ln\sigma}>\frac{\ln p}{\ln q}, \tag{42}$$

then equality (37) is true. In view of (41) inequality (42) is true provided $\eta^*>\ln p/\ln q$, that is $q>p^{1/\eta^*}$. The first part of Theorem 4 is proved.

For the proof of second part we choose $\alpha_0(x)\equiv\ln\ln\alpha(x)$ and $\beta_0(x)\equiv\ln x$ for $x\ge x_0$. Then $\alpha_0\in L_{si}$, and condition (34) is equivalent to the condition $\frac{x}{\ln^c\alpha(x)}\uparrow+\infty$ as $x\to+\infty$. It is easy to verify that the previous condition holds if $\frac{x\alpha'(x)}{\alpha(x)\ln\alpha(x)}\to 0$ as $x\to+\infty$, that is $\ln\alpha(x)\in L_{si}$. Since $\alpha(e^x)\in L^0$ and $\ln\ln\alpha(x)=o(\ln x)$ as $x\to\infty$ we have

$$\alpha_0\left(\frac{x}{\beta_0^{-1}(c\alpha_0(x))}\right)=\ln\ln\alpha\left(\frac{x}{\ln^c\alpha(x))}\right)=\ln\ln\alpha\left(e^{\ln x-c\ln\ln\alpha(x)}\right)=$$
$$=\ln\ln\alpha\left(e^{(1+o(1))\ln x}\right)=(1+o(1))\ln\ln\alpha\left(e^{\ln x}\right)=(1+o(1))\alpha_0(x).$$

Finally, in view of (38) $\ln\lambda_n-\ln\ln n\ge\ln\lambda_n-(1+o(1))\delta\ln\lambda_n=(1+o(1))(1-\delta)\ln\lambda_n$, $n\to\infty$. Since $\ln\ln\alpha(x)=o(\ln x)$ as $x\to\infty$ we obtain

$$\alpha_0(\lambda_n)=\ln\ln\alpha(\lambda_n)=o(\ln\lambda_n)=o(\ln\lambda_n-\ln\ln n)=o\left(\ln\frac{\lambda_n}{\ln n}\right)=o\left(\beta_0\left(\ln\frac{\lambda_n}{\ln n}\right)\right).$$

Thus, all the assumptions of Lemma 2 hold and equality (36) by Lemma 2 implies the equality

$$\varliminf_{\sigma\to+\infty} \frac{\ln\ln\alpha(\ln M(\sigma,F))}{\ln(1/|\sigma|)} = \eta_* < +\infty. \tag{43}$$

On the other hand, if

$$\varliminf_{\sigma\to+\infty} \frac{\ln\ln\alpha(\ln M(\sigma,F))}{\ln(1/|\sigma|)} < \frac{\ln p}{\ln q} \tag{44}$$

then by Corollary 4 equality (40) is true. Since $q < p^{1/\eta_*}$, (43) implies (44). □

4. Remarks. If in Corollaries 1, 3 and 5 we choose $\Phi(\sigma) = \ln M(\sigma, F)$ then we obtain the corresponding results for Dirichlet series. For example, the following proposition is true.

Proposition 1. *If $q > 1$, $p > 1$ and $\alpha \in L$ then for an entire Dirichlet series the condition*

$$0 < \varliminf_{\sigma\to+\infty} \frac{\alpha(\ln M(q\sigma,F))}{\alpha^p(\ln M(\sigma,F))} \le \varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln M(q\sigma,F))}{\alpha^p(\ln M(\sigma,F))} < +\infty$$

yields the existence of the limits

$$\lim_{\sigma\to+\infty} \frac{\ln\ln\alpha(\ln M(\sigma,F))}{\ln\sigma} = \frac{\ln p}{\ln q}.$$

It is easy to verify that the function $\alpha(x) \equiv \ln x (x \ge x_0)$ satisfies the assumptions of Theorem 4 and the second part of Theorem 2 and the function $\alpha(x) \equiv x (x \ge x_0)$ satisfies the assumptions of the second part of Theorem 3. Therefore, for example, the following proposition is true.

Proposition 2. *Let $A = +\infty$ and $p > 1$. If $\varlimsup\limits_{n\to\infty} \frac{\lambda_n \ln\ln\lambda_n}{-\ln|a_n|} = \eta^* > 0$ then*

$$\varlimsup_{\sigma\to+\infty} \frac{\ln M(\sigma+h,F)}{\ln^p M(\sigma,F)} = +\infty$$

for each $h > \ln p/\eta^*$.

