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HOMOTOPY COOPERADS

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The theory of 2-monads is used as a ground to study non-symmetric cooperads. We give a new definition of homotopy cooperads. Ordinary and homotopy cooperads are placed in lax $\mathcal{C}at$-operads which are lax algebras over the free-operad strict 2-monad. We give an example of a homotopy cooperad cofree with respect to ordinary cooperads.

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Теория 2-монад используется как основа для изучения несимметрических кооперад. Дано новое определение гомотопических кооперад. Обычные и гомотопические кооперады помещены в расслабленые $\mathcal{C}at$-операды, которые являются расслаблеными алгебрами над строгой 2-монадой свободной операды. Приведен пример гомотопической кооперады косвободной по отношению к обычным кооперадам.

1. Introduction. We use the theory of 2-monads as a ground to (non-symmetric) cooperads. The main tool is an explicit construction of colax morphism classifier for the free-operad strict 2-monad $\mathbb{T}\colon \underline{\mathcal{C}at}^{\mathbb{N}} \to \underline{\mathcal{C}at}^{\mathbb{N}}$, where $\underline{\mathcal{C}at}$ denotes the 2-category of categories and the set of natural numbers is $\mathbb{N} = \{0, 1, 2, \dots\}$.

The choice between 'lax' and 'colax' for monoidal categories made in [2] was dictated by the relationship: a lax monoidal category gives rise to a multicategory. We shall switch the usage to the opposite. The reason is that lax (in the sense opposite to [2]) monoidal categories are lax algebras over the free-monoid 2-monad $\top$. Similarly colax monoidal categories are colax algebras over the same 2-monad. This way the present article becomes closer to many other works (in which often 'oplax' is used instead of 'colax').

Strict/strong=pseudo/lax/colax $\mathbb{T}$-algebras and $\mathbb{T}$-morphisms are renamed to corresponding $\mathcal{C}at$-operads and $\mathcal{C}at$-multifunctors. In sequel article a (co)operad will mean a non-symmetric (co)operad. (Co)operads in a lax $\mathcal{C}at$-operad C are defined as (co)lax $\mathcal{C}at$-multifunctors $\mathsf{1} \to \mathsf{C}$, where $\mathsf{1}$ is the terminal $\mathcal{C}at$-operad. Applying to the latter the colax morphism classifier Q_c we get $Q_c\mathsf{1} = \mathbf{TR}^{\mathrm{op}}$, where the $\mathcal{C}at$-operad of trees $\mathbf{TR}$ is introduced by T. Leinster ([12]). This motivates the definition of homotopy cooperads in a strong $\mathcal{C}at$-operad C as lax $\mathcal{C}at$-multifunctors $\mathbf{TR}^{\mathrm{op}} \to \mathsf{C}$. We discuss in detail an example of a homotopy cooperad, so called "cofree cooperad", which is different from the familiar notion of cofree conilpotent cooperad. This is the reason for introducing the notion of a homotopy cooperad.

Let $\perp\colon \mathcal{C} \to \mathcal{C}$ be a comonad. The category of $\perp$-coalgebras is denoted $\mathcal{C}_\perp$. The following lemma is well known.

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Lemma 1 (Kleisli). *The forgetful functor* $\mathcal{C}_\perp \to \mathcal{C}$ *has the right adjoint* $\perp\colon \mathcal{C} \to \mathcal{C}_\perp$*: for any* $V \in \operatorname{Ob}\mathcal{C}$ *and any* $\perp$*-coalgebra* (X, δ) *there are mutually inverse bijections*

$$\begin{aligned} \mathcal{C}(X,V) &\longleftrightarrow \mathcal{C}_\perp(X, \perp V), \\ f &\longmapsto \hat{f} = \delta \cdot (\perp f), \\ \check{g} = g \cdot \varepsilon &\longleftarrow g. \end{aligned}$$

This is generalized to multicategories in [2, Lemma 5.3]. The composition of $h\colon U \to V$ and $k\colon V \to W$ is denoted $h \cdot k = kh = k \circ h\colon U \to W$. $\mathcal{O}_{\mathsf{sk}}$ denotes the category of finite totally ordered sets $\mathbf{n} = \{1, 2, \dots, n\}$, $n \geqslant 0$, and their non-decreasing maps (denoting this category by Δ would introduce the risk of confusing it with its full subcategory that does not contain the empty set as an object).

In Section 2 we recall definitions of algebras of various kind over a strict 2-monad.

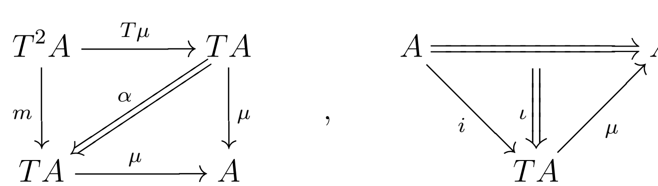
In Section 3 we concentrate on the free-monoid strict 2-monad $\top\colon \underline{\mathcal{C}at} \to \underline{\mathcal{C}at}$ and strict/strong=pseudo/lax/colax algebras over it, which are corresponding monoidal categories. We recall the related notion of homotopy comonoid due to T. Leinster. An example of homotopy comonad is given. Coalgebras over it are ordinary coalgebras in a strong monoidal category. We discuss also m-cluster trees. A 2-cluster tree is a family of trees which can be substituted into internal vertices of a given tree. 2-cluster trees are morphisms of the category $\mathbf{TR}$. m-cluster trees are composable sequences of $m-1$ morphisms of $\mathbf{TR}$.

The fourth section is devoted to the free-operad strict 2-monad $\mathbb{T}\colon \underline{\mathcal{C}at}^{\mathbb{N}} \to \underline{\mathcal{C}at}^{\mathbb{N}}$. $\mathbb{T}$-algebras, $\mathbb{T}$-morphisms and $\mathbb{T}$-transformations are called $\mathcal{C}at$-operads, $\mathcal{C}at$-multifunctors and $\mathcal{C}at$-transformations. There is a colax morphism classifier 2-functor $Q_c\colon {}_p\mathbb{T}\text{-}\mathrm{Alg}_c \to {}_p\mathbb{T}\text{-}\mathrm{Alg}_p$, left biadjoint to the inclusion ${}_p\mathbb{T}\text{-}\mathrm{Alg}_p \hookrightarrow {}_p\mathbb{T}\text{-}\mathrm{Alg}_c$ (pseudoalgebras with pseudomorphisms are included into pseudoalgebras with colax morphisms). The new possibility exploited in this article is to define (co)operads inside a strong $\mathcal{C}at$-operad C as (co)lax $\mathcal{C}at$-multifunctors $1 \to \mathsf{C}$. We identify augmented cooperads with non-counital cooperads. There is a comonad $\perp\!\!\!\perp$, whose coalgebras constitute part of non-counital cooperads. Under certain condition on categories $\mathsf{C}(n)$ we show that the category of conilpotent non-counital cooperads is isomorphic to the category of $\perp\!\!\!\perp$-coalgebras. Summing up, there is a way to obtain augmented cooperads in the form $\perp\!\!\!\perp_\circ X = \mathbb{1} \oplus \perp\!\!\!\perp X$.

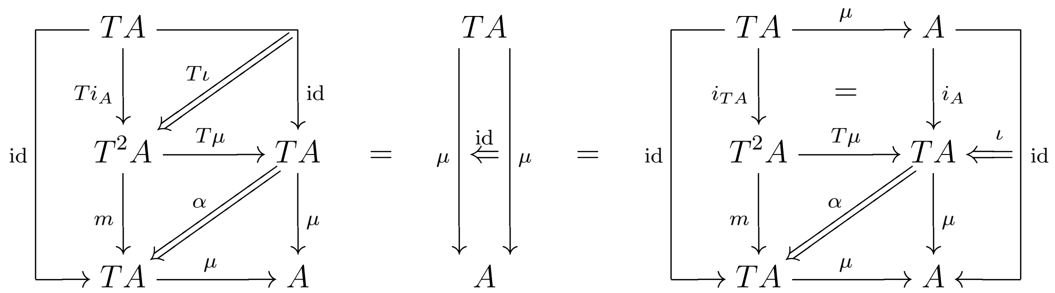
Homotopy cooperads are introduced in the fifth section as lax $\mathcal{C}at$-multifunctors $Q_c 1 = \mathbf{TR}^{\mathrm{op}} \to \mathsf{C}$. There is an example of a homotopy cooperad, "completion" $\tilde{\perp\!\!\!\perp}_\circ X$ of $\perp\!\!\!\perp_\circ X$. In a sense $\tilde{\perp\!\!\!\perp}_\circ X$ is cofree with respect to the ordinary cooperads in C. We describe a wide class of morphisms $\tilde{\perp\!\!\!\perp}_\circ X \to \tilde{\perp\!\!\!\perp}_\circ Y$ of homotopy cooperads.

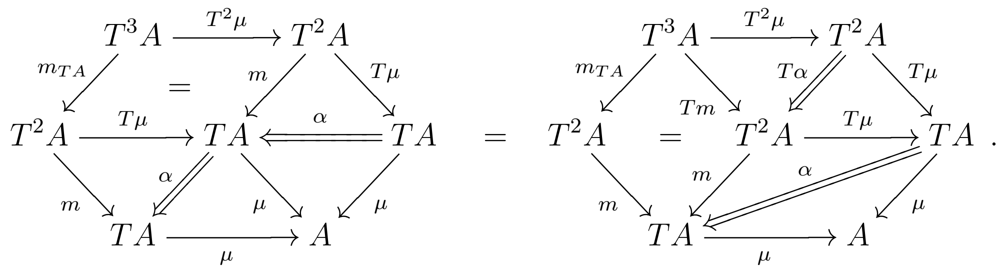
2. 2-monads. For the sake of simplicity we use the notation as if $(\mathcal{S}\mathrm{et}, \times, \mathbf{1})$ were a symmetric strict monoidal category (instead of weak=strong one). A 2-category $\mathcal{K}$ is viewed as a $\mathcal{C}at$-category. A strict 2-monad $(T, m, i)\colon \mathcal{K} \to \mathcal{K}$ is a monad enriched over $\mathcal{C}at$.

Recall [19, 1], [7, Definition 4.1] that a *lax T-algebra* for a strict 2-monad (T, m, i) in a 2-category $\mathcal{K}$ is the quadruple $(A, \mu\colon TA \to A, \alpha, \iota)$, where 2-morphisms



($\Longrightarrow$ means identity 1-morphism) satisfy the equations


(1)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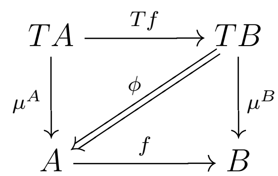
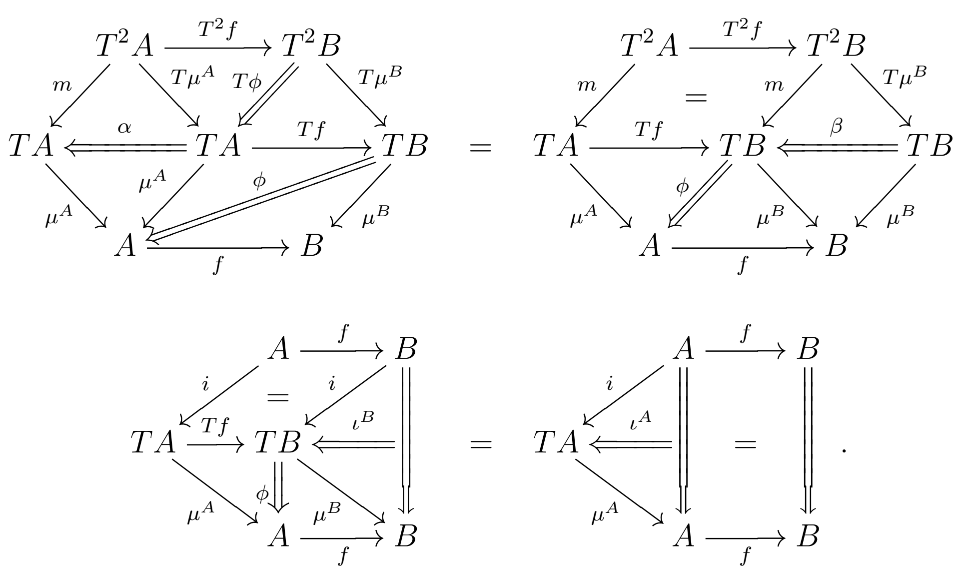
Colax T-algebras are the same notion with the direction of 2-morphisms α, ι reversed. Pseudo (=weak =strong) T-algebras are lax ones with invertible α, ι. Strict T-algebras are those with $\alpha = \mathrm{id}$, $\iota = \mathrm{id}$.

Let A, B be lax T-algebras for a 2-monad (T, m, i) in a 2-category $\mathcal{K}$.

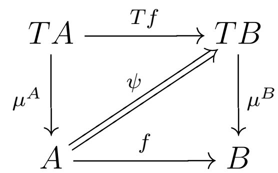
Definition 1 (*e. g.* [9, Section 4.1])**.** A *lax T-morphism* $(f, \phi) : (A, \mu^A, \alpha, \iota^A) \to (B, \mu^B, \beta, \iota^B)$ is a 1-morphism $f : A \to B \in \mathcal{K}$ and a 2-morphism



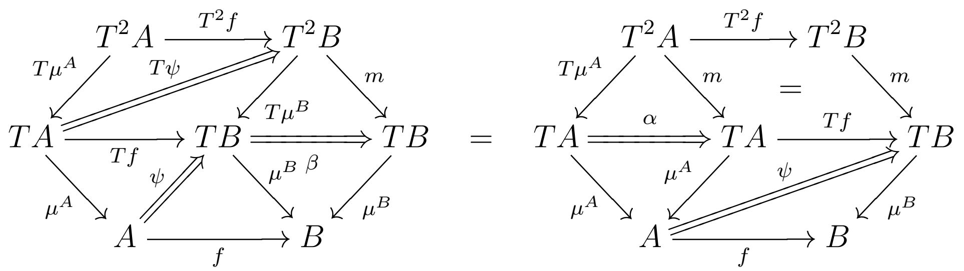
such that

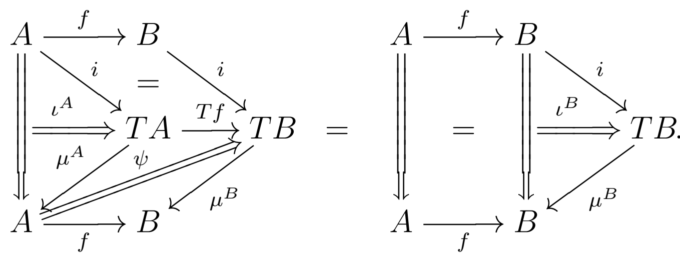

(3)

Definition 2. A *colax T-morphism* $(f, \psi)\colon (A, \mu^A, \alpha, \iota^A) \to (B, \mu^B, \beta, \iota^B)$ is a 1-morphism $f\colon A \to B \in \mathcal{K}$ and a 2-morphism

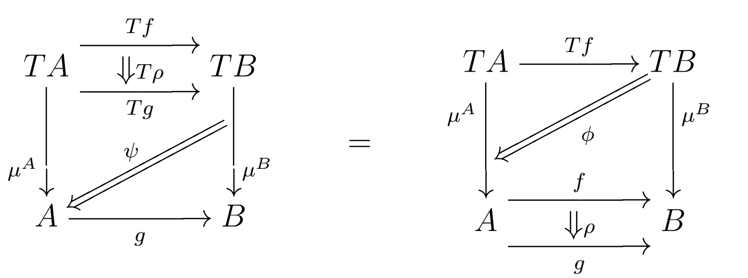


such that

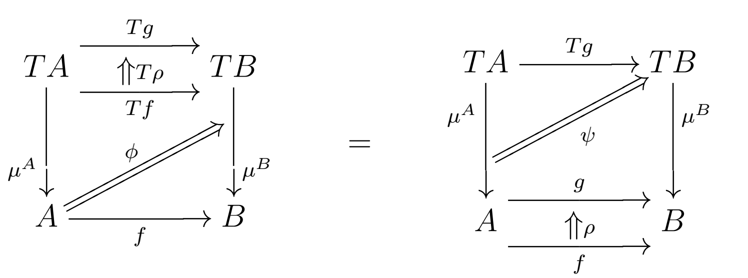




Definition 3 (*e.g.* [9, Section 4.1])**.** A *T-transformation* between lax T-morphisms $\rho\colon (f, \phi) \to (g, \psi)\colon (A, \mu^A, \alpha, \iota^A) \to (B, \mu^B, \beta, \iota^B)$ is a 2-morphism $\rho\colon f \to g\colon A \to B$ in $\mathcal{K}$ such that



A *T-transformation* between colax T-morphisms $\rho\colon (f, \phi) \to (g, \psi)\colon (A, \mu^A, \alpha, \iota^A) \to (B, \mu^B, \beta, \iota^B)$ is a 2-morphism $\rho\colon f \to g\colon A \to B$ in $\mathcal{K}$ such that



Let s, p, l, c stand for strict, pseudo, lax and colax. Denote by ${}_lT\text{-Alg}_l$ (resp. ${}_pT\text{-Alg}_p$, ${}_pT\text{-Alg}_c$) the 2-category of lax T-algebras (resp. pseudo (=weak =strong) T-algebras), lax T-morphisms (resp. pseudo (=weak =strong) T-morphisms, colax T-morphisms), and their T-transformations, see *e.g.* [9, Section 4.1]. Similarly for the other reasonable combinations ${}_xT\text{-Alg}_y$ with $x, y \in \{s, p, l, c\}$.

Exercise 1. Let $\mathcal{K}$ be a 2-category with a 2-monad $(T, m, i)\colon \mathcal{K} \to \mathcal{K}$. For any $B \in \operatorname{Ob}\mathcal{K}$ and any lax T-algebra $(C, \mu^C, \gamma, \iota^C)$ there is a functor

$$E\colon \mathcal{K}(B,C) \longrightarrow {}_lT\text{-Alg}_l(TB,C),\ (P\colon B \to C) \longmapsto \big(TB \xrightarrow{TP} TC \xrightarrow{\mu^C} C, \phi\big), \tag{4}$$

$$\phi = \begin{array}{ccccc} T^2B & \xrightarrow{T^2P} & T^2C & \xrightarrow{T\mu^C} & TC \\ {\scriptstyle m}\big\downarrow & = & {\scriptstyle m}\big\downarrow & \overset{\gamma}{\swarrow} & \big\downarrow{\scriptstyle \mu^C} \\ TB & \xrightarrow{TP} & TC & \xrightarrow{\mu^C} & C \end{array}$$

with the obvious value on 2-morphisms. (*Hint*: use associativity (2) for γ and unitality (1) for γ and ι.)

The following statement is already known.

Lemma 2 (Kleisli lemma for 2-categories)**.** *Let $\mathcal{K}$ be a 2-category with a 2-monad $(T, m, i)\colon \mathcal{K} \to \mathcal{K}$. For any $B \in \operatorname{Ob}\mathcal{K}$ and any strong T-algebra $(C, \mu^C, \gamma, \iota^C)$ the functor $E\colon \mathcal{K}(B,C) \to {}_pT\text{-Alg}_p(TB,C)$ given by* (4) *is an equivalence.*

Proof. Introduce the functor

$$G\colon {}_pT\text{-Alg}_p(TB,C) \longrightarrow \mathcal{K}(B,C), \quad \big((g,\psi)\colon TB \to C\big) \longmapsto \big(B \xrightarrow{i} TB \xrightarrow{g} C\big). \tag{5}$$

Then invertibility of ι implies that $E \cdot G \cong \mathrm{Id}_{\mathcal{K}(B,C)}$. Due to (3) there is a natural transformation $G \cdot E \to \mathrm{Id}_{{}_pT\text{-Alg}_p(TB,C)}$ involving ψ from $(g,\psi) \in {}_pT\text{-Alg}_p(TB,C)$. This natural transformation is invertible thanks to the invertibility of ψ. $\square$

Recall that a 2-monad T is said to have a rank if it preserves α-filtered colimits for some regular cardinal α ([1]). Assume that $\mathcal{K}$ is complete and cocomplete and T has a rank. It is shown in [1] that the inclusion 2-functor ${}_sT\text{-Alg}_s \hookrightarrow {}_sT\text{-Alg}_p$ (resp. ${}_sT\text{-Alg}_s \hookrightarrow {}_sT\text{-Alg}_l$) admits a left adjoint, which could be called a pseudo/strict (resp. lax/strict) morphism classifier. Furthermore, C. Hermida ([7, Theorem 6.1]) proved that under certain assumptions the inclusion 2-functor ${}_pT\text{-Alg}_p \hookrightarrow {}_pT\text{-Alg}_c$ admits a left biadjoint $Q_c\colon {}_pT\text{-Alg}_c \to {}_pT\text{-Alg}_p$, which could be called a colax/pseudo morphism classifier. We shall encounter examples of such Q_c below.

3. Coalgebras as colax morphisms. As an example of a 2-monad consider the free-monoid strict 2-monad $\top\colon \underline{\mathcal{C}at} \to \underline{\mathcal{C}at}$,

$$\top\mathcal{C} = \coprod_{k\in\mathbb{N}} \mathcal{C}^k.$$

A lax $\top$-algebra $(\mathcal{C}, \mu, \alpha, \iota)$ is the same as a lax monoidal category [12, Section 3.1] or a lax monoidal category $(\mathcal{C}, \otimes^I, \lambda^f, \rho)$ (Definition 2.5 [2] applied to $\mathcal{C}^{\mathrm{op}}$): $\otimes^I = \mu\big|_{\mathcal{C}^I}$, $\lambda^f = \alpha\big|_{\prod_{j\in J}\mathcal{C}^{f^{-1}j}}$ for any $f\colon I \to J \in \mathcal{O}_{\mathsf{sk}}$, $\iota = \rho\colon X \to \otimes^{\mathbf{1}} X$. A strong $\top$-algebra is the same as a strong monoidal category [*ibid.*] or an unbiased monoidal category [12, Definition 3.1.1]. A (co)lax $\top$-morphism $(F,\phi)\colon (\mathcal{C}, \otimes, \lambda, \rho) \to (\mathcal{D}, \otimes, \lambda, \rho)$ between lax $\top$-algebras is the same as a (co)lax monoidal functor between lax monoidal categories ([12, Definition 3.1.3] or Definition 2.6 ([2]) with appropriate arrows reversed). A $\top$-transformation is the same as a monoidal transformation ([12, Definition 3.1.4] or [2, Definition 2.7] mutatis mutandis).

The free-monoid monad $\top\colon \mathcal{S}et \to \mathcal{S}et$ preserves small filtered colimits, thus is finitary. This follows from the property of $\mathcal{S}et$ that filtered colimits commute with finite limits, see Example 9. This is extended further to a monad $\top\colon \mathcal{C}at \to \mathcal{C}at$ being finitary and to 2-monad $\top\colon \underline{\mathcal{C}at} \to \underline{\mathcal{C}at}$ being finitary. The monad $\top\colon \mathcal{S}et \to \mathcal{S}et$ as well as $\top\colon \mathcal{C}at \to \mathcal{C}at$ is cartesian [12, Example 4.1.4].

Remark 1. As noticed in [7, Section 11] the 2-category $\underline{\mathcal{C}at}$ admits a calculus of bimodules. The 2-functor $\top\colon \underline{\mathcal{C}at} \to \underline{\mathcal{C}at}$ preserves pullbacks and comma-categories. It preserves also coidentifiers $f^* \circ g_\sharp \twoheadrightarrow f^* \bullet g_\sharp$, the notation and definitions are in [7, Section 2]. In fact, for $\underline{\mathcal{C}at}$ we have $f^* \bullet g_\sharp = \int_{\operatorname{dom} g} f^* \times g_\sharp$, the above projections are induction maps for the coend. The rest is the direct inspection.

A coalgebra (=comonoid) in a lax monoidal category $(\mathcal{M}, \otimes, \lambda, \rho)$ is defined as an algebra in the colax monoidal category $(\mathcal{M}^{\mathrm{op}}, \otimes, \lambda^{\mathrm{op}})$, or as a colax monoidal functor $C\colon \mathit{1} \to \mathcal{M}$, *cf.* [2, Definition 2.25] where $\mathit{1}$ is the terminal (one-morphism) category. Equivalently, it is an object C of $\mathcal{M}$ equipped with a morphism $\Delta_I\colon C \to C^{\otimes I}$ for each $I \in \operatorname{Ob} \mathcal{O}_{\mathsf{sk}}$ such that $\Delta_{\mathbf{1}} = \mathrm{id}$ and for every map $f\colon I \to J \in \mathcal{O}_{\mathsf{sk}}$ the following equation holds

$$\Delta_I = \bigl(C \xrightarrow{\Delta_J} C^{\otimes J} \xrightarrow{\otimes^{j\in J}\Delta_{f^{-1}j}} \otimes^{j\in J} C^{\otimes f^{-1}j} \xrightarrow{\lambda^f} C^{\otimes I}\bigr).$$

The following result of [2] treats algebras in colax monoidal categories, however, we cite it in dual form.

Proposition 1 (Proposition 2.27 of [2])**.** *A coalgebra C in a lax monoidal category $\mathcal{M}$ defines a lax monoidal functor*

$$(F, \phi^I)\colon (\mathcal{O}_{\mathsf{sk}}^{\mathrm{op}}, \sqcup_I, \mathrm{id}) \to (\mathcal{M}, \otimes^I, \lambda^f), \quad F(J) = C^{\otimes J},$$

$$(f^{\mathrm{op}}\colon I \to J) \in \mathcal{O}_{\mathsf{sk}}^{\mathrm{op}} \leftrightarrow (f\colon J \to I) \in \mathcal{O}_{\mathsf{sk}} \mapsto \Delta_C^f = \bigl(C^{\otimes I} \xrightarrow{\otimes^{i\in I}\Delta_{f^{-1}i}} \otimes^{i\in I} C^{\otimes f^{-1}i} \xrightarrow{\lambda^f} C^{\otimes J}\bigr).$$

Let $n_i \in \mathbb{N} = \operatorname{Ob} \mathcal{O}_{\mathsf{sk}}^{\mathrm{op}}$ for $i \in I \in \operatorname{Ob} \mathcal{O}_{\mathsf{sk}}$. The natural transformation $\phi^I\colon \otimes^{i\in I} C^{\otimes n_i} \to C^{\otimes \sum_{i\in I} n_i}$ is defined as $\lambda^g\colon \otimes^{i\in I} C^{\otimes g^{-1}i} \to C^{\otimes N}$ for $N = \sqcup_{i\in I}\mathbf{n}_i$, $g\colon N \to I \in \mathcal{O}_{\mathsf{sk}}$ such that $|g^{-1}i| = n_i$. If $\mathcal{M}$ is strong monoidal (λ is invertible), then (F, ϕ) is strong monoidal.

If $\mathcal{M}$ is a strict monoidal category then a coalgebra C in $\mathcal{M}$ gives rise to a strict monoidal functor, as proven by S. Mac Lane ([16, Proposition VII.5.1]).

The previous statement of the proposition admits a stronger version: the described functor

$$\mathcal{M}\mathrm{on}_c(\mathit{1}, \mathcal{M}) \equiv {}_p\top\text{-}\mathrm{Alg}_c(\mathit{1}, \mathcal{M}) \to {}_p\top\text{-}\mathrm{Alg}_p(\mathcal{O}_{\mathsf{sk}}^{\mathrm{op}}, \mathcal{M}) \equiv \mathcal{M}\mathrm{on}(\mathcal{O}_{\mathsf{sk}}^{\mathrm{op}}, \mathcal{M})$$

is an equivalence ([14, Propositions 1.11, 1.12]). Further strengthening is given by Proposition 4.

Proposition 2. *There are a 2-functor $Q_c\colon {}_s\top\text{-}\mathrm{Alg}_c \to {}_s\top\text{-}\mathrm{Alg}_s$ and a 2-natural isomorphism ${}_s\top\text{-}\mathrm{Alg}_s(Q_c\mathcal{B}, \mathcal{C}) \xrightarrow{\cong} {}_s\top\text{-}\mathrm{Alg}_c(\mathcal{B}, \mathcal{C})$, which turn Q_c into a left adjoint to the inclusion ${}_s\top\text{-}\mathrm{Alg}_s \hookrightarrow {}_s\top\text{-}\mathrm{Alg}_c$.*

Since 2-monad $\top$ is finitary, this statement follows from [1, Theorem 3.13]. Also it follows from [7, Theorem 6.1.1] whose hypothesis is satisfied due to Remark 1. We leave the other proofs in this section to the reader since the results follow from analogous statements of Section 4.

For the moment we recall that every unbiased monoidal category is equivalent to a strict monoidal category ([12, Theorem 3.1.6]). We reformulate this as follows.

Proposition 3. *The embedding 2-functor* ${}_s\top\text{-}\mathrm{Alg}_s \hookrightarrow {}_p\top\text{-}\mathrm{Alg}_p$ *admits a left adjoint* $L\colon {}_p\top\text{-}\mathrm{Alg}_p \to {}_s\top\text{-}\mathrm{Alg}_s$ *such that the unit of the adjunction* $\mathcal{B} \to L\mathcal{B}$ *is an equivalence.*

A more general result is given by C. Hermida ([7, Corollary 7.5]).

Corollary 1. *The embedding 2-functor* ${}_s\top\text{-}\mathrm{Alg}_s \hookrightarrow {}_p\top\text{-}\mathrm{Alg}_p$ *is a biequivalence with a quasi-inverse* L.

Proposition 4. *There are a 2-functor* $Q_c\colon \mathcal{M}\mathrm{on}_c \to \mathcal{M}\mathrm{on}$ *and a 2-natural equivalence*

$$\mathcal{M}\mathrm{on}(Q_c\mathcal{B}, \mathcal{C}) \xrightarrow{\simeq} \mathcal{M}\mathrm{on}_c(\mathcal{B}, \mathcal{C}),$$

which turn Q_c *into left biadjoint to the inclusion* $\mathcal{M}\mathrm{on} \hookrightarrow \mathcal{M}\mathrm{on}_c$.

This follows from Hermida's Theorem 6.1 ([7]) applicable due to Remark 1.

Remark 2. The 2-functor Q_c from Proposition 2 is the restriction of the 2-functor Q_c from Proposition 4

$$\big({}_s\top\text{-}\mathrm{Alg}_c \xrightarrow{Q_c} {}_s\top\text{-}\mathrm{Alg}_s \hookrightarrow {}_p\top\text{-}\mathrm{Alg}_p\big) = \big({}_s\top\text{-}\mathrm{Alg}_c \hookrightarrow {}_p\top\text{-}\mathrm{Alg}_c \xrightarrow{Q_c} {}_p\top\text{-}\mathrm{Alg}_p\big). \tag{6}$$

Corollary 2 (to Propositions 2, 4)**.** *There are a 2-functor* $Q_l\colon {}_s\top\text{-}\mathrm{Alg}_l \to {}_s\top\text{-}\mathrm{Alg}_s$ *(resp.* $Q_l\colon {}_p\top\text{-}\mathrm{Alg}_l \to {}_p\top\text{-}\mathrm{Alg}_p$*) and a 2-natural isomorphism (resp. equivalence)*

$${}_s\top\text{-}\mathrm{Alg}_s(Q_l\mathcal{B}, \mathcal{C}) \xrightarrow{\cong} {}_s\top\text{-}\mathrm{Alg}_l(\mathcal{B}, \mathcal{C}), \quad \textit{resp.} \quad {}_p\top\text{-}\mathrm{Alg}_p(Q_l\mathcal{B}, \mathcal{C}) \xrightarrow{\simeq} {}_p\top\text{-}\mathrm{Alg}_l(\mathcal{B}, \mathcal{C}),$$

which turn Q_l *into a left adjoint (resp. biadjoint) to the inclusion* ${}_s\top\text{-}\mathrm{Alg}_s \hookrightarrow {}_s\top\text{-}\mathrm{Alg}_l$ *(resp.* ${}_p\top\text{-}\mathrm{Alg}_p \hookrightarrow {}_p\top\text{-}\mathrm{Alg}_l$*). 2-functors* Q_l *agree similarly to* (6).

The proof and the construction of Q_l is made by dualising the results for Q_c using opposite monoidal categories. Thus, $Q_l\mathcal{B}$ is the universal strict monoidal category generated over $\top\mathcal{B}$ by the morphisms $\xi^I\colon (X_i)_{i\in I} \to \otimes_{\mathcal{B}}^{i\in I} X_i$ subject to naturality, normalisation and multiplicativity.

In particular, if $\mathcal{B} = \mathit{1}$ then the category $Q_c\mathit{1}$ is $\mathcal{O}_{\mathsf{sk}}^{\mathrm{op}}$. This is the monoid classifier for monoidal categories ([7, Corollary 9.1, Theorem 9.2]). The classical description of the monoid classifier for monoidal categories is usually credited to Lawvere.

3.1. Trees. We use conventions, terminology and notation for trees from [15, § 2.1]. A *rooted tree* t can be defined as a *parent map* $P_t\colon V(t) \to V(t)$, where $V(t)$ is a finite set (of vertices), such that $|\operatorname{Im}(P_t^k)| = 1$ for some $k \in \mathbb{N}$. The only element $r \in \operatorname{Im}(P_t^k)$ is called the root. An oriented graph without loops G is constructed out of P, whose set of vertices is $V(t)$ and arrows are $v \to P_t(v)$ if vertex v is not the root. Since G is a connected graph, whose number of edges is one less than the number of vertices, it is a tree. Thus the rooted tree is oriented towards the root. The only oriented path connecting a vertex v with the root consists of v, $P(v)$, $P^2(v)$, …, the root. This gives a partial ordering on the set of all vertices $V(t)$, namely, $u \preccurlyeq v$ if and only if v lies on the oriented path connecting u with the root. For each vertex $p \in V(t)$ the non-negative number $|p| = |P_t^{-1}(p) \setminus \{\mathrm{root}\}|$ of its children is called its arity.

A *planar rooted tree* is a rooted tree with a chosen total ordering $\leqslant$ of the set of incoming edges for each vertex. A *rooted tree with inputs* is a rooted tree t with a chosen subset $\operatorname{Inp}(t)$ of the set $L(t)$ of leaves, vertices without incoming edges. For instance, a 1-vertex tree has one leaf — the root. It gives rise to two rooted trees with inputs: $\tau[0] = \bullet = (\mathbf{1}; \operatorname{Inp} \bullet = \varnothing)$ and $\theta_0 = \circ = (\mathbf{1}; \operatorname{Inp} \circ = \mathbf{1})$. The *set of internal vertices* is defined by $\mathrm{v}(t) = V(t) - \operatorname{Inp}(t)$.

The total ordering $\trianglelefteqslant$ of the set $V(t)$ of vertices of a planar rooted tree t is defined as follows. For any two vertices $u, v \in V(t)$ either they are comparable with respect to $\preccurlyeq$ and we set $u \trianglelefteqslant v$ iff $u \preccurlyeq v$, or they are not $\preccurlyeq$-comparable. Then there are unique $n, m \in \mathbb{N}_{>0}$ such that $P^n(u) = P^m(v)$ and both $P^{n-1}(u) \neq P^{m-1}(v)$ are distinct from the root. We set $u \triangleleft v$ iff $P^{n-1}(u) < P^{m-1}(v)$ in the set of descendants (incoming edges) of $P^n(u)$.

Let $\mathbf{tr}$ be the set of isomorphism classes of planar rooted trees with inputs. It splits up, $\mathbf{tr} = \bigsqcup_{n\geqslant 0} \mathbf{tr}(n)$, in components $\mathbf{tr}(n) = \{t \in \mathbf{tr} \mid |\operatorname{Inp} t| = n\}$. We view $\mathbf{tr} = (\mathbf{tr}(n))_{n\geqslant 0}$ as a collection of sets. Thus, for any finite set S the notation $\mathbf{tr}(S)$ means $\mathbf{tr}(|S|)$. Also for any internal vertex p the notation $\mathbf{tr}\,|p|$ means $\mathbf{tr}(|p|)$.

For each tree $t \in \mathbf{tr}$ there is an operation of substituting trees into internal vertices

$$I_t \colon \prod_{p\in \mathrm{v}(t)} \mathbf{tr}\,|p| \to \mathbf{tr}(\operatorname{Inp} t) \tag{7}$$

which takes a family $(t_p)_{p\in\mathrm{v}(t)}$ with $|\operatorname{Inp} t_p| = |p|$ to the tree $I_t(t_p \mid p \in \mathrm{v}(t))$ obtained from t by replacing each internal vertex $p \in \mathrm{v}(t)$ with the tree t_p. If $t = \circ$, then $I_\circ() = \circ$. If $t \neq \circ$, then $\tau = I_t(t_p \mid p \in \mathrm{v}(t))$ has

$$V(\tau) = \bigsqcup_{p\in\mathrm{v}(t)} V(t_p)/\sim . \tag{8}$$

Notice that the projection map

$$\pi \colon \bigsqcup_{p\in(\mathrm{v}(t),\trianglelefteqslant)} (V(t_p), \trianglelefteqslant) \longrightarrow (V(\tau), \trianglelefteqslant) \tag{9}$$

is not necessarily non-decreasing, where the first set is lexicographically totally ordered ($p \triangleleft q$ for $p, q \in \mathrm{v}(t)$ implies $(p, x) \triangleleft (q, y)$ for all $x \in \mathrm{v}(t_p)$, $y \in \mathrm{v}(t_q)$).

The set $\operatorname{Inp}\tau$ of input vertices of τ by definition consists of single elements (u, x), $u \in \mathrm{v}(t)$, $x \in \operatorname{Inp} t_u \setminus \phi_u\big(\mathrm{v}(t) \cap (P_t^{-1}u \setminus \operatorname{root}(t))\big)$. Clearly, $\operatorname{Inp}\tau \subset L(\tau)$ and $\operatorname{Inp}\tau = V(\tau) - \mathrm{v}(\tau)$, where

$$\mathrm{v}(\tau) = \mathrm{v}\big(I_t(t_p \mid p \in \mathrm{v}(t))\big) \simeq \bigsqcup_{p\in\mathrm{v}(t)} \mathrm{v}(t_p). \tag{10}$$

The set of *staged trees* $\mathbf{str}(m)$ consists of sequences in $\mathcal{O}_{\mathsf{sk}}$

$$t = \big(t(0) \xrightarrow{t_1} t(1) \xrightarrow{t_2} t(2) \xrightarrow{t_3} \dots\ t(m-1) \xrightarrow{t_m} t(m) = \mathbf{1}\big),$$
$$\mathbf{str}(m) = \{t \in \mathcal{C}at([m], \mathcal{O}_{\mathsf{sk}}) \mid t(m) = \mathbf{1}\}.$$

It is partitioned into subsets $\mathbf{str}(n, m)$ of staged trees t of height m with $t(0) = \mathbf{n}$. The set $\mathbf{str}(n, m)$ is naturally embedded into $\mathbf{tr}(n)$ so that $\operatorname{Inp} t = t(0)$, respectively, $\mathrm{v}(t) = \bigcup_{j=1}^{m} t(j)$.

3.2. Homotopy comonoids.

Definition 4. A *homotopy comonoid* in a lax monoidal category $(\mathcal{M}, \otimes, \lambda)$ is a lax monoidal functor $(F, \phi^I) \colon (\mathcal{O}_{\mathsf{sk}}^{\mathrm{op}}, \sqcup_I, \mathrm{id}) \to (\mathcal{M}, \otimes^I, \lambda^f)$.

If $(\mathcal{M}, \otimes, \lambda)$ is a strong monoidal category then the notion of homotopy (co)monoid is due to T. Leinster ([10, Definition 2.2]) with detailed exposition in [11]. Based on this, Leinster defined homotopy monoidal categories ([12, Definition 3.3.7]). Examples of homotopy comonoids are given in [14].

Definition 5. A *homotopy comonad* is a homotopy comonoid in the strict monoidal category of endofunctors $\mathcal{M} = \operatorname{End}\mathcal{C}$ for some category $\mathcal{C}$.

The necessity to consider such structures is motivated by the following example.

Example 1. We take a strong monoidal category $\mathcal{V}$ for $\mathcal{C}$ and consider the functor $F\colon \mathcal{O}_{\mathsf{sk}}^{\mathrm{op}} \to \mathcal{M} = \operatorname{End}\mathcal{V}$, $F(m)(X) = \prod_{t\in\mathbf{str}(m)} X^{\otimes t(0)}$. A morphism $f^{\mathrm{op}}\colon J \to I \in \mathcal{O}_{\mathsf{sk}}^{\mathrm{op}}$ which corresponds to the map $f\colon I \to J \in \mathcal{O}_{\mathsf{sk}}$ is taken by F to the morphism

$$F(f^{\mathrm{op}})\colon \prod_{r\in\mathbf{str}(J)} X^{\otimes r(0)} \to \prod_{t\in\mathbf{str}(I)} X^{\otimes t(0)},$$

$$F(f^{\mathrm{op}})\cdot \mathrm{pr}_t = \mathrm{pr}_{[J]\xrightarrow{[f]}[I]\xrightarrow{t}\mathcal{O}_{\mathsf{sk}}}\colon \prod_{r\in\mathbf{str}(J)} X^{\otimes r(0)} \to X^{\otimes t(0)}.$$

Recall that $t \in \mathbf{str}(I)$ is a functor $t\colon [I] \to \mathcal{O}_{\mathsf{sk}}$ such that $t(|I|) = \mathbf{1}$. The functor $[f]\colon [J] \to [I]$ is described in [2, Section 2.1]. Notice that $[f](0) = 0$ and $[f](|J|) = |I|$.

If $(F, \zeta)\colon \mathcal{O}_{\mathsf{sk}}^{\mathrm{op}} \to \operatorname{End}\mathcal{C}$ is a homotopy comonad then it makes sense to define F-coalgebras as follows.

Definition 6. An *F-coalgebra* is an object C of $\mathcal{C}$ together with a coaction morphism $\delta\colon C \to F(1)(C)$ such that $\big[C \xrightarrow{\delta} F(1)(C) \xrightarrow{F(1)(!^{\mathrm{op}})} F(0)(C)\big] = \zeta^0$, with $!^{\mathrm{op}}\colon \mathbf{1} \to \mathbf{0} \in \mathcal{O}_{\mathsf{sk}}^{\mathrm{op}}$ corresponding to the only map $!\colon \varnothing \to \mathbf{1} \in \mathcal{O}_{\mathsf{sk}}$ and

$$\begin{array}{ccccc} C & \xrightarrow{\delta} & F(1)(C) & \xrightarrow{F(1)(\delta)} & F(1)F(1)(C) \\ \Big\downarrow{\scriptstyle\delta} & & = & & \Big\downarrow{\scriptstyle\zeta^2_{1,1}} \\ F(1)(C) & & \xrightarrow{\quad F(\mathsf{V}^{\mathrm{op}})\quad} & & F(2)(C) \end{array}$$

with $\mathsf{V}^{\mathrm{op}}\colon \mathbf{1} \to \mathbf{2} \in \mathcal{O}_{\mathsf{sk}}^{\mathrm{op}}$ corresponding to the only map $\mathsf{V}\colon \mathbf{2} \to \mathbf{1} \in \mathcal{O}_{\mathsf{sk}}$.