If $\varkappa_n \nearrow +\infty$, $\ln\ln\lambda_{n+1} = (1+o(1))\ln\ln\lambda_n$, $\ln n = o(\lambda_n \ln\ln\lambda_n)$ as $n \to \infty$ and $\varliminf\limits_{n\to\infty} \frac{\lambda_n \ln\ln\lambda_n}{-\ln|a_n|} = \eta_* < +\infty$ then $\varliminf\limits_{\sigma\to+\infty} \frac{\ln M(\sigma+h,F)}{\ln^p M(\sigma,F)} = 0$ for each $h < \ln p/\eta_*$.

For entire Dirichlet series (1) let $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \geqslant 0\}$ be the maximal term. The following problem is opened: to investigate the asymptotic behaviour of the function $\frac{\alpha(\ln M(\sigma+h,F))}{\alpha^p(\ln\mu(\sigma,F))}$ and similar to it. A partial solution to this problem is possible to get using the results from [7]. Here we, for example, will consider the case where $\alpha(x) \equiv x (x \ge x_0)$ and use Proposition 2 and the following result from [8]: *if $\varphi \in L$, $\ln|a_n| \le -\lambda_n\varphi(\lambda_n)$ for $n \ge n_0$ and* $\ln n = O(\varphi(\lambda_n))$ *as $n \to \infty$ then* $\ln M(\sigma, F) = (1 + o(1)) \ln\mu(\sigma, F)$ *as $\sigma \to +\infty$*. Setting $\varphi(x) = \ln\ln x (x \ge x_0)$, we obtain the following proposition.

Proposition 3. *Let $A = +\infty$, $p > 1$, $\ln n = O(\ln\ln\lambda_n)$ as $n \to \infty$ and $\varlimsup\limits_{n\to\infty} \frac{\lambda_n \ln\ln\lambda_n}{-\ln|a_n|} < +\infty$. If $\varlimsup\limits_{n\to\infty} \frac{\lambda_n \ln\ln\lambda_n}{-\ln|a_n|} = \eta^* > 0$ then $\varlimsup\limits_{\sigma\to+\infty} \frac{\ln M(\sigma+h,F)}{\ln^p \mu(\sigma,F)} = +\infty$ for each $h > \ln p/\eta^*$.*

If $\varkappa_n \nearrow +\infty$, $\ln\ln\lambda_{n+1} = (1+o(1))\ln\ln\lambda_n$ and $\varliminf\limits_{n\to\infty} \frac{\lambda_n \ln\ln\lambda_n}{-\ln|a_n|} = \eta_* < +\infty$ then $\varliminf\limits_{\sigma\to+\infty} \frac{\ln M(\sigma+h,F)}{\ln^p \mu(\sigma,F)} = 0$ for each $h < \ln p/\eta_*$.

REFERENCES

1. Goodstein R.L. Complex functions. – New York, 1965.
2. Singh S.K. *On the maximum modulus and the means of an entire function*// Matem. Visnyk. – 1976. – V.13(28). – P. 211–213. (in Ukrainian)
3. Sheremeta M.N. *Asymptotic behaviour of Mittag-Leffler type functions and its application 1*// Teoriya funcii, func. anal., ich prilozh. – 1970. – V.10. – P. 97–111. (in Russian)
4. Sheremeta M.N. *Asymptotic behaviour of Mittag-Leffler type functions and its application 2*// Teoriya funcii, func. anal., ich prilozh. – 1971. – V.11. – P. 41–54. (in Russian)
5. Pjanylo Ja.D., Sheremeta M.N. *On the growth of entire functions represented by Dirichlet series*// Izv. vuzov. Matem. – 1975. – №10. – P. 91–93. (in Russian)
6. Hal' Yu.M., Sheremeta M.N. *On the growth of analytic in a half-plane functions represented by Dirichlet series*// Dokl. AN USSR. Ser. A. – 1978. – №12. – P. 1065–1067. (in Russian)
7. Filevych P.V. *To the Sheremeta theorem concerning relations between the maximal term and the maximum modulus of entire Dirichlet series*// Mat. Stud. – 2000. – V.13, №2. – P. 139–144.
8. Sheremeta M.N. *Full equivalence of the logarithms of the maximum modulus and the maximal term of an entire Dirichlet series*// Matem. Zametki. – 1990. – V.47, №6. – P. 119–123. (in Russian)

Ivan Franko National University of Lviv
m_m_sheremeta@list.ru

Received 11.08.2015