In Example 1 an F-coalgebra is a map $\delta = (\Delta_n)_{n\geqslant 0}\colon C \to \prod_{n\in\mathbb{N}} C^{\otimes n}$ such that $\Delta_1 = \mathrm{id}\colon C \to C^{\otimes 1}$ and

$$\begin{array}{ccccc} C & \xrightarrow{\delta} & \prod_{k\in\mathbb{N}} C^{\otimes k} & \xrightarrow{\prod_k \delta^{\otimes k}} & \prod_{k\in\mathbb{N}} \bigotimes_{i\in\mathbf{k}} \prod_{n_i\in\mathbb{N}} C^{\otimes n_i} \\ \Big\downarrow{\scriptstyle\delta} & & = & & {\scriptstyle\prod\zeta}\ \Big\downarrow{\scriptstyle\zeta^2_{1,1}} \\ \prod_{n\in\mathbb{N}} C^{\otimes n} & & \xrightarrow{\quad F(\mathsf{V}^{\mathrm{op}})\quad} & & \prod_{k\in\mathbb{N}} \prod_{(n_i)\in\mathbb{N}^k} C^{\otimes(n_1+\dots+n_k)}. \end{array}$$

The comultiplication $F(\mathsf{V}^{\mathrm{op}})$ in the homotopy comonad F is determined by $F(\mathsf{V}^{\mathrm{op}})\cdot \mathrm{pr}_{k;n_1,\dots,n_k} = \mathrm{pr}_{n_1+\dots+n_k}$. The above commutative diagram is equivalent to the following equation

$$\big[C \xrightarrow{\Delta_k} C^{\otimes k} \xrightarrow{\Delta_{n_1}\otimes\dots\otimes\Delta_{n_k}} C^{\otimes n_1} \otimes \dots \otimes C^{\otimes n_k} \xrightarrow{\cong} C^{\otimes(n_1+\dots+n_k)}\big] = \Delta_{n_1+\dots+n_k}. \qquad (11)$$

This amounts to ordinary coalgebra C in $\mathcal{V}$ with the associative comultiplication Δ_2 and the counit Δ_0, use equation (2.25.1) and Proposition 2.28 of [2].

Definition 7. A *morphism of F-coalgebras* is a morphism $f\colon C \to D \in \mathcal{C}$ such that

$$\begin{array}{ccc} C & \xrightarrow{\delta} & F(1)(C) \\ {\scriptstyle f}\downarrow & = & \downarrow{\scriptstyle F(1)(f)} \\ D & \xrightarrow{\delta} & F(1)(D) \end{array}$$

In Example 1 a morphism of F-coalgebras is the same as a morphism of ordinary coalgebras in $\mathcal{V}$. Thus, in this case the category of F-coalgebras is isomorphic to the category of coalgebras in $\mathcal{V}$.

Example 2. In a non-counital version of Example 1 we consider the functor $F'\colon \varDelta_{\text{sur}}^{\text{op}} \to \operatorname{End}\mathcal{V}$, $F'(m)(X) = \prod_{t\in\mathbf{sstr}(m)} X^{\otimes t(0)}$, where $\varDelta_{\text{sur}} \subset \mathcal{O}_{\mathsf{sk}}$ is the subcategory consisting of non-empty totally ordered sets $\mathbf{n}$, $n > 0$, and surjective non-decreasing maps, and $\mathbf{sstr}(m) \subset \mathbf{str}(m)$ consists of sequences of surjective maps (thus $\mathbf{sstr} \subset N(\varDelta_{\text{sur}})$). The remaining structure is that of F. An F'-coalgebra is a map $\delta = (\Delta_n)_{n>0}\colon C \to \prod_{n>0} C^{\otimes n}$ such that $\Delta_1 = \mathrm{id}$ and (11) holds for $k, n_1, \dots, n_k > 0$. Thus the category of F'-coalgebras is the category of non-counital coassociative coalgebras in $\mathcal{V}$.

3.3. Cluster trees.

Definition 8. An *m-cluster tree* $(n; t; (t_{p_1}); (t_{p_1,p_2}); \dots; (t_{p_1,p_2,\dots,p_{m-1}}))$, $m \geqslant 0$, is a collection consisting of an integer $n \in \mathbb{N}$, a tree t (a planar rooted tree with n inputs), a tree $t_{p_1} \in \mathbf{tr}\,|p_1|$ for each internal vertex $p_1 \in \mathrm{v}(t)$, a tree $t_{p_1,p_2} \in \mathbf{tr}\,|p_2|$ for each internal vertex $p_2 \in \mathrm{v}(t_{p_1})$, *etc.*, a tree $t_{p_1,p_2,\dots,p_{m-1}} \in \mathbf{tr}\,|p_{m-1}|$ for each internal vertex $p_{m-1} \in \mathrm{v}(t_{p_1,p_2,\dots,p_{m-2}})$.

By definition, a 0-cluster tree (n) has a single parameter, the number of inputs $n \in \mathbb{N}$. A 1-cluster tree is just a tree. For a 2-cluster tree $(t; (t_p)_{p\in\mathrm{v}(t)})$ one can substitute trees t_p into vertices $p \in \mathrm{v}(t)$ and get the resulting tree $I(t; (t_p)_{p\in\mathrm{v}(t)}) \overset{\text{def}}{=} I_t(t_p \mid p \in \mathrm{v}(t))$. Thus $I(t; (t_p)) \in \mathbf{tr}$ is obtained from a 2-cluster tree via multiplication in the operad $\mathbf{tr}$. Similarly, $I(t; (t_{p_1}); (t_{p_1,p_2}); \dots; (t_{p_1,p_2,\dots,p_{m-1}})) \in \mathbf{tr}$ is obtained from an m-cluster tree by applying multiplication in $\mathbf{tr}$ $m-1$ times. An m-cluster tree is precisely a datum needed for performing multiplication in $\mathbf{tr}$ $m-1$ times. In the particular case of $m = 1$ the tree $I(t)$ is t, and if $m = 0$, then $I(n)$ is the corolla $\tau[n] = \{\mathbf{n} \to \mathbf{1} \mid \mathrm{Inp} = \mathbf{n}\}$.

One can imagine a 2-cluster tree as a circled planar tree of [13, Appendix C.2.3]. More generally, an m-cluster tree $(t; (t_{p_1}); \dots; (t_{p_1,\dots,p_{m-1}}))$ as an ordinary tree $I(t; (t_{p_1}); \dots; (t_{p_1,\dots,p_{m-1}}))$, whose internal vertices are partitioned into clusters is subtrees $I(t_{p_1}; (t_{p_1,p_2})_{p_2}; \dots)$, $p_1 \in \mathrm{v}(t)$, which are in turn partitioned into clusters of the second level *etc.* Presence of $\circ$ spoils this picture, however, there is another presentation of cluster trees, more important for applications.

With a rooted tree t we associate the partially ordered set $(V(t), \preccurlyeq)$. Recall that $x \preccurlyeq y$ means that y lies on the oriented path connecting x with the root. The poset $(V(t), \preccurlyeq)$ admits suprema of arbitrary non-empty families. It suffices to notice that it admits joins $x \vee y = \sup_{\preccurlyeq}\{x, y\}$, the least upper bound of x and y.

If t is a planar rooted tree then the set $V(t)$ is equipped also with a total ordering $\trianglelefteq$ such that $x \preccurlyeq y$ implies $x \trianglelefteq y$, see Section 3.1.

Definition 9. The set $\mathbf{tr}$ of planar rooted trees with inputs is the set of objects of a category $\mathbf{Tr}$ whose morphisms $r \to t$ are mappings $f\colon V(r) \to V(t)$ such that

(i) the map $f\colon (V(r),\preccurlyeq)\to(V(t),\preccurlyeq)$ is non-decreasing;

(ii) the map f respects the partition into inputs and internal vertices, $f(\operatorname{Inp}(r))\subset\operatorname{Inp}(t)$, $f(\mathrm{v}(r))\subset\mathrm{v}(t)$;

(iii) the restriction $f|\colon \operatorname{Inp}(r)\to\operatorname{Inp}(t)$ is bijective;

(iv) $f(L(r))\supset L(t)$;

(v) f preserves joins $\vee$, $f(x\vee y)=f(x)\vee f(y)$.

Proposition 5. *Morphisms $f\colon r\to t$ are in bijection with 2-cluster trees $(t;(t_p)_{p\in\mathrm{v}(t)})$. The bijection is established by assigning $t_p=\circ$ if $p\notin\operatorname{Im} f$ and t_p is a subtree of r with $\mathrm{v}(t_p)=f^{-1}(p)$ if $p\in\operatorname{Im} f$, all vertices of t_p given by $V(t_p)=P_r^{-1}(\mathrm{v}(t_p))\cup\mathrm{v}(t_p)$. The inverse bijection is given by $r=I(t;(t_p)_{p\in\mathrm{v}(t)})$.*

The category **Tr** of trees whose morphisms are 2-cluster trees was first described by T. Leinster ([12, Section 7.3]). He calls ordinary trees by the name of 2-pasting diagrams (in other terms, 3-opetopes), while 2-cluster trees are maps of 2-pasting diagrams. Subcategories of **Tr** with only surjective maps were used by many authors, see *e.g.* V. L. Ginzburg and M. M. Kapranov ([6]), M. Kontsevich and Yu. I. Manin ([8]), R. E. Borcherds ([5]) and Ya. S. Soibelman ([18]).

Proof. Given a tree $t\neq\circ$ and a tree $t_p\in\mathbf{tr}\,|p|$ for each $p\in\mathrm{v}(t)$, let us construct a map

$$g\colon \bigsqcup_{p\in\mathrm{v}(t)} V(t_p)\longrightarrow V(t).$$

For any $v\in\mathrm{v}(t_p)$ impose $g(v)=p$.

Let $q\in\mathrm{v}(t)$, $q\neq\operatorname{root}(t)$, $p=P_tq$. There is the order preserving bijection

$$\phi_p\colon (P_t^{-1}(p)\setminus\{\operatorname{root}(t)\},\leqslant)\to(\operatorname{Inp} t_p,\trianglelefteq).$$

In $\tau=I(t;(t_p)_{p\in\mathrm{v}(t)})$ vertices $\operatorname{root}(t_q)$ and $\phi_p(q)$ are glued by equivalence relation (8). We impose

$$g(\phi_p(q))=g(\operatorname{root}(t_q)). \tag{12}$$

If $\operatorname{root}(t_q)\in\mathrm{v}(t_q)$, then we already know $g(\operatorname{root}(t_q))$. Otherwise, $t_q=\circ$ and $|q|=1$. Proceeding we include q in a maximal string of consecutive vertices $Pv\neq P^2v\neq\dots\neq P^mv$ such that $t_{Pv}=t_{P^2v}=\dots=t_{P^mv}=\circ$, $m\geqslant 1$, either $v\notin\mathrm{v}(t)$ or $t_v\neq\circ$, and either $P^{m+1}v=P^mv$ is the root of t or $t_{P^{m+1}v}\neq\circ$. Iterating equation (12) we conclude that $g(\phi_p(q))=g(\operatorname{root}(t_v))$ which is already known if $t_v\neq\circ$. It remains to define g on elements (u,x), $u\in\mathrm{v}(t)$, $x\in\operatorname{Inp} t_u\setminus\phi_u\big(\mathrm{v}(t)\cap(P_t^{-1}u\setminus\operatorname{root}(t))\big)$. The set of such elements is in bijection with $\operatorname{Inp} t$. We choose g on the set of (u,x) to be the only $\trianglelefteq$-order preserving bijection with $\operatorname{Inp} t$. This determines g completely.

By construction, g factors through canonical projection (9) and determines a unique map f as in

$$g=\Big[\bigsqcup_{p\in\mathrm{v}(t)} V(t_p)\xrightarrow{\pi} V(I(t;(t_p)))\xrightarrow{f} V(t)\Big]. \tag{13}$$

Equip now $\mathrm{v}(t)$, $V(t_p)$, $V(I(t;(t_p)))$ and $V(t)$ with the partial order $\preccurlyeq$. Equip the source of (13) with the lexicographic order. Any pair of elements $y\prec z\in V(I(t;(t_p)))$ lifts to a pair $w\prec x\in\bigsqcup_{p\in(\mathrm{v}(t),\preccurlyeq)}(V(t_p),\preccurlyeq)$ such that $\pi(w)=y$, $\pi(x)=z$. Since by construction g

preserves the order $\preccurlyeq$, so is f. Thus f satisfies condition (i) of Definition 9. Clearly, conditions (ii)–(iv) are also satisfied. By induction on the size of $V(r)$ we prove also (v).

Let $f\colon V(r) \to V(t)$ satisfy all the conditions of Definition 9. For $p \in \mathrm{v}(t)$ let us construct a tree t_p. The set $\mathrm{v}(t_p) \overset{\mathrm{def}}{=} f^{-1}(p)$ is closed with respect to joins $\vee$. For all vertices $u, w \in f^{-1}(p)$ and any vertex $v \in V(r)$ inequalities $u \preccurlyeq v \preccurlyeq w$ imply that $v \in f^{-1}(p)$. Therefore, $f^{-1}(p)$ is a subtree of r, possibly empty. If $f^{-1}(p) = \varnothing$, then $|p| = 1$ by (iv) and (v). In this case set $t_p = \circ$, otherwise set $V(t_p) = P_r^{-1}(f^{-1}(p)) \cup f^{-1}(p)$, this gives all vertices of subtree t_p of r. Let $v \in V(t_p) - \mathrm{v}(t_p) = \mathrm{Inp}(t_p)$. There is $k = k(v) \in \mathbb{N}$ such that $P_t^k f(v) \ne p$ and $P_t^{k+1} f(v) = p$. The map $\mathrm{Inp}\, t_p \to P_t^{-1} p \setminus \{\mathrm{root}(t)\}$, $v \mapsto P_t^{k(v)} f(v)$, is injective by (v) and surjective by (iv). Therefore, $|\mathrm{Inp}\, t_p| = |p|$.

Let $f\colon r \to t$ satisfy conditions of Definition 9. If $t = \circ$ then the only possible f is $\mathrm{id}_\circ$ and $I_\circ() = \circ = r$. If $t \ne \circ$, then $\sim$-equivalent points of $\bigsqcup_{p\in \mathrm{v}(t)} V(t_p)$ represent the same vertex of r. Hence, $I(t; (t_p)_{p\in\mathrm{v}(t)}) = r$. By construction, the map $I(t; (t_p)_{p\in\mathrm{v}(t)}) \to t$ coincides with f.

Starting with a 2-cluster tree $(t; (t_p)_{p\in\mathrm{v}(t)})$ construct f. By construction the subsets $f^{-1}(p)$ and $P_r^{-1}(f^{-1}(p)) \cup f^{-1}(p)$ determine the tree t_p we have started with. Thus, the described maps are inverse to each other. □

It follows that recursive definition of a morphism of trees [12, Section 7.3] is equivalent to Definition 9.

Remark 3. The category $\mathbf{Tr}$ is a disjoint union of its subcategories $\mathbf{Tr}(n)$, each of which has the terminal object $\tau[n]$. Hence, we obtain the following corollary.

Hence, we obtain the following corollary.

Corollary 3. *m-cluster trees are in bijection with sequences $r(0) \to r(1) \to \cdots \to r(m-1) \to r(m)$ of m composable morphisms in $\mathbf{Tr}$ such that $r(m)$ is a corolla, namely, $r(m) = \tau[\mathrm{Inp}\, r(0)]$.*

Denote the set of such sequences by

$$\mathbf{Ctr}(n,m) = \bigl\{t = \bigl(t(0) \xrightarrow{f_1} t(1) \xrightarrow{f_2} \cdots \to t(m-1) \xrightarrow{f_m} t(m)\bigr) \mid t(m) = \tau[n],\ \forall i\ f_i \in \mathbf{Tr}(n)\bigr\},$$
$$\mathbf{Ctr}(n,m) = \{t \in \mathcal{C}at([m], \mathbf{Tr}(n)) \mid t(m) = \tau[n]\}.$$

It is shown above that $\mathbf{Ctr}(n,m)$ is in bijection with $\mathbf{ctr}(n,m)$. Namely, an m-cluster tree $(n; t; (t_{p_1}); (t_{p_1,p_2}); \dots; (t_{p_1,p_2,\dots,p_{m-1}}))$ is taken to the sequence

$$I(t; (t_{p_1}); (t_{p_1,p_2}); \dots; (t_{p_1,p_2,\dots,p_{m-1}})) \to \dots \to I(t; (t_{p_1}); (t_{p_1,p_2})) \to I(t; (t_{p_1})) \to t \to \tau[n]. \quad (14)$$

Define the linear tree of height m as $\theta_m = \bigl(\theta_m(0) = \mathbf{1} \to \mathbf{1} \to \mathbf{1} \to \dots \to \mathbf{1} \to \mathbf{1} = \theta_m(m)\bigr)$ with $\mathrm{Inp}\,\theta_m = \theta_m(0) = \mathbf{1}$. In particular, $\theta_0 = \circ = (\mathbf{1}; \mathrm{Inp}\circ = \mathbf{1})$. The full and faithful functor $\theta\colon \mathcal{O}_{\mathsf{sk}} \to \mathbf{Tr}(1)$, $n \mapsto \theta_n$, induces the injection

$$\begin{array}{ccc}
\mathbf{str}(m) & \hookrightarrow & \mathbf{Ctr}(1,m) \\
\downarrow & & \downarrow \\
\mathcal{C}at([m], \mathcal{O}_{\mathsf{sk}}) & \xrightarrow{\mathcal{C}at([m],\theta)} & \mathcal{C}at([m], \mathbf{Tr}(1))
\end{array}$$
$$\bigl(p(0) \to p(1) \to \cdots \to p(m)\bigr) \longmapsto \bigl(\theta_{p(0)} \to \theta_{p(1)} \to \cdots \to \theta_{p(m)}\bigr).$$

Each cluster tree $t \in \mathbf{Ctr}(n,m)$ has another form

$$(n;t;(t_{p_1});(t_{p_1,p_2});\dots;(t_{p_1,p_2,\dots,p_{m-1}})) \in \mathbf{ctr}(n,m).$$

The internal vertices of $t(m-k)$, $0 \leqslant k \leqslant m$, are found from the latter presentation as

$$\mathrm{v}(t(m-k)) = \bigsqcup_{p_1\in \mathrm{v}(t)} \bigsqcup_{p_2\in \mathrm{v}(t_{p_1})} \cdots \bigsqcup_{p_{k-1}\in \mathrm{v}(t_{p_1,\dots,p_{k-2}})} \mathrm{v}(t_{p_1,p_2,\dots,p_{k-1}}). \tag{15}$$

In particular, $\mathrm{v}(t(m)) = \mathbf{1}$, $\mathrm{v}(t(m-1)) = \mathrm{v}(t)$.

Let $s = m-k$, $0 \leqslant s \leqslant m$, and $j \in \mathrm{v}(t(s))$. The same point j is identified with

$$(p_1,p_2,\dots,p_k) = (f_{m-1}\cdots f_{s+2}f_{s+1}(p_k),\dots,f_{s+2}f_{s+1}(p_k),f_{s+1}(p_k),p_k).$$

Define $t^{|j}_{[q,s]}$ as the $(s-q)$-cluster subtree of t formed by

$$\big(|p_k|;t_{p_1,p_2,\dots,p_k};(t_{p_1,\dots,p_k,p_{k+1}})_{p_{k+1}};\dots;(t_{p_1,\dots,p_k,p_{k+1},\dots,p_{m-q-1}})_{p_{k+1},\dots,p_{m-q-1}}\big) \in \mathbf{ctr}(|p_k|,s-q).$$

As explained by T. Leinster [12, Section 7.3] Proposition 5 implies the following enforcement of Example 3, see also Description 1 of $\mathcal{C}at$-operads.

Corollary 4. *The collection* **TR** *of categories* $\mathbf{Tr}(n)$*,* $n \in \mathbb{N}$*, equipped with functors* I_t *from* (7) *and identity transformations* α*,* ι *is a strict* $\mathcal{C}at$*-operad — (strict algebra over the strict 2-monad* $\mathbb{T}\colon \underline{\mathcal{C}at}^{\mathbb{N}} \to \underline{\mathcal{C}at}^{\mathbb{N}}$*).*

4. $\mathcal{C}at$**-operads.** Consider the free-operad monad

$$\mathbb{T}\colon\ \mathcal{S}\mathrm{et}^{\mathbb{N}} \to \mathcal{S}\mathrm{et}^{\mathbb{N}}, X = (X(n))_{n\geqslant 0} \mapsto \mathbb{T}X, \quad (\mathbb{T}X)(n) = \coprod_{t\in \mathbf{tr}(n)} \prod_{p\in \mathrm{v}(t)} X|p|,$$

on the category of collections of sets. For any $p \in \mathrm{v}(t)$ the notation $X|p|$ means $X(|p|)$. The multiplication $m\colon \mathbb{T}^2X \to \mathbb{T}X$ is taking identically a summand $\prod_{p\in\mathrm{v}(t)}\prod_{q\in\mathrm{v}(t_p)} X|q|$ indexed by a 2-cluster tree $(t;(t_p)_{p\in\mathrm{v}(t)})$ to the summand $\prod_{v\in\mathrm{v}(I(t;(t_p)))} X|v|$ indexed by the tree $I(t;(t_p)_{p\in\mathrm{v}(t)}) = I_t(t_p \mid p \in \mathrm{v}(t))$ from (7). The identity mapping here is due to (10):

$$\mathrm{v}\big(I(t;(t_p)_{p\in\mathrm{v}(t)})\big) = \bigsqcup_{p\in\mathrm{v}(t)} \mathrm{v}(t_p).$$

This monad is described by T. Leinster [12, Example 4.1.11] as the monad substituting trees into vertices, see also [13, Section 5.9.5] and (7). Algebras over this monad are (non-symmetric) operads in $\mathcal{S}$et. Equivalence of this definition of an operad and the conventional ones is shown in [13, Section 5.9]. Similar monad using abstract (non-planar) labeled trees is described in [17, Section 1.12], a reduced monad is in [13, Section 5.6.1]. This is the free-symmetric-operad monad. Equivalence of algebras over it and usual definitions of symmetric operads is proven in [17, Theorem 1.105], see also [13, Chapter 5]. The version for graphs instead of trees is in [3, Proposition 1.7.1].

The free-operad monad $\mathbb{T}\colon\ \mathcal{S}\mathrm{et}^{\mathbb{N}} \to \mathcal{S}\mathrm{et}^{\mathbb{N}}$ preserves small filtered colimits, thus is finitary. This follows from the property of $\mathcal{S}$et that filtered colimits commute with finite limits, see Example 9. This extends further to monad $\mathbb{T}\colon \mathcal{C}at^{\mathbb{N}} \to \mathcal{C}at^{\mathbb{N}}$ being finitary. The monad $\mathbb{T}\colon\ \mathcal{S}\mathrm{et}^{\mathbb{N}} \to \mathcal{S}\mathrm{et}^{\mathbb{N}}$ as well as $\mathbb{T}\colon \mathcal{C}at^{\mathbb{N}} \to \mathcal{C}at^{\mathbb{N}}$ is cartesian ([12, Example 6.5.5]).

This monad lifts to a free-operad strict 2-monad

$$\mathbb{T}\colon \underline{\mathcal{C}at}^{\mathbb{N}} \to \underline{\mathcal{C}at}^{\mathbb{N}}, P = (P(n))_{n\geqslant 0} \mapsto \mathbb{T}P, \quad (\mathbb{T}P)(n) = \coprod_{t\in \mathbf{tr}(n)} \prod_{v\in \mathrm{v}(t)} P|v|.$$

Due to the above remarks the 2-monad $\mathbb{T}$ is finitary.

Remark 4. As noticed in [7, Section 11], the 2-category $\underline{\mathcal{C}at}^{\mathbb{N}}$ admits a calculus of bimodules. The 2-functor $\mathbb{T}\colon \underline{\mathcal{C}at}^{\mathbb{N}} \to \underline{\mathcal{C}at}^{\mathbb{N}}$ preserves pullbacks, comma-categories and coidentifiers $f^* \circ g_\sharp \twoheadrightarrow f^* \bullet g_\sharp$ similarly to Remark 1.

Strict $\mathbb{T}$-algebras are operads in $\mathcal{C}at$ (see also [3, Definition 4.5.1]). We describe lax algebras over the 2-monad $\mathbb{T}\colon \underline{\mathcal{C}at}^{\mathbb{N}} \to \underline{\mathcal{C}at}^{\mathbb{N}}$ named lax $\mathcal{C}at$-operads. These are collections of categories $\mathsf{C} = (\mathsf{C}(n))_{n\geqslant 0}$ equipped with a family of functors

$$\mu_t\colon \prod_{p\in \mathrm{v}(t)} \mathsf{C}|p| \to \mathsf{C}(\mathrm{Inp}\, t), \qquad t \in \mathbf{tr}\,.$$

Equations for μ are replaced with natural transformations indexed by a 2-cluster tree

$$\begin{array}{ccc}
\prod_{p\in \mathrm{v}(t)} \prod_{q\in \mathrm{v}(t_p)} \mathsf{C}|q| & \xrightarrow{\prod_{p\in \mathrm{v}(t)} \mu_{t_p}} & \prod_{p\in \mathrm{v}(t)} \mathsf{C}|p| \\
\cong\big\downarrow & \swarrow\!\!\swarrow \alpha_{t;(t_p)_p} & \big\downarrow \mu_t \\
\prod_{x\in \mathrm{v}(I_t(t_p|p\in \mathrm{v}(t)))} \mathsf{C}|x| & \xrightarrow{\mu_{I_t(t_p|p\in \mathrm{v}(t))}} & \mathsf{C}(\mathrm{Inp}\, t)
\end{array} \tag{16}$$

We are interested in pseudo (=weak=strong) $\mathbb{T}$-algebras for which $\alpha_{t;(t_p)_p}$ are invertible.

Let us introduce some technical notation. For an arbitrary $t \in \mathbf{tr}$ we denote by ${}^{>}t$ the $|\,\mathrm{Inp}\, t|$-corolla, ${}^{>}t = (\mathrm{Inp}\, t \to \mathbf{1}; \mathrm{Inp}\, t)$ if $\mathrm{Inp}\, t$ is not empty and ${}^{>}t = \tau[0] = (\mathbf{1}; \varnothing)$ if $\mathrm{Inp}\, t = \varnothing$. For any $p \in \mathrm{v}(t)$ denote by $t_p^{>}$ the $|p|$-corolla, $t_p^{>} = (P_t^{-1}p \to \mathbf{1}; P_t^{-1}p)$ if p is not a leaf and $t_p^{>} = \tau[0]$ if p is a leaf. In particular, for $t = \circ = (\mathbf{1}; \mathbf{1})$ we have ${}^{>}t = \tau[1]$ and there are no $t_p^{>}$ for $\mathrm{v}(t) = \varnothing$. For $t = \tau[0] = \bullet$ we have ${}^{>}t = \tau[0] = t = t_1^{>}$.

The notion of a lax T-algebra for $T = \mathbb{T}$ is realized as follows.

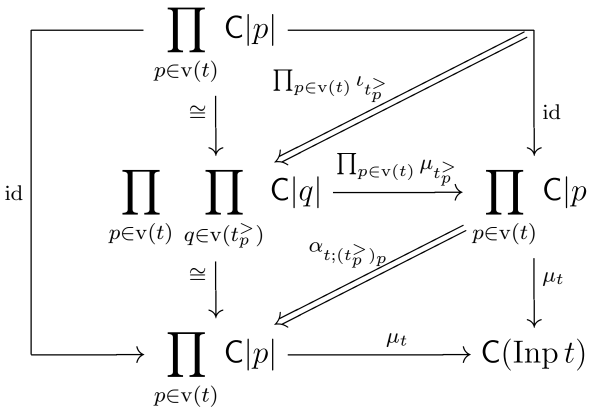
Description 1. A *lax $\mathcal{C}at$-operad* (=lax $\mathbb{T}$-algebra) consists of

— a collection of categories $\mathsf{C} = (\mathsf{C}(n))_{n\geqslant 0}$,
— functors $\mu_t\colon \prod_{p\in \mathrm{v}(t)} \mathsf{C}|p| \to \mathsf{C}(\mathrm{Inp}\, t)$ for $t \in \mathbf{tr}$,
— natural transformations (16) for $t \in \mathbf{tr}$ and for family $t_p \in \mathbf{tr}\,|p|$, $p \in \mathrm{v}(t)$,
— and natural transformations for $n \geqslant 0$,

$$\begin{array}{ccc}
\mathsf{C}(n) & \xrightarrow{\quad\mathrm{Id}\quad} & \mathsf{C}(n) \\
 & \Downarrow \iota_{\tau[n]} & \\
i' \searrow\ \cong & \mathsf{C}(n)^{\mathbf{1}} & \nearrow \mu_{\tau[n]}
\end{array}$$

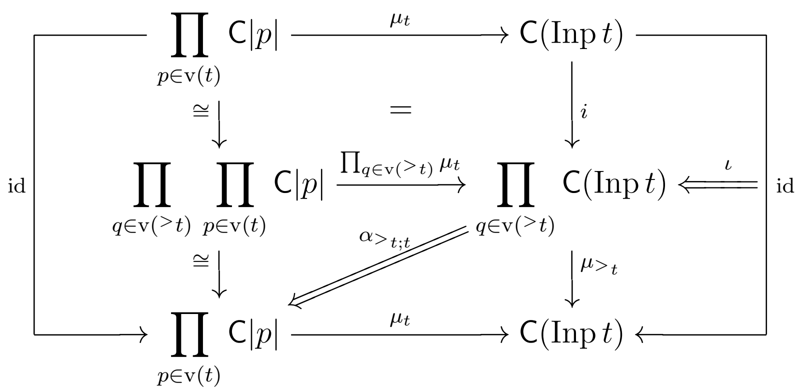
such that

(i) for every $t \in \mathbf{tr}$ unitality-1 holds



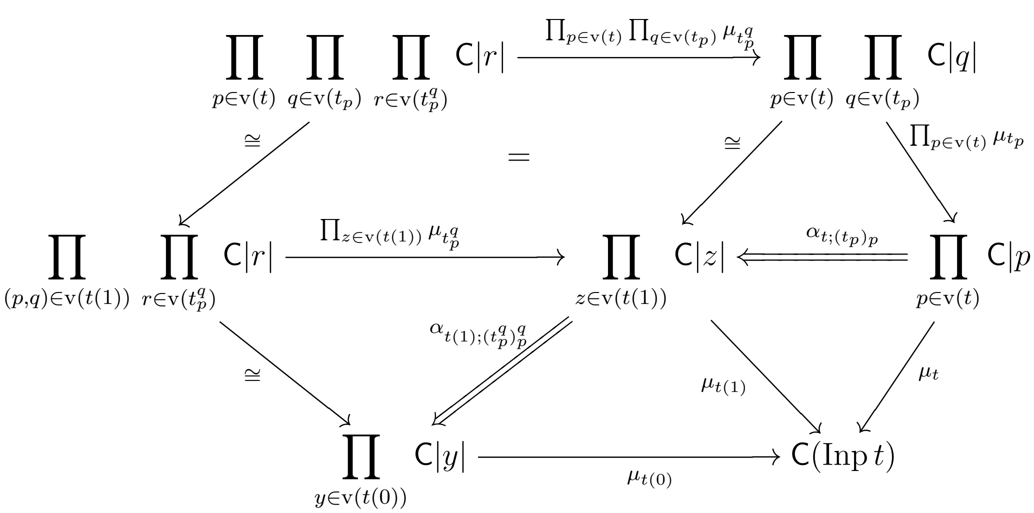
$= \mathrm{id}_{\mu_t};$ (17)

(ii) for every $t \in \mathbf{tr}$ unitality-2 holds



$= \mathrm{id}_{\mu_t};$ (18)

(iii) for every $t \in \mathbf{tr}$, every family $t_p \in \mathbf{tr}\,|p|$, $p \in v(t)$, and all families $t_p^q \in \mathbf{tr}\,|q|$, $q \in v(t_p)$, $p \in v(t)$, that is, for any 3-cluster tree $(t; (t_p)_{p\in v(t)}; ((t_p^q)_{q\in v(t_p)})_{p\in v(t)})$, associativity equation (2) holds, which is equality of two natural transformations



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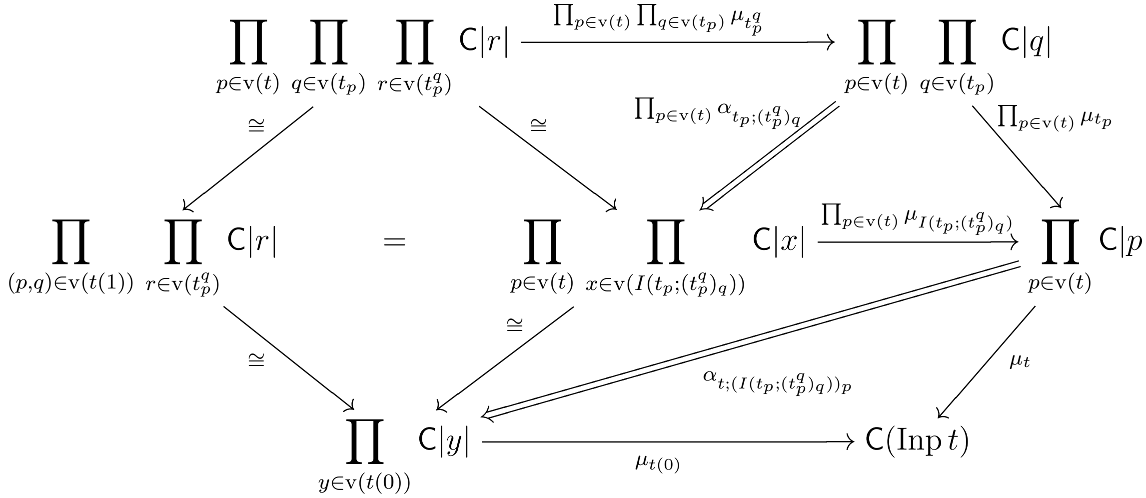


(19)

A *strong (=weak =pseudo) $\mathcal{C}at$-operad* C is a lax $\mathcal{C}at$-operad $(\mathsf{C},\mu_t,\alpha,\iota)$ such that α, ι are invertible. A *strict $\mathcal{C}at$-operad* C is an operad enriched over $\mathcal{C}at$ (of the form $(\mathsf{C},\mu_t,\mathrm{id},\mathrm{id})$).

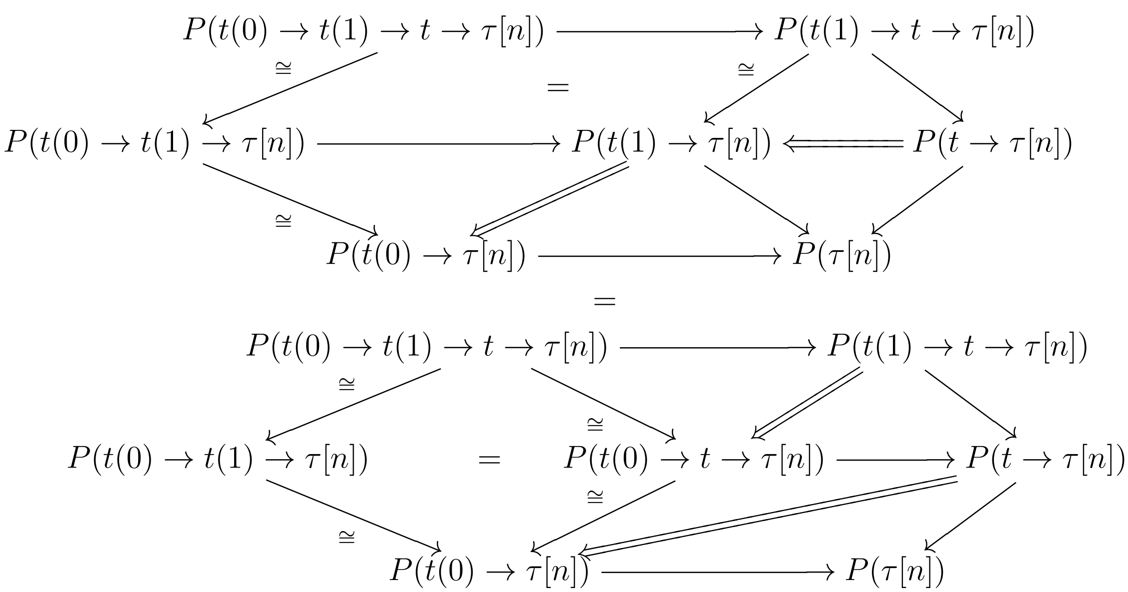
Associate with an m-cluster tree, $(n;t;(t_{p_1});(t_{p_1,p_2});\dots;(t_{p_1,p_2,\dots,p_{m-1}}))$ or equivalently with $t(0)\to t(1)\to\dots\to t(m-1)\to\tau[n]$ from (14) the product

$$P(t(0)\to\dots\to\tau[n])=\prod_{p_1\in \mathrm{v}(t)}\ \prod_{p_2\in \mathrm{v}(t_{p_1})}\cdots\prod_{p_{k-1}\in \mathrm{v}(t_{p_1,\dots,p_{k-2}})}\mathsf{C}|p_m|.$$

For $m=3$ the cluster tree is

$$(t(0)\to t(1)\to t(2)\to t(3))=(I(t;(t_p);(t_p^q))\to I(t;(t_p))\to t\to\tau[n]).$$

Then equation (19) can be schematically written as



in order to stress cubical origin of this equation.

We have not expanded the definition of a colax $\mathcal{C}at$-operad. Nevertheless, given a lax $\mathcal{C}at$-operad $\mathsf{C}=(\mathsf{C}(n),\mu_t,\alpha,\iota)$, one can produce the opposite colax $\mathcal{C}at$-operad $\mathsf{C}^{\mathrm{op}}=(\mathsf{C}(n)^{\mathrm{op}},\mu_t^{\mathrm{op}},\alpha^{\mathrm{op}},\iota^{\mathrm{op}})$. If α and ι are invertible then one has also the opposite strong $\mathcal{C}at$-operad $(\mathsf{C}(n)^{\mathrm{op}},\mu_t^{\mathrm{op}},\alpha^{\mathrm{op}-1},\iota^{\mathrm{op}-1})$. If C is strict then so is C^{op}.

Example 3. The discrete category **tr** equipped with functors I_t from (7) and identity transformations α, ι is a strict $\mathcal{C}at$-operad — strict algebra over the 2-monad $\mathbb{T}\colon \underline{\mathcal{C}at}^{\mathbb{N}} \to \underline{\mathcal{C}at}^{\mathbb{N}}$.

Example 4. Let $(\mathcal{V}, \otimes^I, \lambda^f, \rho)$ be a lax symmetric monoidal category, which is the opposite to a colax symmetric monoidal category ([2, Definition 2.5]). Note that a weak(=strong) monoidal category is the same as an unbiased monoidal category ([12, Definition 3.1.1]). It gives rise to a lax $\mathcal{C}at$-operad C with $\mathsf{C}(n) = \mathcal{V}$, $\mu_t = \otimes^{\mathrm{v}(t)}\colon \mathcal{V}^{\mathrm{v}(t)} \to \mathcal{V}$ being the tensor product, $\alpha_{t;(t_p)_p} = \lambda^f\colon \otimes^{p\in \mathrm{v}(t)} \circ \otimes^{\mathrm{v}(t_p)} \to \otimes^{\mathrm{v}(I_t(t_p|p\in\mathrm{v}(t)))}\colon \mathcal{V}^{\mathrm{v}(I_t(t_p|p\in\mathrm{v}(t)))} \to \mathcal{V}$, where

$$f = \bigl(\mathrm{v}(I_t(t_p \mid p \in \mathrm{v}(t))) \xrightarrow{\cong} \bigsqcup_{p\in\mathrm{v}(t)} \mathrm{v}(t_p) \to \mathrm{v}(t)\bigr)$$

is the natural “projection on the index” map, $\iota_{\tau[n]} = i' \cdot \rho\colon \mathrm{Id} = i'i'^{-1} \to i' \cdot \otimes^{\mathbf{1}}\colon \mathcal{V} \to \mathcal{V}$ for each $n \in \mathbb{N}$. The total ordering of $\mathrm{v}(t)$ is the canonical one. The first factor of f is not necessarily order-preserving, see (9), while the second is. Equality (i), (ii) follow from properties (2.5.1), (2.5.2) of [2]. Equality (iii) follows from equality (2.5.4) [*ibid.*], written for the pair of “projection on the index” maps

$$\bigsqcup_{(p,q)\in\sqcup_{r\in\mathrm{v}(t)}\mathrm{v}(t_r)} \mathrm{v}(t_p^q) \xrightarrow{f} \bigsqcup_{p\in\mathrm{v}(t)} \mathrm{v}(t_p) \xrightarrow{g} \mathrm{v}(t).$$

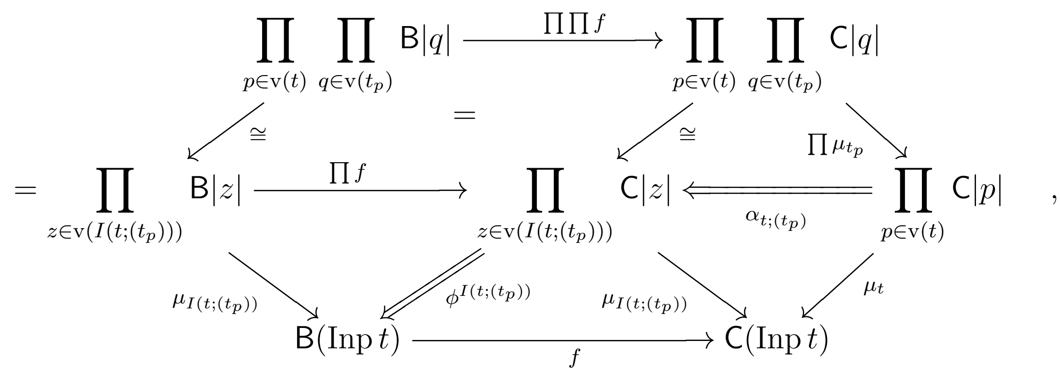
For $T = \mathbb{T}\colon \underline{\mathcal{C}at}^{\mathbb{N}} \to \underline{\mathcal{C}at}^{\mathbb{N}}$ (co)lax $\mathbb{T}$-morphisms are also called (co)lax $\mathcal{C}at$-multifunctors. (An operad is a particular case of a multicategory; a morphism of operads is a particular case of a multifunctor.) The notion of lax $\mathbb{T}$-morphisms is equivalent to coherent lax morphisms of strict operads in categories ([3, Definition 4.5.2]). Let us describe it in detail.

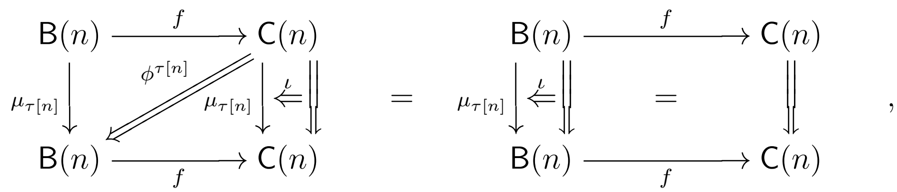
Description 2. A *lax $\mathcal{C}at$-multifunctor* (=lax $\mathbb{T}$-morphism) $f\colon (\mathsf{B}, \mu_t, \beta, \iota) \to (\mathsf{C}, \mu_t, \alpha, \iota)$ consists of a morphism of collections $f\colon \mathsf{B} \to \mathsf{C} \in \underline{\mathcal{C}at}^{\mathbb{N}}$ and a natural transformation

$$\begin{array}{ccc}
\prod_{p\in\mathrm{v}(t)} \mathsf{B}|p| & \xrightarrow{\prod_{p\in\mathrm{v}(t)} f} & \prod_{p\in\mathrm{v}(t)} \mathsf{C}|p| \\
{\scriptstyle\mu_t}\big\downarrow & \overset{\phi^t}{\swarrow\!\!\!\swarrow} & \big\downarrow{\scriptstyle\mu_t} \\
\mathsf{B}(\operatorname{Inp} t) & \xrightarrow{f} & \mathsf{C}(\operatorname{Inp} t)
\end{array}$$

for each tree $t \in \mathbf{tr}$ such that 2-cluster tree $(t;(t_p))$

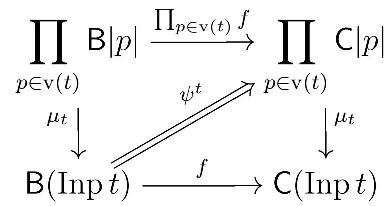
$$\begin{array}{ccccc}
 & \prod_{p\in\mathrm{v}(t)}\prod_{q\in\mathrm{v}(t_p)} \mathsf{B}|q| & \xrightarrow{\prod\prod f} & \prod_{p\in\mathrm{v}(t)}\prod_{q\in\mathrm{v}(t_p)} \mathsf{C}|q| & \\
\swarrow{\scriptstyle\cong} & & \searrow{\scriptstyle\prod\mu_{t_p}} \quad \overset{\prod\phi^{t_p}}{\swarrow\!\!\!\swarrow} & & \searrow{\scriptstyle\prod\mu_{t_p}} \\
\prod_{z\in\mathrm{v}(I(t;(t_p)))} \mathsf{B}|z| & \xLeftarrow{\beta_{t;(t_p)}} & \prod_{p\in\mathrm{v}(t)} \mathsf{B}|p| & \xrightarrow{\prod f} & \prod_{p\in\mathrm{v}(t)} \mathsf{C}|p| \\
\searrow{\scriptstyle\mu_{I(t;(t_p))}} & & \swarrow{\scriptstyle\mu_t} \quad \overset{\phi^t}{\swarrow\!\!\!\swarrow} & & \swarrow{\scriptstyle\mu_t} \\
 & \mathsf{B}(\operatorname{Inp} t) & \xrightarrow{f} & \mathsf{C}(\operatorname{Inp} t) &
\end{array}$$



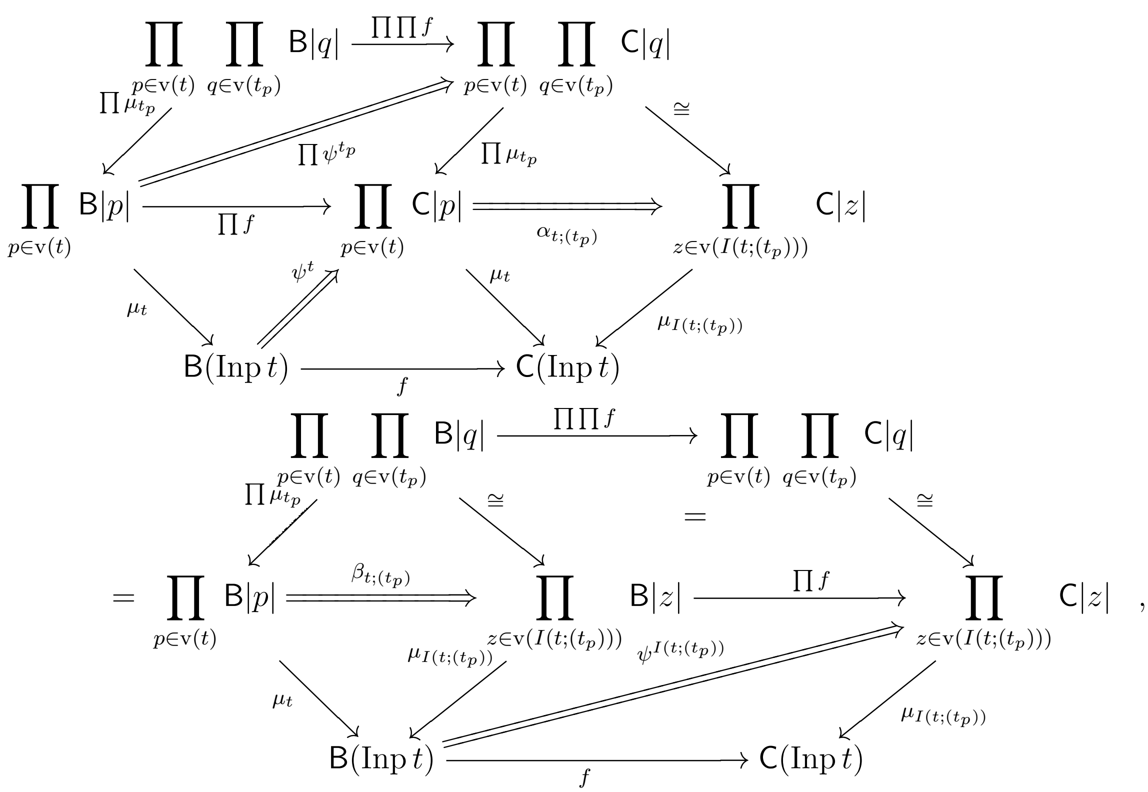


where we identify $\prod_{p\in \mathrm{v}(\tau[n])} X$ with X.

Description 3. A *colax* $\mathcal{C}at$*-multifunctor* (=colax $\mathbb{T}$-morphism) $f\colon (\mathsf{B},\mu_t,\beta,\iota) \to (\mathsf{C},\mu_t,\alpha,\iota)$ consists of a morphism of collections $f\colon \mathsf{B}\to\mathsf{C}\in \underline{\mathcal{C}at}^{\mathbb{N}}$ and a natural transformation



for each tree $t\in \mathbf{tr}$ such that for each 2-cluster tree $(t;(t_p))$



(20)

$$\begin{array}{ccc} \mathsf{B}(n) & \xrightarrow{\quad f \quad} & \mathsf{C}(n) \\ \Big\Downarrow \overset{\iota}{\Rightarrow} \Big\downarrow \mu_{\tau[n]} & \overset{\psi^{\tau[n]}}{\nearrow\!\!\nearrow} & \Big\downarrow \mu_{\tau[n]} \\ \mathsf{B}(n) & \xrightarrow[\quad f \quad]{} & \mathsf{C}(n) \end{array} \quad = \quad \begin{array}{ccc} \mathsf{B}(n) & \xrightarrow{f} & \mathsf{C}(n) \\ \Big\Downarrow & = & \Big\Downarrow \overset{\iota}{\Rightarrow} \Big\downarrow \mu_{\tau[n]} \\ \mathsf{B}(n) & \xrightarrow[f]{} & \mathsf{C}(n) \end{array} . \tag{21}$$

Strong (=weak =pseudo) $\mathcal{C}at$-multifunctors are colax $\mathcal{C}at$-multifunctors (f,ψ) with invertible ψ. *Strict $\mathcal{C}at$-multifunctors* are those enriched over $\mathcal{C}at$ (of the form (f,id)).

Example 5. For any lax $\mathcal{C}at$-operad C the category $\mathsf{C}(1)$ has a canonical lax monoidal structure. It is due to the linear trees θ_n, $\mathrm{v}(\theta_n) \cong \mathbf{n}$. Namely, the tensor product functors are $\otimes^{\mathbf{n}} = \mu_{\theta_n} \colon \mathsf{C}(1)^n \to \mathsf{C}(1)$ and the natural transformations are $\lambda^f = \alpha_{\theta_J;(\theta_{f^{-1}j})_{j\in J}} \colon \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i \to \otimes^{i\in I} X_i$ for each $f \colon I \to J \in \mathcal{O}_{\mathsf{sk}}$, and $\rho = \iota_{\tau[1]} \colon \mathrm{Id} \to \otimes^{\mathbf{1}} \colon \mathsf{C}(1) \to \mathsf{C}(1)$. The functor lax-$\mathcal{C}at$-$\mathrm{Op}_x \to$ lax-mono-$\mathcal{C}at_x$, $\mathsf{C} \mapsto \mathsf{C}(1)$, has a left adjoint $\mathcal{C} \mapsto \mathsf{C} = (\varnothing, \mathcal{C}, \varnothing, \varnothing, \dots)$, where $\mathsf{C}(1) = \mathcal{C}$, $\mathsf{C}(n) = \varnothing$ for $n \neq 1$, whichever version of morphisms $x \in \{$lax, colax, weak, strict$\}$ we choose.

Furthermore, for any lax $\mathcal{C}at$-operad C the category $\mathsf{C}(0)$ is a module over the lax monoidal category $\mathsf{C}(1)$ in the obvious sense. This is due to linear trees without inputs $\bar\theta_m = (\mathbf{1} \to \mathbf{1} \to \dots \to \mathbf{1} \mid \mathrm{Inp}\, \bar\theta_m = \varnothing)$ $(1+m$ singletons$)$, $m \geqslant 0$. Namely, the action is $\otimes^{1+m} = \mu_{\bar\theta_m} \colon \mathsf{C}(0) \times \mathsf{C}(1)^m \to \mathsf{C}(0)$, the natural transformations are

$$\alpha_{\bar\theta_J;(\bar\theta_{f^{-1}0},(\theta_{f^{-1}j})_{j\in J})} \colon \otimes^{j\in[J]} \otimes^{i\in f^{-1}j} X_i \to \otimes^{i\in[I]} X_i, \quad X_0 \in \mathsf{C}(0), \quad X_i \in \mathsf{C}(1) \text{ for } i \in I,$$

for each $f \colon [I] = 0 \sqcup I \to 0 \sqcup J = [J] \in \mathcal{O}$ such that $f(0) = 0$, and $\iota_{\tau[0]} \colon \mathrm{Id} \to \mu_{\bar\theta_0} \colon \mathsf{C}(0) \to \mathsf{C}(0)$. Axioms for module categories and their morphisms can be read from Descriptions 1, 2, 3. Given a module category, one can construct a lax $\mathcal{C}at$-operad. In fact, the functor lax-$\mathcal{C}at$-$\mathrm{Op}_x \to$ lax-mono-$\mathcal{C}at$-Mod_x, $\mathsf{C} \mapsto (\mathsf{C}(0), \mathsf{C}(1))$, has a left adjoint $(\mathcal{M}, \mathcal{C}) \mapsto \mathsf{C} = (\mathcal{M}, \mathcal{C}, \varnothing, \varnothing, \dots)$, where $\mathsf{C}(n) = \varnothing$ for $n > 1$.

Example 6. The terminal collection $\mathsf{1} = (\mathit{1})_{n\geqslant 0}$ consisting of terminal (1-morphism) categories $\mathit{1}$ is a strict $\mathcal{C}at$-operad.

Definition 10. A (co)operad in a lax $\mathcal{C}at$-operad C is a (co)lax $\mathcal{C}at$-multifunctor (=(co)lax $\mathbb{T}$-morphism) $\mathsf{1} \to \mathsf{C}$.

In detail, a cooperad (C, Δ_t) in a lax $\mathcal{C}at$-operad $(\mathsf{C}, \mu_t, \alpha, \iota)$ is the following data:

— a collection of objects $C(n) \in \mathsf{C}(n)$, $n \geqslant 0$,
— morphisms $\Delta_t \colon C(\mathrm{Inp}\, t) \to \mu_t(C|p| \mid p \in \mathrm{v}(t)) \in \mathsf{C}(\mathrm{Inp}\, t)$ for $t \in \mathbf{tr}$,

such that

(i) for any family $t \in \mathbf{tr}$, $t_p \in \mathbf{tr}\,|p|$, $p \in \mathrm{v}(t)$, and the tree $\tau = I_t(t_p \mid p \in \mathrm{v}(t))$

$$\big\langle C(n) \xrightarrow{\Delta_t} \mu_t(C|p| \mid p \in \mathrm{v}(t)) \xrightarrow{\mu_t(\Delta_{t_p} | p\in \mathrm{v}(t))} \mu_t(\mu_{t_p}(C|q| \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t)) \\ \xrightarrow{\alpha} \mu_\tau(C|r| \mid r \in \mathrm{v}(\tau))\big\rangle = \Delta_\tau; \tag{22}$$

(ii) for all $n \in \mathbb{N}$ $\Delta_{\tau[n]} = \iota^{\mathsf{C}}_{\tau[n]} \colon C(n) \to \mu_{\tau[n]}(C(n))$.

In the main for us Example 4 a cooperad in a lax symmetric monoidal category $\mathcal{V}$ is the following data:

— a collection of objects $C(n) \in \mathcal{V}$, $n \geqslant 0$,
— morphisms $\Delta_t : C(\operatorname{Inp} t) \to \otimes^{p\in \mathrm{v}(t)} C|p| \in \mathcal{V}$ for $t \in \mathbf{tr}$, $\mathrm{v}(t) = (\mathrm{v}(t), \trianglelefteq)$,

such that

(i) for any family $t \in \mathbf{tr}$, $t_p \in \mathbf{tr}\,|p|$, $p \in \mathrm{v}(t)$, and the tree $\tau = I_t(t_p \mid p \in \mathrm{v}(t))$

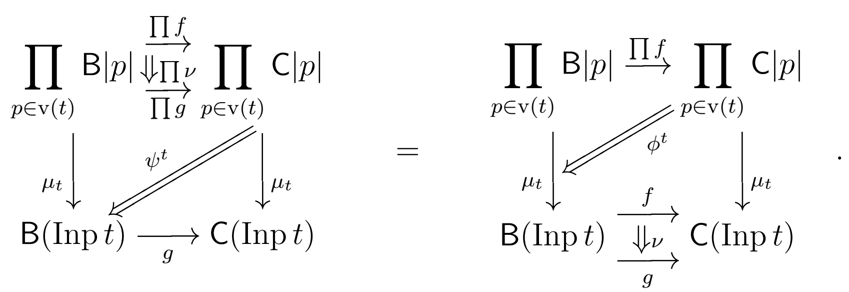
$$\big\langle C(n) \xrightarrow{\Delta_t} \otimes^{p\in \mathrm{v}(t)} C|p| \xrightarrow{\otimes^{p\in \mathrm{v}(t)}\Delta_{t_p}} \otimes^{p\in \mathrm{v}(t)} \otimes^{q\in \mathrm{v}(t_p)} C|q| \xrightarrow{\lambda} \otimes^{r\in \mathrm{v}(\tau)} C|r| \big\rangle = \Delta_\tau;$$

(ii) for all $n \in \mathbb{N}$ $\Delta_{\tau[n]} = \rho : C(n) \to \otimes^{\mathbf{1}} C(n)$.

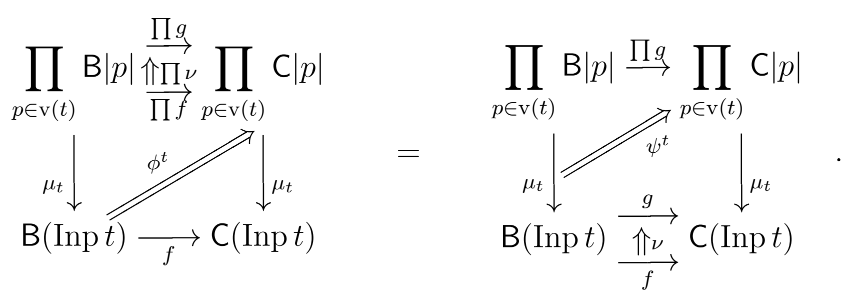
Conditions (i), (ii) are precisely equations (2.2.2), (2.2.3) of [15], at least in the case of strongly symmetric monoidal $\mathcal{V}$. Therefore, in this case the same notion can be presented as a collection equipped with coassociative counital comultiplication Δ_t for staged trees t of height 2 [*ibid.*, Corollary 2.2.2]. Other ways to put this definition are equations (2.4.1)–(2.4.3) [*ibid.*], Proposition 2.4.4 [*ibid.*] (a cooperad is a coalgebra in a lax monoidal category $(\mathcal{V}^{\mathbb{N}}, \bar{\odot}^I)$, or most traditional definition via binary comultiplication(1)–(4) [*ibid.*].

Equivalence of Definition 10 for operads and more traditional definitions is similar.

Description 4. A $\mathcal{C}at$-*transformation* between lax $\mathcal{C}at$-multifunctors $\nu : (f, \phi) \to (g, \psi) : (\mathsf{B}, \mu_t, \beta, \iota) \to (\mathsf{C}, \mu_t, \alpha, \iota)$ is a collection of natural transformations $\nu : f \to g : \mathsf{B}(n) \to \mathsf{C}(n)$, $n \in \mathbb{N}$, such that for any tree $t \in \mathbf{tr}$



A $\mathcal{C}at$-*transformation* between colax $\mathcal{C}at$-multifunctors $\nu : (f, \phi) \to (g, \psi) : (\mathsf{B}, \mu_t, \beta, \iota) \to (\mathsf{C}, \mu_t, \alpha, \iota)$ is a collection of natural transformations $\nu : f \to g : \mathsf{B}(n) \to \mathsf{C}(n)$, $n \in \mathbb{N}$, such that for any tree $t \in \mathbf{tr}$



(23)

Morphisms of (co)operads in a lax $\mathcal{C}at$-operad are $\mathcal{C}at$-transformations. For instance, morphisms of cooperads are those collections of morphisms that agree with comultiplications Δ_t.

Example 7. Given a category $\mathcal{C}$ we describe its endomorphism strict $\mathcal{C}at$-operad $\mathcal{E} = \mathcal{E}nd\,\mathcal{C}$. This $\mathcal{C}at$-operad is introduced by Borisov and Manin [3, Example 4.2.2]. By definition, categories $\mathcal{E}(n) = \underline{\mathcal{C}at}(\mathcal{C}^n, \mathcal{C})$ consist of functors $\mathcal{C}^n \to \mathcal{C}$ and natural transformations between them. Multiplication functors

$$\mu_t \colon \prod_{p\in \mathrm{v}(t)} \underline{\mathcal{C}at}(\mathcal{C}^{|p|}, \mathcal{C}) \to \underline{\mathcal{C}at}(\mathcal{C}^{\mathrm{Inp}\, t}, \mathcal{C})$$

are compositions of functors with several arguments. In particular, $\mu_\circ \colon \mathbb{1} \to \underline{\mathcal{C}at}(\mathcal{C}, \mathcal{C})$, $* \mapsto \mathrm{Id}_{\mathcal{C}}$. Since compositions are strict, we choose $\alpha = \mathrm{id}$ and $\iota = \mathrm{id}$.

Example 8. Category of (co)operads in the strict $\mathcal{C}at$-operad $\mathcal{E}nd\,\mathcal{C}$ is isomorphic to the category of (co)lax monoidal structures of the category $\mathcal{C}$ and their (co)lax monoidal functors of the form $(\mathrm{Id}_{\mathcal{C}}, \phi)$. In fact, a colax $\mathcal{C}at$-multifunctor $\mathbf{1} \to \mathcal{E}nd\,\mathcal{C}$ consists of a functor $\otimes^n \colon \mathcal{C}^n \to \mathcal{C}$ (the image of $* \in \mathrm{Ob}\,\mathbf{1}(n)$) for each $n \in \mathbb{N}$ and a natural transformation

$$\psi^t \colon \otimes^{\mathrm{Inp}\, t} \to \underset{p\in \mathrm{v}(t)}{\bullet} \otimes^{|p|}$$

for each tree t. Condition (21) says that $\psi^{\tau[n]} = \mathrm{id}$ for all $n \in \mathbb{N}$. Denote $\rho = \psi^\circ \colon \otimes^{\mathbf{1}} \to \mathrm{Id}_{\mathcal{C}}$ and $\lambda^f = \psi^{(I \xrightarrow{f} J \xrightarrow{\triangleright} \mathbf{1})} \colon \otimes^I \to \otimes^{j\in J}(\otimes^{i\in f^{-1}j} -)$ for every map $f \colon I \to J \in \mathcal{O}_{\mathsf{sk}}$. Then (20) for the 2-cluster staged tree $t = (I \xrightarrow{\mathrm{id}} I \xrightarrow{\triangleright} \mathbf{1})$, $t_i = \circ$ for $i \in t(1) = I$, $t_* = \tau[I]$ for $* \in t(2) = \mathbf{1}$, implies $\lambda^{\mathrm{id}_I} \cdot \otimes^I(\rho) = \mathrm{id}$, see [2, (2.5.1)]. Relation (20) for the 2-cluster staged tree $t = (I \xrightarrow{\triangleright} \mathbf{1} \xrightarrow{\mathrm{id}} \mathbf{1})$, $t_* = \tau[I]$ for $* \in t(1) = \mathbf{1}$, $t_\star = \circ$ for $\star \in t(2) = \mathbf{1}$, implies equation $\lambda^{\triangleright \colon I \to \mathbf{1}} \cdot \rho = \mathrm{id}$, see [2, (2.5.2)]. Equation (20) for the 2-cluster staged trees $t = (I \xrightarrow{f} J \xrightarrow{\triangleright} \mathbf{1})$, $t_j = \tau[f^{-1}j]$ for $j \in t(1) = J$, $t_* = (J \xrightarrow{g} K \xrightarrow{\triangleright} \mathbf{1})$ for $* \in t(2) = \mathbf{1}$, and $t = (I \xrightarrow{fg} K \xrightarrow{\triangleright} \mathbf{1})$, $t_k = (f^{-1}g^{-1}k \xrightarrow{f|} g^{-1}k \xrightarrow{\triangleright} \mathbf{1})$ for $k \in t(1) = K$, $t_* = \tau[K]$ for $* \in t(2) = \mathbf{1}$ implies the qualities $\lambda^f \cdot \lambda^g = \psi^\theta = \lambda^{fg} \cdot \otimes^{k\in K}(\lambda^{f| \colon f^{-1}g^{-1}k \to g^{-1}k})$, where $\theta = I(t; (t_p)) = (I \xrightarrow{f} J \xrightarrow{g} K \xrightarrow{\triangleright} \mathbf{1})$, see [2, (2.5.4)]. Thus a cooperad in $\mathcal{E}nd\,\mathcal{C}$ defines a colax monoidal structure on $\mathcal{C}$. Vice versa, equation (20) allows to express an arbitrary ψ^t via $\rho = \psi^\circ$ and $\lambda^f = \psi^{(I \xrightarrow{f} J \xrightarrow{\triangleright} \mathbf{1})}$.

4.1. Morphism classifiers.

Proposition 6. *There are a 2-functor $Q_c \colon {}_s\mathbb{T}\text{-}\mathrm{Alg}_c \to {}_s\mathbb{T}\text{-}\mathrm{Alg}_s$ and a $\mathcal{C}at$-natural isomorphism*

$${}_s\mathbb{T}\text{-}\mathrm{Alg}_s(Q_c\mathsf{B}, \mathsf{C}) \xrightarrow{\cong} {}_s\mathbb{T}\text{-}\mathrm{Alg}_c(\mathsf{B}, \mathsf{C}),$$

which turn Q_c into a left adjoint to the inclusion ${}_s\mathbb{T}\text{-}\mathrm{Alg}_s \hookrightarrow {}_s\mathbb{T}\text{-}\mathrm{Alg}_c$.

Since 2-monad $\mathbb{T}$ is finitary, this statement follows from [1, Theorem 3.13]. Also it follows from [7, Theorem 6.1.1] whose hypothesis is satisfied due to Remark 4. We give a proof mostly in order to describe colax/strict morphism classifier Q_c explicitly.

Proof. Let $(\mathsf{B}, \mu, \mathrm{id}, \mathrm{id})$ be a strict $\mathcal{C}at$-operad. Define $Q_c\mathsf{B}$ as the universal strict $\mathcal{C}at$-operad generated over $\mathbb{T}\mathsf{B}$ by the morphisms

$$\xi^t = \xi^t_{(X_p)} \colon \big(\tau[n]; \mu_t(X_p \mid p \in \mathrm{v}(t))\big) \to \big(t; (X_p)_{p\in \mathrm{v}(t)}\big), \tag{24}$$

$t \in \mathbf{tr}(n)$, $n \in \mathbb{N}$, $X_p \in \mathrm{Ob}\,\mathsf{B}|p|$, subject to

— naturality: for all families $f_p\colon X_p \to Y_p \in \mathsf{B}|p|$, $p \in \mathrm{v}(t)$, $t \in \mathbf{tr}(n)$, $n \in \mathbb{N}$,

$$\begin{array}{ccc} \big(\tau[n];\mu_t(X_p \mid p \in \mathrm{v}(t))\big) & \xrightarrow{\xi^t_{(X_p)}} & \big(t;(X_p)_{p\in\mathrm{v}(t)}\big) \\ {\scriptstyle (\tau[n];\mu_t(f_p|p\in\mathrm{v}(t)))}\downarrow & = & \downarrow{\scriptstyle (t;(f_p)_{p\in\mathrm{v}(t)})} \\ \big(\tau[n];\mu_t(Y_p \mid p \in \mathrm{v}(t))\big) & \xrightarrow{\xi^t_{(Y_p)}} & \big(t;(Y_p)_{p\in\mathrm{v}(t)}\big) \end{array} \tag{25}$$

— normalisation: for $t = \tau[n]$, $X \in \operatorname{Ob}\mathsf{B}(n)$, we have $\xi^{\tau[n]}_X = \mathrm{id}\colon \big(\tau[n];\mu_{\tau[n]}(X)\big) \to (\tau[n];X)$;

— multiplicativity: for a 2-cluster tree $(n;t;(t_p))$, $t \in \mathbf{tr}(n)$, $t_p \in \mathbf{tr}\,|p|$, $p \in \mathrm{v}(t)$, and objects $X^q_p \in \mathsf{B}|q|$, $q \in \mathrm{v}(t_p)$, we have

$$\begin{array}{ccc} \big(\tau[n];\mu_t(\mu_{t_p}(X^q_p \mid q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t))\big) & \xrightarrow{\xi^t} & \big(t;(\mu_{t_p}(X^q_p \mid q\in\mathrm{v}(t_p))_{p\in\mathrm{v}(t)}\big) \\ {\scriptstyle (\tau[n];\beta_{t;(t_p)})}\downarrow & = & \downarrow{\scriptstyle (\xi^{t_p})_{p\in\mathrm{v}(t)}} \\ \big(\tau[n];\mu_{I(t;(t_p))}(X^q_p \mid q\in\mathrm{v}(t_p), p\in\mathrm{v}(t))\big) & \xrightarrow{\xi^{I(t;(t_p))}} & \big(I(t;(t_p));((X^q_p)_{q\in\mathrm{v}(t_p)})_{p\in\mathrm{v}(t)}\big) \end{array} \tag{26}$$

(where $\beta = \mathrm{id}$ for strict B).

There is a canonical colax $\mathcal{C}at$-multifunctor $(e,\xi)\colon \mathsf{B} \to Q_c\mathsf{B}$, $e = \big(\mathsf{B} \xhookrightarrow{\mathrm{in}_{\tau[-]}} \mathbb{T}\mathsf{B} \to Q_c\mathsf{B}\big)$. Naturality (25) implies that ξ is a 2-morphism; normalisation for ξ is the same as relation (21); multiplicativity (26) implies equality (20).

Precomposition with $e = (e,\xi)$ gives a functor

$${}_s\mathbb{T}\text{-}\mathrm{Alg}_c(e,\mathsf{C})\colon {}_s\mathbb{T}\text{-}\mathrm{Alg}_s(Q_c\mathsf{B},\mathsf{C}) \to {}_s\mathbb{T}\text{-}\mathrm{Alg}_c(\mathsf{B},\mathsf{C}), \quad (F,\mathrm{id}) \mapsto (F\big|_{\mathsf{B}}, F\xi^t). \tag{27}$$

Let us prove that this is an isomorphism of categories. First of all, ${}_s\mathbb{T}\text{-}\mathrm{Alg}_s(\mathbb{T}\mathsf{B},\mathsf{C}) \cong \underline{\mathcal{C}at}^{\mathbb{N}}(\mathsf{B},\mathsf{C})$ by enriched in $\mathcal{C}at$ version of Kleisli Lemma 1. Therefore, $F\big|_{\mathbb{T}\mathsf{B}}$ is determined by $G = F\big|_{\mathsf{B}}$. The remaining datum

$$\psi^t_{(X_p)} = F\xi^t_{(X_p)}\colon F\mu_t(X_p \mid p\in\mathrm{v}(t)) \to \mu_t(FX_p \mid p\in\mathrm{v}(t))$$

is precisely the one that makes (G,ψ) into a colax $\mathcal{C}at$-multifunctor. In fact, F applied to naturality (25) is naturality of ψ; F applied to normalisation condition $\xi^{\tau[n]} = \mathrm{id}_e$ gives $\psi^{\tau[n]} = F\xi^{\tau[n]} = \mathrm{id}_F$; F applied to multiplicativity (26) is the equation

$$\begin{aligned} &\big\langle F\mu_t(\mu_{t_p}(X^q_p \mid q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t)) \xrightarrow{F\xi^t} \mu_t(F\mu_{t_p}(X^q_p \mid q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t)) \\ &\qquad\qquad \xrightarrow{\mu_t(F\xi^{t_p}|p\in\mathrm{v}(t))} \mu_t(\mu_{t_p}(FX^q_p \mid q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t))\big\rangle \\ &\quad = \big\langle F\mu_{I(t;(t_p))}(X^q_p \mid q\in\mathrm{v}(t_p),\, p\in\mathrm{v}(t)) \xrightarrow{F\xi^{I(t;(t_p))}} \mu_{I(t;(t_p))}(FX^q_p \mid q\in\mathrm{v}(t_p),\, p\in\mathrm{v}(t))\big\rangle, \end{aligned}$$

which is (20) for $\psi^t = F\xi^t$. Thus ${}_s\mathbb{T}\text{-}\mathrm{Alg}_c(e,\mathsf{C})$ is bijective on objects.

A 2-morphism $\chi\colon (F,\mathrm{id}) \to (G,\mathrm{id})\colon (Q_c\mathsf{B},\mu_t,\mathrm{id},\mathrm{id}) \to (\mathsf{C},\mu_t,\mathrm{id},\mathrm{id})$ is taken by ${}_s\mathbb{T}\text{-}\mathrm{Alg}_c(e,\mathsf{C})$ to $\nu = e\cdot\chi\colon (F\big|,\phi) \to (G\big|,\psi)\colon (\mathsf{B},\mu_t,\mathrm{id},\mathrm{id}) \to (\mathsf{C},\mu_t,\mathrm{id},\mathrm{id})$, $\phi = F\xi$, $\psi = G\xi$. In particular, $\chi_{(\tau[n];X)} = \nu_X$ for $X \in \operatorname{Ob}\mathsf{B}(n)$. To restore χ from ν we have to write $\chi_{(t;(X_p)_{p\in\mathrm{v}(t)})} = \mu_t(\nu_{X_p} \mid p\in\mathrm{v}(t))\colon \mu_t(FX_p \mid p\in\mathrm{v}(t)) \to \mu_t(GX_p \mid p\in\mathrm{v}(t))$.

For an arbitrary collection of natural transformations $\nu_X(n)$ this gives a transformation χ natural with respect to $\mathbb{T}\mathsf{B}$. Naturality of χ with respect to $\xi^t_{(X_p)}$ is the equation

$$\begin{array}{ccc} F\mu_t(X_p \mid p\in \mathrm{v}(t)) & \xrightarrow{\quad\nu\quad} & G\mu_t(X_p \mid p\in \mathrm{v}(t)) \\ {\scriptstyle \phi^t}\downarrow & = & \downarrow{\scriptstyle \psi^t} \\ \mu_t(FX_p \mid p\in \mathrm{v}(t)) & \xrightarrow{\mu_t(\nu_{X_p}\mid p\in \mathrm{v}(t))} & \mu_t(GX_p \mid p\in \mathrm{v}(t)) \end{array}$$

which is (23). Naturality of χ with respect to $(\xi^{t_p})_{p\in\mathrm{v}(t)}$ does not impose extra conditions since $\mu^{Q_c\mathsf{B}}_{I(t;(t_p))} = \mu^{Q_c\mathsf{B}}_t(\mu^{Q_c\mathsf{B}}_{t_p} \mid p\in \mathrm{v}(t))$. Thus, (27) is bijective on morphisms as well. $\square$

Proposition 7. *The embedding 2-functor ${}_s\mathbb{T}\text{-Alg}_s \hookrightarrow {}_p\mathbb{T}\text{-Alg}_p$ admits a left biadjoint $L\colon {}_p\mathbb{T}\text{-Alg}_p \to {}_s\mathbb{T}\text{-Alg}_s$ such that the unit of the adjunction $\mathsf{B}\to L\mathsf{B}$ is an equivalence.*

Proof. Any strong $\mathcal{C}at$-operad C is strongly isomorphic to a strong $\mathcal{C}at$-operad B=$(\mathsf{B},\mu_t,\beta,\mathrm{id})$ which coincides with C as a collection of categories, has the same multiplications μ_t for $t\neq\tau[n]$ and such that $\mu_{\tau[n]} = \mathrm{Id}$ and $\iota_{\tau[n]} = \mathrm{id}\colon \mathrm{Id}\to\mu_{\tau[n]}$ for $n\in\mathbb{N}$. It suffices to describe L on the full 2-subcategory of ${}_p\mathbb{T}\text{-Alg}_p$ formed by B's with the above property. We define $L\mathsf{B}$ as the universal strict $\mathcal{C}at$-operad generated over $\mathbb{T}\mathsf{B}$ by the morphisms ξ^t from (24) and their inverse morphisms $\eta^t = (\xi^t)^{-1}$. The morphisms ξ^t are subject to naturality (25), multiplicativity (26) and normalisation as in the proof of Proposition 6.

There is a canonical strong $\mathcal{C}at$-multifunctor $(e,\xi^t)\colon \mathsf{B}\to L\mathsf{B}$, $e = \big(\mathsf{B}\hookrightarrow \mathbb{T}\mathsf{B}\to \mathbb{T}\mathsf{B}\langle\xi,\eta\rangle \to L\mathsf{B}\big)$. The functor e is essentially surjective on objects. Let us construct a one-sided inverse to it. First of all there is the action 1-morphism $\mu\colon \mathbb{T}\mathsf{B}\to\mathsf{B}$, $\big(t;(X_p)_{p\in\mathrm{v}(t)}\big)\mapsto \mu_t(X_p \mid p\in\mathrm{v}(t))$.

Of course, $\big(\mathsf{B}\hookrightarrow\mathbb{T}\mathsf{B}\xrightarrow{\mu}\mathsf{B}\big) = \mathrm{Id}$. Further we extend μ to a 1-morphism $\mathbb{T}\mathsf{B}\langle\xi^t\rangle_{t\neq\tau[n]}\to\mathsf{B}$ from the strict $\mathcal{C}at$-operad generated over $\mathbb{T}\mathsf{B}$ by $\xi^t_{(X_p)}$, $t\neq\tau[n]$. Namely, for any 2-cluster tree $(t;(t_p))$

$$\begin{aligned}(\xi^{t_p})_{p\in\mathrm{v}(t)}\colon \big(t;(\mu_{t_p}(X^q_p \mid q\in\mathrm{v}(t_p)))_{p\in\mathrm{v}(t)}\big) \to \mu^{\mathbb{T}\mathsf{B}}_t\big((t_p;(X^q_p)_{q\in\mathrm{v}(t_p)}) \mid p\in\mathrm{v}(t)\big) =\\ = \big(I(t;(t_p));((X^q_p)_{q\in\mathrm{v}(t_p)})_{p\in\mathrm{v}(t)}\big),\end{aligned} \tag{28}$$

(where $\xi^{\tau[n]}$ denotes id) is taken to

$$\beta_{t;(t_p)}\colon \mu_t(\mu_{t_p}(X^q_p \mid q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t)) \to \mu_{I(t;(t_p))}(X^q_p \mid q\in\mathrm{v}(t_p), p\in\mathrm{v}(t)).$$

Note that this recipe includes particular cases in which all but one t_p are corollas and ξ^{t_p} are identity morphisms. In order to verify that this assignment is correct we check that defining identities satisfied by (28) are taken to valid equations. (The identities follow from commutation relations between morphisms (28) in which all but one t_p are corollas.) Namely, for each 2-cluster tree $(t;(t_p)_{p\in\mathrm{v}(t)})$ and each decomposition $\mathrm{v}(t) = I\sqcup J$ we get $(\xi^{t_p})_{p\in\mathrm{v}(t)} = (\xi^{\tau_p})_{p\in\mathrm{v}(t)}\cdot(\xi^{\tau'_y})_{y\in\mathrm{v}(I(t;(\tau_p)))}$, where $\tau_p = t_p$ for $p\in I$, $\tau_p = \tau[|p|]$ for $p\in J$, $\tau'_y = t_y$ for $y\in J\subset\mathrm{v}(I(t;(\tau_p)))$ and $\tau'_y = \tau[|y|]$ for $y\in\mathrm{v}(I(t;(\tau_p)))-J$. In detail

$$\begin{aligned}(\xi^{t_p})_{p\in\mathrm{v}(t)} = \Big\langle \big(t;(\mu_{t_p}(X^q_p \mid q\in\mathrm{v}(t_p)))_{p\in\mathrm{v}(t)}\big) \xrightarrow{(\xi^{\tau_p})_{p\in\mathrm{v}(t)}} \\ \mu^{\mathbb{T}\mathsf{B}}_t\big[(t_p;(X^q_p)_{q\in\mathrm{v}(t_p)})_{p\in I}, \big(\tau[|p|];(\mu_{t_p}(X^q_p \mid q\in\mathrm{v}(t_p)))_{p\in J}\big]\\ = \big(I(t;(\tau_p));(\mu_{\tau'_y}(X^r_{\mathrm{pr}_1 y} \mid r\in\mathrm{v}(\tau'_y)))_{y\in\mathrm{v}(I(t;(\tau_p)))}\big) \xrightarrow{(\xi^{\tau'_y})_{y\in\mathrm{v}(I(t;(\tau_p)))}} \\ \mu^{\mathbb{T}\mathsf{B}}_{I(t;(\tau_p))}\big((\tau'_y;(X^r_{\mathrm{pr}_1 y})_{r\in\mathrm{v}(\tau'_y)}) \mid y\in\mathrm{v}(I(t;(\tau_p)))\big) = \big(I(t;(t_p));((X^q_p)_{q\in\mathrm{v}(t_p)})_{p\in\mathrm{v}(t)}\big)\Big\rangle.\end{aligned} \tag{29}$$

Here $\mathrm{pr}_1\, y = p$ for $y = (p, q)$. Respectively, identity $\beta_{t;(t_p)_{p\in \mathrm{v}(t)}} = \beta_{t;(\tau_p)_{p\in \mathrm{v}(t)}} \cdot \beta_{I(t;(\tau_p));(\tau'_y)_{y\in \mathrm{v}(I(t;(\tau_p)))}}$ holds in $\mathsf{B}(\mathrm{Inp}\, t)$ due to (19) written for the 3-cluster tree

$$(t; (\tau_p)_{p\in \mathrm{v}(t)}; (\tau'_y)_{y\in \mathrm{v}(I(t;(\tau_p)))}) = (t; (\tau_p)_{p\in \mathrm{v}(t)}; ((t^q_p)_{q\in \mathrm{v}(\tau_p)})_{p\in \mathrm{v}(t)}),$$

where $t^q_p = t_p = \tau'_{(p,q)}$ for $p \in J$, $q \in \mathrm{v}(\tau_p) \cong \mathbf{1}$ and $t^q_p = \tau[|q|] = \tau'_{(p,q)}$ for $p \in I$, $q \in \mathrm{v}(t_p)$. In fact, $\beta_{\tau_p;(t^q_p)_q} = \mathrm{id}$ and $I(\tau_p; (t^q_p)_q) = t_p$ for all $p \in I \sqcup J = \mathrm{v}(t)$.

Since β are invertible, the 1-morphism is extended to $\mathbb{T}\mathsf{B}\langle \xi^t, (\xi^t)^{-1}\rangle_{t\neq\tau[n]} \to \mathsf{B}$. It remains to check that defining relations for $L\mathsf{B}$ are taken to identities valid in B.

Naturality. The defining relation of $L\mathsf{B}$

$$(\xi^{t_p})_{p\in \mathrm{v}(t)} \cdot \big(I(t; (t_p)); ((f^q_p)_{q\in \mathrm{v}(t_p)})_{p\in \mathrm{v}(t)}\big) = \big(t; (\mu_{t_p}(f^q_p \mid q \in \mathrm{v}(t_p)))_{p\in \mathrm{v}(t)}\big) \cdot (\xi^{t_p})_{p\in \mathrm{v}(t)}$$

is taken to the valid identity

$$\beta_{t;(t_p)} \cdot \mu_{I(t;(t_p))}(f^q_p \mid q \in \mathrm{v}(t_p), p \in \mathrm{v}(t)) = \mu_t(\mu_{t_p}(f^q_p \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t)) \cdot \beta_{t;(t_p)},$$

which is nothing else but naturality of β.

Normalisation. We have to check that in the case of all $t_p = \tau[|p|]$ the identity morphism $(\xi^{t_p})_{p\in \mathrm{v}(t)}$ is taken to the identity morphism. Indeed, $\beta_{t;(\tau[|p|])_p} = \mathrm{id}$ by (17).

Multiplicativity. The defining relation (26) of $L\mathsf{B}$ is taken to the identity $\mathrm{id} \cdot \beta_{t;(t_p)} = \beta_{t;(t_p)} \cdot \mathrm{id}$. Thus $L\mathsf{B} \to \mathsf{B}$ is constructed.

Clearly $(\mathsf{B} \xrightarrow{e} L\mathsf{B} \to \mathsf{B}) = \mathrm{Id}$. On the other hand, $\xi\colon (L\mathsf{B} \to \mathsf{B} \xrightarrow{e} L\mathsf{B}) \to \mathrm{Id}_{L\mathsf{B}}$ is a natural transformation, as seen from naturality relation (25) and from multiplicativity (26). Since ξ is invertible, the functor $L\mathsf{B} \to \mathsf{B}$ is quasi-inverse to $e\colon \mathsf{B} \to L\mathsf{B}$.

Precomposition with the functor $(e, \xi^t)\colon \mathsf{B} \to L\mathsf{B}$ gives a functor

$$R = {}_p\mathbb{T}\text{-}\mathrm{Alg}_p(e, \mathsf{D})\colon {}_s\mathbb{T}\text{-}\mathrm{Alg}_s(L\mathsf{B}, \mathsf{D}) \to {}_p\mathbb{T}\text{-}\mathrm{Alg}_p(\mathsf{B}, \mathsf{D}), \quad (G, \mathrm{id}) \mapsto (F, \phi), \quad F = G\big|_{\mathsf{B}},$$
$$\phi^t = \big(F\mu_t(X_p \mid p \in \mathrm{v}(t)) = G\mu_t(X_p \mid p \in \mathrm{v}(t)) \xrightarrow{G\xi^t} G(X_p)_{p\in \mathrm{v}(t)} = \mu_t(FX_p \mid p \in \mathrm{v}(t))\big).$$

Let us prove that it is an equivalence. A quasi-inverse functor

$$S\colon {}_p\mathbb{T}\text{-}\mathrm{Alg}_p(\mathsf{B}, \mathsf{D}) \to {}_s\mathbb{T}\text{-}\mathrm{Alg}_s(L\mathsf{B}, \mathsf{D}),\ (F, \phi) \mapsto (G, \mathrm{id}), \tag{30}$$

is constructed as follows: $GX = FX$, $Gf = Ff$,

$$G(\xi^t) = \phi^t\colon F\mu_t(X_p \mid p \in \mathrm{v}(t)) \to \mu_t(FX_p \mid p \in \mathrm{v}(t)). \tag{31}$$

Since we plan G to be a strict $\mathcal{C}at$-multifunctor, we conclude that

$$G((X_p)_{p\in \mathrm{v}(t)}) = \mu_t(FX_p \mid p \in \mathrm{v}(t)), \quad G((f_p)_{p\in \mathrm{v}(t)}) = \mu_t(Ff_p \mid p \in \mathrm{v}(t)),$$
$$G((\xi^{t_p})_{p\in \mathrm{v}(t)}) = \mu_t(\phi^{t_p} \mid p \in \mathrm{v}(t))\colon \mu_t(F\mu_{t_p}(X^q_p \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t))$$
$$\to \mu_t(\mu_{t_p}(FX^q_p \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t)) = \mu_{I(t;(t_p))}(FX^q_p \mid q \in \mathrm{v}(t_p),\, p \in \mathrm{v}(t)).$$

Applying G to relations (naturality, normalisation, multiplicativity) imposed on ξ gives valid identities in D. For instance, applying G to multiplicativity (26) gives (20). The transformation part of the $\mathcal{C}at$-multifunctor $(G, \text{-})$ under construction may be chosen equal to identity morphism since for any 2-cluster tree $(t;(t_p))$

$$G\big(((X_p^q)_{q\in \mathrm{v}(t_p)})_{p\in \mathrm{v}(t)}\big) = \mu_{I(t;(t_p))}(FX_p^q \mid q\in \mathrm{v}(t_p),\, p\in \mathrm{v}(t)) = \\ = \mu_t(\mu_{t_p}(FX_p^q \mid q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t)) = \mu_t\big(G((X_p^q)_{q\in \mathrm{v}(t_p)}) \mid p\in \mathrm{v}(t)\big).$$

Obviously, this identity transformation is natural and satisfies necessary conditions, making (G, id) into a $\mathcal{C}at$-multifunctor.

Clearly R and S are mutually inverse functors, thereby, isomorphisms of categories. □

Corollary 5. *The embedding 2-functor* ${}_s\mathbb{T}\text{-Alg}_s \hookrightarrow {}_p\mathbb{T}\text{-Alg}_p$ *is a biequivalence with a quasi-inverse* L.

Proof. Assume that B is a strict $\mathcal{C}at$-operad. Applying (30) to Id_B gets a strict $\mathcal{C}at$-multifunctor $\Gamma = S(\mathrm{Id}_\mathsf{B})\colon L\mathsf{B} \to \mathsf{B}$. Explicit description (31) of Γ shows that

$$\big(\mathsf{B} \xrightarrow{(e,\xi)} L\mathsf{B} \xrightarrow{(\Gamma,\mathrm{id})} \mathsf{B}\big) = (\mathrm{Id}_\mathsf{B}, \mathrm{id}).$$

Therefore, Γ is an equivalence, quasi-inverse to e. The composition of two equivalences

$${}_s\mathbb{T}\text{-Alg}_s(\mathsf{B},\mathsf{D}) \xrightarrow[\Gamma\cdot-]{\simeq} {}_s\mathbb{T}\text{-Alg}_s(L\mathsf{B},\mathsf{D}) \xrightarrow[e\cdot-]{\cong} {}_s\mathbb{T}\text{-Alg}_p(\mathsf{B},\mathsf{D}), \quad F\mapsto F, \quad \nu\mapsto\nu,$$

is the inclusion (and an equivalence). □

Proposition 8. *There are a 2-functor* $Q_c\colon {}_p\mathbb{T}\text{-Alg}_c \to {}_p\mathbb{T}\text{-Alg}_p$ *and a 2-natural equivalence* ${}_p\mathbb{T}\text{-Alg}_p(Q_c\mathsf{B},\mathsf{C}) \xrightarrow{\simeq} {}_p\mathbb{T}\text{-Alg}_c(\mathsf{B},\mathsf{C})$, *which turn* Q_c *into left biadjoint to the inclusion* ${}_p\mathbb{T}\text{-Alg}_p \hookrightarrow {}_p\mathbb{T}\text{-Alg}_c$.

This follows from Hermida's Theorem 6.1 [7], applicable due to Remark 4. We give a proof mostly in order to describe colax/pseudo morphism classifier Q_c and related functors explicitly.

Proof. Let B, C be strong $\mathcal{C}at$-operads. We replace them with isomorphic strong $\mathcal{C}at$-operads $\mathsf{B} = (\mathsf{B}, \mu_t, \beta, \mathrm{id})$, $\mathsf{C} = (\mathsf{C}, \mu_t, \alpha, \mathrm{id})$ as in the proof of Proposition 7. Define $Q_c\mathsf{B}$ to be the universal strict $\mathcal{C}at$-operad generated over $\mathbb{T}\mathsf{B}$ by the morphisms ξ^t from (24) subject to naturality (25), normalisation and multiplicativity (26) as in the proof of Proposition 6.

There is a canonical colax $\mathcal{C}at$-multifunctor $(e,\xi)\colon \mathsf{B} \to Q_c\mathsf{B}$, $e = \big(\mathsf{B} \xhookrightarrow{\mathrm{in}_{\tau[-]}} \mathbb{T}\mathsf{B} \to Q_c\mathsf{B}\big)$. Precomposition with $e = (e,\xi)$ gives a functor

$$R = {}_p\mathbb{T}\text{-Alg}_c(e,\mathsf{C})\colon {}_p\mathbb{T}\text{-Alg}_p(Q_c\mathsf{B},\mathsf{C}) \to {}_p\mathbb{T}\text{-Alg}_c(\mathsf{B},\mathsf{C}), \quad (G,\psi)\mapsto (F,\phi), \quad F = G\big|_\mathsf{B},$$

$$\phi^t = \big(F\mu_t(X_p \mid p\in \mathrm{v}(t)) = G\mu_t(X_p \mid p\in \mathrm{v}(t)) \xrightarrow{G\xi^t} G((X_p)_{p\in \mathrm{v}(t)}) \xrightarrow{\psi^t} \mu_t(FX_p \mid p\in \mathrm{v}(t))\big).$$

Let us prove that R is an equivalence. We construct a quasi-inverse functor

$$S\colon {}_p\mathbb{T}\text{-Alg}_c(\mathsf{B},\mathsf{C}) \to {}_p\mathbb{T}\text{-Alg}_p(Q_c\mathsf{B},\mathsf{C}), \quad (F,\phi)\mapsto (G,\psi), \tag{32}$$

by (31) extended by assignment

$$G((X_p)_{p\in \mathrm{v}(t)}) = \mu_t(FX_p \mid p\in \mathrm{v}(t)), \quad G((f_p)_{p\in \mathrm{v}(t)}) = \mu_t(Ff_p \mid p\in \mathrm{v}(t)),$$
$$G((\xi^{t_p})_{p\in \mathrm{v}(t)}) = \big\langle \mu_t(F\mu_{t_p}(X_p^q \mid q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t)) \xrightarrow{\mu_t(\phi^{t_p} \mid p\in \mathrm{v}(t))}$$
$$\mu_t(\mu_{t_p}(FX_p^q \mid q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t)) \xrightarrow{\alpha_{t;(t_p)}} \mu_{I(t;(t_p))}(FX_p^q \mid q\in \mathrm{v}(t_p),\, p\in \mathrm{v}(t))\big\rangle. \tag{33}$$

In order to prove that G is a well-defined functor we follow the reasoning from the proof of Proposition 7 extending G from $\mathbb{T}\mathsf{B}$ to $\mathbb{T}\mathsf{B}\langle\xi\rangle$ and further to $Q_c\mathsf{B}$. It suffices to show that relations (26) and (29) are taken by G to valid identities. We combine these relations into a single one: for any 3-cluster tree $(t;(t_p);(t_p^q))$

$$\begin{array}{ccc}
\big(t;(\mu_{t_p}(\mu_{t_p^q}(X_{pr}^q \mid r\in \mathrm{v}(t_p^q)) \mid q\in\mathrm{v}(t_p)))_{p\in\mathrm{v}(t)}\big) & \xrightarrow{(\xi^{t_p})_{p\in\mathrm{v}(t)}} & \big(I(t;(t_p));(\mu_{t_p^q}(X_{pr}^q \mid r\in\mathrm{v}(t_p^q)))_{p\in\mathrm{v}(t)}^{q\in\mathrm{v}(t_p)}\big) \\
\Big\downarrow{\scriptstyle (t;(\beta_{t_p;(t_p^q)})_{p\in\mathrm{v}(t)})} & = & \Big\downarrow{\scriptstyle (\xi^{t_p^q})_{p\in\mathrm{v}(t)}^{q\in\mathrm{v}(t_p)}} \\
\big(t;(\mu_{I(t_p;(t_p^q)_q)}(X_{pr}^q \mid r\in\mathrm{v}(t_p^q), q\in\mathrm{v}(t_p)))_{p\in\mathrm{v}(t)}\big) & \xrightarrow{(\xi^{I(t_p;(t_p^q)_q)})_{p\in\mathrm{v}(t)}} & \big(I(t;(t_p);(t_p^q));(X_{pr}^q)_{p\in\mathrm{v}(t),r\in\mathrm{v}(t_p^q)}^{q\in\mathrm{v}(t_p)}\big)
\end{array}$$

Applying G to this equation we get the diagram

$$\begin{array}{ll}
\mu_t\big(F\mu_{t_p}(\mu_{t_p^q}(X_{pr}^q \mid r\in\mathrm{v}(t_p^q)) \mid q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t)\big) & \\
\qquad\searrow{\scriptstyle \mu_t(\phi^{t_p}\mid p\in\mathrm{v}(t))} & \\
 & \mu_t\big(\mu_{t_p}(F\mu_{t_p^q}(X_{pr}^q \mid r\in\mathrm{v}(t_p^q)) \mid q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t)\big) \\
\Big\downarrow{\scriptstyle \mu_t(F\beta_{t_p;(t_p^q)_q}\mid p\in\mathrm{v}(t))} & \Big\downarrow{\scriptstyle \alpha_{t;(t_p)}} \\
\mu_t\big(F\mu_{I(t_p;(t_p^q)_q)}(X_{pr}^q \mid r\in\mathrm{v}(t_p^q), q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t)\big) & \\
 & \mu_{I(t;(t_p))}\big(F\mu_{t_p^q}(X_{pr}^q \mid r\in\mathrm{v}(t_p^q)) \mid q\in\mathrm{v}(t_p), p\in\mathrm{v}(t)\big) \\
\Big\downarrow{\scriptstyle \mu_t(\phi^{I(t_p;(t_p^q)_q}\mid p\in\mathrm{v}(t))} & \Big\downarrow{\scriptstyle \mu_{I(t;(t_p))}(\phi^{t_p^q}\mid q\in\mathrm{v}(t_p),p\in\mathrm{v}(t))} \\
\mu_t\big(\mu_{I(t_p;(t_p^q)_q)}(FX_{pr}^q \mid r\in\mathrm{v}(t_p^q), q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t)\big) & \\
 & \mu_{I(t;(t_p))}\big(\mu_{t_p^q}(FX_{pr}^q \mid r\in\mathrm{v}(t_p^q)) \mid q\in\mathrm{v}(t_p), p\in\mathrm{v}(t)\big) \\
\Big\downarrow{\scriptstyle \alpha_{t;(I(t_p;(t_p^q)_q))_p}} & \qquad\swarrow{\scriptstyle \alpha_{I(t;(t_p));(t_p^q)_p^q}} \\
\mu_{I(t;(t_p);(t_p^q))}\big(FX_{pr}^q \mid r\in\mathrm{v}(t_p^q), q\in\mathrm{v}(t_p), p\in\mathrm{v}(t)\big) &
\end{array}$$

Due to naturality of $\alpha_{t;(t_p)}$ the two maps in the second column can be replaced with

$$\xrightarrow{\mu_t(\mu_{t_p}(\phi^{t_p^q}\mid q\in\mathrm{v}(t_p))\mid p\in\mathrm{v}(t))} \mu_t\big(\mu_{t_p}(\mu_{t_p^q}(FX_{pr}^q \mid r\in\mathrm{v}(t_p^q)) \mid q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t)\big) \xrightarrow{\alpha_{t;(t_p)}} .$$

The above middle term can be related by the map $\mu_t(\alpha_{t_p;(t_p^q)_q} \mid p\in\mathrm{v}(t))$ with

$$\mu_t\big(\mu_{I(t_p;(t_p^q)_q)}(FX_{pr}^q \mid r\in\mathrm{v}(t_p^q), q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t)\big)$$

in the first column. This decomposes the diagram into a commutative pentagon due to (20) and a commutative square (19).

The transformation part ψ of the $\mathcal{C}at$-multifunctor (G,ψ) under construction is chosen as follows for any 2-cluster tree $(t;(t_p))$

$$\begin{aligned}\psi^t = \big\langle G(t;((X_p^q)_{q\in\mathrm{v}(t_p)})_{p\in\mathrm{v}(t)}) = \mu_{I(t;(t_p))}(FX_p^q \mid q\in\mathrm{v}(t_p),\, p\in\mathrm{v}(t)) \xrightarrow{\alpha_{t;(t_p)}^{-1}} \\
\mu_t(\mu_{t_p}(FX_p^q \mid q\in\mathrm{v}(t_p)) \mid p\in\mathrm{v}(t)) = \mu_t\big(G(t_p;(X_p^q)_{q\in\mathrm{v}(t_p)}) \mid p\in\mathrm{v}(t)\big)\big\rangle.\end{aligned} \tag{34}$$

The necessary property: for any 3-cluster tree $(t;(t_p);(t_p^q))$

$$
\begin{array}{ccc}
G\mu_t^{\mathbb{T}\mathsf{B}}\big(t_p;(\mu_{t_p^q}(X_{pr}^q \mid r\in \mathrm{v}(t_p^q)))_{q\in \mathrm{v}(t_p)} \mid p\in \mathrm{v}(t)\big) & \xrightarrow{\psi^t} & \mu_t\big(G(t_p;(\mu_{t_p^q}(X_{pr}^q \mid r\in \mathrm{v}(t_p^q)))_{q\in \mathrm{v}(t_p)}) \mid p\in \mathrm{v}(t)\big) \\
\Big\downarrow {\scriptstyle G\mu_t^{\mathbb{T}\mathsf{B}}((\xi^{t_p^q})_{q\in \mathrm{v}(t_p)} \mid p\in \mathrm{v}(t))} & & \Big\downarrow {\scriptstyle \mu_t(G((\xi^{t_p^q})_{q\in \mathrm{v}(t_p)}) \mid p\in \mathrm{v}(t))} \\
G\mu_t^{\mathbb{T}\mathsf{B}}\big(I(t_p;(t_p^q)_q);(X_{pr}^q)_{r\in \mathrm{v}(t_p^q)}^{q\in \mathrm{v}(t_p)} \mid p\in \mathrm{v}(t)\big) & \xrightarrow{\psi^t} & \mu_t\big(G(I(t_p;(t_p^q)_q);(X_{pr}^q)_{r\in \mathrm{v}(t_p^q)}^{q\in \mathrm{v}(t_p)}) \mid p\in \mathrm{v}(t)\big)
\end{array}
$$

is verified below

$$
\begin{array}{ccc}
\mu_{I(t;(t_p))}\big(F\mu_{t_p^q}(X_{pr}^q \mid r\in \mathrm{v}(t_p^q)) \mid q\in \mathrm{v}(t_p), p\in \mathrm{v}(t)\big) & \xleftarrow{\alpha_{t;(t_p)}} & \mu_t\big(\mu_{t_p}(F\mu_{t_p^q}(X_{pr}^q \mid r\in \mathrm{v}(t_p^q)) \mid q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t)\big) \\
\Big\downarrow {\scriptstyle \mu_{I(t;(t_p))}(\phi^{t_p^q} \mid q\in \mathrm{v}(t_p), p\in \mathrm{v}(t))} & & \Big\downarrow {\scriptstyle \mu_t(\mu_{t_p}(\phi^{t_p^q} \mid q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t))} \\
\mu_{I(t;(t_p))}\big(\mu_{t_p^q}(FX_{pr}^q \mid r\in \mathrm{v}(t_p^q)) \mid q\in \mathrm{v}(t_p), p\in \mathrm{v}(t)\big) & \xleftarrow{\alpha_{t;(t_p)}} & \mu_t\big(\mu_{t_p}(\mu_{t_p^q}(FX_{pr}^q \mid r\in \mathrm{v}(t_p^q)) \mid q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t)\big) \\
\Big\downarrow {\scriptstyle \alpha_{I(t;(t_p));(t_p^q)_p^q}} & & \Big\downarrow {\scriptstyle \mu_t(\alpha_{t_p;(t_p^q)_q} \mid p\in \mathrm{v}(t))} \\
\mu_{I(t;(t_p)_p;(t_p^q)_p^q)}\big(FX_{pr}^q \mid r\in \mathrm{v}(t_p^q), q\in \mathrm{v}(t_p), p\in \mathrm{v}(t)\big) & \xleftarrow{\alpha_{t;(I(t_p;(t_p^q)_q))_p}} & \mu_t\big(\mu_{I(t_p;(t_p^q)_q)}(FX_{pr}^q \mid r\in \mathrm{v}(t_p^q), q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t)\big)
\end{array}
$$

The top parallelogram commutes due to naturality of $\alpha_{t;(t_p)}$ and the bottom is equation (19). The functor S is constructed on objects with obvious extension to morphisms.

Clearly $R\circ S = \mathrm{Id}$. Let us construct an isomorphism of $\mathcal{C}at$-multifunctors $\Psi\colon \mathrm{Id}\to S\circ R$. Start with $(G,\psi)\in \mathrm{Ob}\,{}_p\mathbb{T}\text{-}\mathrm{Alg}_p(Q_c\mathsf{B},\mathsf{D})$, and denote $(H,\chi) = SR(G,\psi)\colon Q_c\mathsf{B}\to\mathsf{D}$. Then $H(X_p)_{p\in \mathrm{v}(t)} = \mu_t(GX_p \mid p\in \mathrm{v}(t))$ and the $\mathcal{C}at$-transformation $\Psi\colon (G,\psi)\to(H,\chi)$ is taken in the form $\psi^t\colon G((X_p)_{p\in \mathrm{v}(t)}) \to \mu_t(GX_p \mid p\in \mathrm{v}(t)) = H(X_p)_{p\in \mathrm{v}(t)}$. Naturality of Ψ with respect to the morphism $(\xi^{t_p})_{p\in \mathrm{v}(t)}$ from (28) is the exterior of

$$
\begin{array}{ccc}
G((\mu_{t_p}(X_p^q \mid q\in \mathrm{v}(t_p)))_{p\in \mathrm{v}(t)}) & \xrightarrow{\psi^t} & \mu_t(G\mu_{t_p}(X_p^q \mid q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t)) \\
\Big\downarrow {\scriptstyle G(\xi^{t_p})_{p\in \mathrm{v}(t)}} & = & \Big\downarrow {\scriptstyle \mu_t(G\xi^{t_p} \mid p\in \mathrm{v}(t))} \\
G(((X_p^q)_{q\in \mathrm{v}(t_p)})_{p\in \mathrm{v}(t)}) & \xrightarrow{\psi^t} & \mu_t\big(G((X_p^q)_{q\in \mathrm{v}(t_p)}) \mid p\in \mathrm{v}(t)\big) \\
\Big\downarrow {\scriptstyle \psi^{I(t;(t_p))}} & (20) & \Big\downarrow {\scriptstyle \mu_t(\psi^{t_p} \mid p\in \mathrm{v}(t))} \\
\mu_{I(t;(t_p))}(GX_p^q \mid q\in \mathrm{v}(t_p),\, p\in \mathrm{v}(t)) & \xleftarrow{\alpha_{t;(t_p)}} & \mu_t(\mu_{t_p}(GX_p^q \mid q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t)).
\end{array}
$$

Commutativity is proven via naturality of ψ^t and property (20). The family $\Psi\colon (G,\psi)\to(H,\chi)$ is a $\mathcal{C}at$-transformation again due to (20). Clearly the $\mathcal{C}at$-transformation Ψ is natural with respect to change of (G,ψ). □

Remark 5. 2-functor Q_c constructed in Proposition 6 is the restriction of 2-functor Q_c from Proposition 8

$$\big({}_s\mathbb{T}\text{-Alg}_c \xrightarrow{Q_c} {}_s\mathbb{T}\text{-Alg}_s \hookrightarrow {}_p\mathbb{T}\text{-Alg}_p\big) = \big({}_s\mathbb{T}\text{-Alg}_c \hookrightarrow {}_p\mathbb{T}\text{-Alg}_c \xrightarrow{Q_c} {}_p\mathbb{T}\text{-Alg}_p\big). \tag{35}$$

Corollary 6 (to Propositions 6, 8)**.** *There are a 2-functor* $Q_l\colon {}_s\mathbb{T}\text{-Alg}_l \to {}_s\mathbb{T}\text{-Alg}_s$ *(resp.* $Q_l\colon {}_p\mathbb{T}\text{-Alg}_l \to {}_p\mathbb{T}\text{-Alg}_p$*) and a 2-natural isomorphism (resp. equivalence)*

$${}_s\mathbb{T}\text{-Alg}_s(Q_l\mathsf{B},\mathsf{C}) \xrightarrow{\cong} {}_s\mathbb{T}\text{-Alg}_l(\mathsf{B},\mathsf{C}), \quad \textit{resp.} \quad {}_p\mathbb{T}\text{-Alg}_p(Q_l\mathsf{B},\mathsf{C}) \xrightarrow{\simeq} {}_p\mathbb{T}\text{-Alg}_l(\mathsf{B},\mathsf{C}),$$

which turn Q_l *into a left adjoint (resp. biadjoint) to the inclusion* ${}_s\mathbb{T}\text{-Alg}_s \hookrightarrow {}_s\mathbb{T}\text{-Alg}_l$ *(resp.* ${}_p\mathbb{T}\text{-Alg}_p \hookrightarrow {}_p\mathbb{T}\text{-Alg}_l$*). 2-functors* Q_l *agree similarly to* (35)*.*

The proof and the construction of Q_l is by dualising the results for Q_c using opposite $\mathcal{C}at$-operads. Thus, $Q_l\mathsf{B}$ is the universal strict $\mathcal{C}at$-operad generated over $\mathbb{T}\mathsf{B}$ by the morphisms $\xi^t\colon (t;(X_p)_{p\in\mathrm{v}(t)}) \to (\tau[n];\mu_t(X_p \mid p\in\mathrm{v}(t)))$ subject to naturality, normalisation and multiplicativity.

Remark 6. It follows from the proof of Proposition 8 that there is a functor

$$S\colon {}_l\mathbb{T}\text{-Alg}_c(\mathsf{B},\mathsf{C}) \to {}_l\mathbb{T}\text{-Alg}_l(Q_c\mathsf{B},\mathsf{C}), \quad (F,\phi)\mapsto (G,\psi),$$

where B, C are lax $\mathbb{T}$-algebras, G is given by (33) and $\psi^t = \alpha_{t;(t_p)}$ is the inverse to (34).

A *non-counital cooperad* (C,Δ_t) in a lax $\mathcal{C}at$-operad $(\mathsf{C},\mu_t,\alpha,\iota)$ consists of

— a collection of objects $C(n)\in\mathsf{C}(n)$, $n\geqslant 0$,
— morphisms $\Delta_t\colon C(\operatorname{Inp} t)\to \mu_t(C|p| \mid p\in\mathrm{v}(t))\in\mathsf{C}(\operatorname{Inp} t)$ for $t\in\mathbf{tr}\smallsetminus\circ$,

such that

(i) for any family $t\in\mathbf{tr}\smallsetminus\circ$, $t_p\in\mathbf{tr}\,|p|\smallsetminus\circ$, $p\in\mathrm{v}(t)$, and the tree $\tau = I_t(t_p \mid p\in\mathrm{v}(t))$ equation (22) holds;

(ii) for all $n\in\mathbb{N}$ $\Delta_{\tau[n]} = \iota^{\mathsf{C}}_{\tau[n]}\colon C(n)\to\mu_{\tau[n]}(C(n))$.

Morphisms of non-counital cooperads are those agreeing with comultiplications Δ_t.

Example 9. Let a strong $\mathcal{C}at$-operad C consist of additive categories $\mathsf{C}(n)$ with countable colimits, countable products and finite limits and additive functors μ_t, which preserve countable colimits. We assume that $\mathsf{C}(n)$ is idempotent-split (Karoubian). For some results we assume that

(FF) for any functor $F\colon C\times D\to\mathsf{C}(n)$ with filtered countable category C and finite category D the canonical morphism $\operatorname{colim}_C\lim_D F\to\lim_D\operatorname{colim}_C F$ is invertible.

This axiom is satisfied for $\mathcal{S}et$, Ab and similar categories but not for the category of topological spaces ([4, Section 2.13]). For any collection $X(n)\in\mathsf{C}(n)$, $n\geqslant 0$, define a collection

$$(\perp\!\!\!\perp X)(n) = \coprod_{t\in\mathbf{tr}(n)\smallsetminus\circ} \mu_t(X|p| \mid p\in\mathrm{v}(t)) \in\mathsf{C}(n)$$

which defines a functor $\perp\!\!\!\perp\colon \prod_{n\in\mathbb{N}} \mathsf{C}(n) \to \prod_{n\in\mathbb{N}} \mathsf{C}(n)$. It admits a comonad structure, namely, $\Delta\colon (\perp\!\!\!\perp X)(n) = \coprod_{\tau\in\mathbf{tr}(n)\smallsetminus\circ} \mu_\tau(X|v| \mid v \in \mathrm{v}(\tau)) \to (\perp\!\!\!\perp\perp\!\!\!\perp X)(n)$ is a family indexed by $\tau \in \mathbf{tr}(n) \smallsetminus \circ$ of morphisms

$$\mu_\tau(X|v| \mid v \in \mathrm{v}(\tau)) \xrightarrow[\text{diag}]{(\text{id})} \prod_{f\colon \tau\twoheadrightarrow t\in\mathbf{Tr}(n)} \mu_\tau(X|v| \mid v \in \mathrm{v}(\tau)) \xrightarrow{\cong} \coprod_{f\colon \tau\twoheadrightarrow t\in\mathbf{Tr}(n)} \mu_\tau(X|v| \mid v \in \mathrm{v}(\tau))$$
$$\hookrightarrow \coprod_{t\in\mathbf{tr}(n)\smallsetminus\circ} \; \coprod_{(t_p)\in\prod_{p\in\mathrm{v}(t)} \mathbf{tr}\,|p|\smallsetminus\circ} \mu_{I(t;(t_p))}(X|q| \mid q \in \mathrm{v}(t_p), p \in \mathrm{v}(t))$$
$$\xrightarrow{\coprod\coprod\alpha^{-1}} \coprod_{t\in\mathbf{tr}(n)\smallsetminus\circ} \; \coprod_{(t_p)\in\prod_{p\in\mathrm{v}(t)} \mathbf{tr}\,|p|\smallsetminus\circ} \mu_t\big(\mu_{t_p}(X|q| \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t)\big)$$
$$\xrightarrow{\cong} \coprod_{t\in\mathbf{tr}(n)\smallsetminus\circ} \mu_t\Big(\coprod_{t_p\in\mathbf{tr}\,|p|\smallsetminus\circ} \mu_{t_p}(X|q| \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t)\Big) = (\perp\!\!\!\perp\perp\!\!\!\perp X)(n).$$

Notice that for a fixed τ the number of surjective morphisms $f\colon \tau \twoheadrightarrow t \in \mathbf{Tr}(n)$ is finite (the map $f|\colon \mathrm{v}(\tau) \to \mathrm{v}(t)$ is surjective), which explains the second arrow. The third arrow is the canonical split embedding, coming from the inclusion of indexing sets

$$\tau\backslash\, \mathbf{surTr}(n) \hookrightarrow \bigsqcup_{t\in\mathbf{tr}(n)\smallsetminus\circ} \; \prod_{p\in\mathrm{v}(t)} \mathbf{tr}\,|p| \smallsetminus \circ, \quad (f: \tau \twoheadrightarrow t) \mapsto (t; (t_p)),$$

described in Proposition 5. The corresponding subtrees $t_p \subset \tau$ are determined by $\mathrm{v}(t_p) = f^{-1}(p)$, $p \in \mathrm{v}(t)$, and $\tau = I(t; (t_p)_{p\in\mathrm{v}(t)})$. Associativity of the comultiplication Δ is the observation that t-partitioning of τ followed by t_p-partitioning of the obtained pieces $\overline{f^{-1}(p)}$ amounts to $I(t; (t_p))$-partitioning of τ. The counit of the comonad is given by

$$(\perp\!\!\!\perp X)(n) \xrightarrow{\mathrm{pr}_{\tau[n]}} \mu_{\tau[n]}(X(n)) \xrightarrow{\iota^{-1}} X(n).$$

An example of $\perp\!\!\!\perp$-coalgebra is the *cofree* $\perp\!\!\!\perp$-coalgebra $\perp\!\!\!\perp X$. Its coaction $\delta = (\Delta_t \cdot \mathrm{in}_t)_{t\in\mathbf{tr}\smallsetminus\circ}\colon \perp\!\!\!\perp X \to \perp\!\!\!\perp\perp\!\!\!\perp X$ is specified by comultiplications for $\mathrm{Inp}\, t \cong \mathrm{Inp}\, \tau$

$$\Delta_t\big|_{\mu_\tau(X|v||v\in\mathrm{v}(\tau))} = \sum_{f\colon \tau\twoheadrightarrow t\in\mathbf{surTr}} \big\langle \mu_\tau(X|v| \mid v \in \mathrm{v}(\tau)) = \mu_{I(t;(t_p))}(X|v| \mid v \in \mathrm{v}(I(t; (t_p))))$$
$$\xrightarrow{\alpha^{-1}_{I(t;(t_p))}} \mu_t\big(\mu_{t_p}(X|q| \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t)\big) \xhookrightarrow{\mu_t(\mathrm{in}_{t_p})} \mu_t((\perp\!\!\!\perp X)|p| \mid p \in \mathrm{v}(t))\big\rangle, \quad (36)$$

where $t_p = \overline{f^{-1}(p)}$.

$\perp\!\!\!\perp$-coalgebras are non-counital cooperads. In fact, a $\perp\!\!\!\perp$-coalgebra $\delta\colon C \to \perp\!\!\!\perp C$ determines morphisms $t \in \mathbf{tr}(n) \smallsetminus \circ$

$$\Delta_t = \Big\langle C(n) \xrightarrow{\delta} \coprod_{\tau\in\mathbf{tr}(n)\smallsetminus\circ} \mu_\tau(C|p| \mid p \in \mathrm{v}(\tau)) \xrightarrow{\mathrm{pr}_t} \mu_t(C|p| \mid p \in \mathrm{v}(t))\Big\rangle.$$

Postcomposing the equation $\delta\cdot\Delta = \delta\cdot\perp\!\!\!\perp\delta$ with $\mathrm{pr}_t \cdot\mu_t(\mathrm{pr}_{t_p} \mid p \in \mathrm{v}(t))$ for an arbitrary family $t \in \mathbf{tr}\smallsetminus\circ$, $t_p \in \mathbf{tr}\,|p| \smallsetminus \circ$, $p \in \mathrm{v}(t)$, we get (22). Equality $\delta \cdot \varepsilon_{\perp\!\!\!\perp} = 1$ implies that $\Delta_{\tau[n]} = \iota^{\mathsf{C}}_{\tau[n]}$ for all $n \in \mathbb{N}$.

Definition 11. A non-counital cooperad (C, Δ_t) is *conilpotent* if it has a filtration $C_1 \subset \cdots \subset C_k \subset C_{k+1} \subset \cdots \subset C$ by subobjects in $\prod_{n\in\mathbb{N}} \mathsf{C}(n)$ such that $\mathrm{colim}_{k>0}\, C_k = C$ and for any $t \in \mathbf{tr}$ with $|\,\mathrm{v}(t)| > k$ we have $\Delta_t\big|_{C_k} = 0$.

Proposition 9. *Assume that all $\mathsf{C}(n)$ satisfy axiom (FF). The full subcategory of* nuCoop *consisting of conilpotent non-counital cooperads is isomorphic to the category of $\perp\!\!\!\perp$-coalgebras.*

Proof. Let (C,δ) be a $\perp\!\!\!\perp$-coalgebra. Then pull-backs C_k from

$$\begin{array}{ccc} C_k & \longrightarrow & \bigoplus_{t\in\mathbf{tr},\,0<|\operatorname{v}(t)|\leqslant k} \mu_t(C|p| \mid p\in\operatorname{v}(t)) \\ \downarrow & \lrcorner & \downarrow \\ C & \xrightarrow{\delta} & \perp\!\!\!\perp C \end{array}$$

define the required filtration due to axiom (FF).

If C is a conilpotent non-counital cooperad with the filtration (C_k), then the morphisms $(\Delta_t|_{C_k})_{t\neq\circ}\colon C_k \to \prod_{t\neq\circ}\mu_t(C|p| \mid p\in\operatorname{v}(t))$ are decomposed as

$$\begin{array}{ccccc} C_k & \xrightarrow{d_k} & \bigoplus_{t\in\mathbf{tr},\,0<|\operatorname{v}(t)|\leqslant k} \mu_t(C|p| \mid p\in\operatorname{v}(t)) & \hookrightarrow & \prod_{t\neq\circ}\mu_t(C|p| \mid p\in\operatorname{v}(t)) \\ \downarrow & & \downarrow & & \| \\ C_{k+1} & \xrightarrow{d_{k+1}} & \bigoplus_{t\in\mathbf{tr},\,0<|\operatorname{v}(t)|\leqslant k+1} \mu_t(C|p| \mid p\in\operatorname{v}(t)) & \hookrightarrow & \prod_{t\neq\circ}\mu_t(C|p| \mid p\in\operatorname{v}(t)). \end{array}$$

Passing to the colimit we decompose $(\Delta_t)_{t\neq\circ}$ into

$$C \xrightarrow{\delta} \perp\!\!\!\perp C \hookrightarrow \prod_{t\neq\circ}\mu_t(C|p| \mid p\in\operatorname{v}(t)).$$

The second arrow is a monomorphism due to axiom (FF). As noticed above, $\perp\!\!\!\perp$-coalgebra property of $(C,\delta\colon C\to\perp\!\!\!\perp C)$ is equivalent to equation (22) and normalization condition (ii). □

For any strong $\mathcal{C}at$-operad C there is a functor $\mu_\circ\colon 1\to\mathsf{C}(1)$, whose image is an object $\mathbb{1}$. This is the unit object of the strong monoidal category $\mathsf{C}(1)$. We shall use the same symbol for the cooperad $\mathbb{1}(1)=\mathbb{1}$, $\mathbb{1}(n)=0$ for $n\neq 1$, whose structure maps are canonical isomorphisms.

Proposition 10 (Proposition 3.2.4 of [15])**.** *The category* augCoop *of augmented cooperads* $\eta\colon(\mathbb{1},\Delta)\to(C,\Delta)\in$ Coop *is equivalent to the category* nuCoop.

Example 10. Using (36) and the above proposition we get an augmented cooperad $\perp\!\!\!\perp_\circ X = \mathbb{1}\oplus\perp\!\!\!\perp X$,

$$(\perp\!\!\!\perp_\circ X)(n) = \coprod_{t\in\mathbf{tr}(n)} \mu_t(X|p| \mid p\in\operatorname{v}(t)) \in \mathsf{C}(n),$$

with the comultiplications for $\operatorname{Inp} t\cong\operatorname{Inp}\tau$

$$\Delta_t\big|_{\mu_\tau(X|v| \mid v\in\operatorname{v}(\tau))} = \sum_{f\colon\tau\to t\in\mathbf{Tr}} \big\langle \mu_\tau(X|v| \mid v\in\operatorname{v}(\tau)) = \mu_{I(t;(t_p))}(X|v| \mid v\in\operatorname{v}(I(t;(t_p))))$$
$$\xrightarrow{\alpha^{-1}_{t;(t_p)}} \mu_t\big(\mu_{t_p}(X|q| \mid q\in\operatorname{v}(t_p)) \mid p\in\operatorname{v}(t)\big) \xrightarrow{\mu_t(\operatorname{in}_{t_p})} \mu_t((\perp\!\!\!\perp_\circ X)|p| \mid p\in\operatorname{v}(t))\big\rangle,$$

where $t_p=\overline{f^{-1}(p)}$ if $p\in\operatorname{Im}(f\colon\operatorname{v}(\tau)\to\operatorname{v}(t))$ and $t_p=\circ$ otherwise. In fact, $N\equiv\operatorname{v}(t)-\operatorname{Im} f\subset\operatorname{u}(t)$.

4.2. Monoidal category of collections. Let a strong $\mathcal{C}at$-operad $(\mathsf{C}, \mu_t, \alpha, \iota)$ consist of categories $\mathsf{C}(n)$ with countable coproducts and functors μ_t, which preserve countable coproducts. Denote $\widehat{\mathsf{C}} = \prod_{n\in\mathbb{N}} \mathsf{C}(n)$. The category $\widehat{\mathsf{C}}$ admits a strong monoidal structure $\odot^m\colon \widehat{\mathsf{C}}^m \to \widehat{\mathsf{C}}$

$$\Bigl(\bigodot_{j\in\mathbf{m}} X_j\Bigr)(n) = \coprod_{t\in\mathbf{str}(n,m)} \mu_t\bigl(X_j|(j,k)| \mid j\in\mathbf{m},\, k\in t(j)\bigr), \qquad |(j,k)| = |t_j^{-1}(k)|,$$

where $p = (p_1, p_2) = (j,k) \in \mathrm{v}(t)$ is an internal vertex of t. The structure isomorphisms $\lambda_{\widehat{\mathsf{C}}}$ for $f\colon I \to J \in \mathcal{O}_{\mathsf{sk}}$ are the compositions

$$\Bigl(\bigodot_{j\in J}\bigodot_{i\in f^{-1}j} X_i\Bigr)(n) = \coprod_{t\in\mathbf{str}(n,J)} \mu_t^{p\in\mathrm{v}(t)}\Bigl(\coprod_{t_p\in\mathbf{str}(|p|,f^{-1}p_1)} \mu_{t_p}^{q\in\mathrm{v}(t_p)}(X_{q_1}|q|)\Bigr) \cong$$

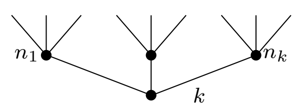
$$\coprod_{t\in\mathbf{str}(n,J)}\ \coprod_{t_p\in\mathbf{str}(|p|,f^{-1}p_1)} \mu_t^{p\in\mathrm{v}(t)}\mu_{t_p}^{q\in\mathrm{v}(t_p)}(X_{q_1}|q|) \xrightarrow[\cong]{\coprod \alpha_{t;(t_p)}} \coprod_{\tau\in\mathbf{str}(n,I)} \mu_\tau^{r\in\mathrm{v}(\tau)}(X_{r_1}|r|) = \Bigl(\bigodot_{i\in I} X_i\Bigr)(n).$$

In fact, $\tau = I(t;(t_p)_{p\in\mathrm{v}(t)})$ runs over all staged trees from $\mathbf{str}(n,I)$. At last,

$$\rho = \bigl\langle X(n) \xrightarrow[\cong]{\iota_{\tau[n]}} \mu_{\tau[n]}(X(n)) = \coprod_{t=\tau[n]} \mu_{\tau[n]}(X(n)) = \bigl(\bigodot_{\mathbf{1}} X\bigr)(n)\bigr\rangle.$$

Operads in C are the same as algebras in $(\widehat{\mathsf{C}}, \odot)$ similarly to [13, Section 5.9.2]. Also the category of coalgebras in $(\widehat{\mathsf{C}}, \odot)$ is contained in the category of cooperads in C.

4.3. Conilpotent augmented cooperads. Let a strong $\mathcal{C}at$-operad $(\mathsf{C}, \mu_t, \alpha, \iota)$ consist of the abelian categories $\mathsf{C}(n)$ with countable limits and colimits and additive functors μ_t, which preserve countable colimits. Denote by $t(n_1, \dots, n_k)$ the two-level tree

$$(\mathbf{n_1} + \cdots + \mathbf{n_k} \xrightarrow{g} \mathbf{k} \xrightarrow{\triangleright} \mathbf{1}) =$$ 

,

where $g^{-1}(j) \cong \mathbf{n_j}$ for all $j \in \mathbf{k}$. Let us consider an augmented cooperad (C, Δ_t, η) in the $\mathcal{C}at$-operad C such that for all $n \geqslant 0$ the morphism

$$(\Delta_{t(n_1,\dots,n_k)})\colon C(n) \to \prod_{n_1+\dots+n_k=n} \mu_{t(n_1,\dots,n_k)}(C(n_1), \dots, C(n_k), C(k))$$

factors through the coproduct

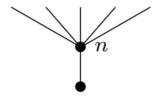
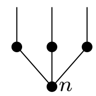
$$\Delta\colon C(n) \to \coprod_{n_1+\dots+n_k=n} \mu_{t(n_1,\dots,n_k)}(C(n_1), \dots, C(n_k), C(k)) = (C \odot C)(n)$$

followed by the extension-by-0 embedding. The augmentation $\eta\colon \mathbb{1} \to C \in \mathrm{Coop}_{\mathsf{C}}$ is reduces to the coalgebra morphism $\eta(1)\colon \mathbb{1} \to C(1) \in \mathsf{C}(1)$, since $\eta(m)$ necessarily vanishes for $m \neq 1$. Similarly to [13, Section 5.8.6] we define another morphism

$$\widehat{\Delta}^1\colon C(n) \to \coprod_{n_1+\dots+n_k=n} \mu_{t(n_1,\dots,n_k)}(C(n_1), \dots, C(n_k), C(k)),$$

$$\widehat{\Delta}^1\big|_{\bar{C}(n)} = \Delta\big|_{\bar{C}(n)} - \bigl\langle \bar{C}(n) \xrightarrow{\cong} \mu_{t(n)}(\bar{C}(n), \mathbb{1}) \xrightarrow{\mu_{t(n)}(\overline{\mathrm{in}}, \eta(1))} \mu_{t(n)}(C(n), C(1))\bigr\rangle \cdot \mathrm{in}_n$$
$$- \bigl\langle \bar{C}(n) \xrightarrow{\cong} \mu_{t({}^n 1)}({}^n\mathbb{1}, \bar{C}(n)) \xrightarrow{\mu_{t({}^n 1)}({}^n\eta(1), \overline{\mathrm{in}})} \mu_{t({}^n 1)}({}^nC(1), C(n))\bigr\rangle \cdot \mathrm{in}_{{}^n 1}\,.$$

The trees occurring here are

$t(n) =$  , $t({}^n1) =$  .

The restriction of $\widehat{\Delta}^1$ to $\mathbb{1} \xhookrightarrow{\eta(1)} C(1)$ vanishes by definition. Define also $\widehat{\Delta}^0 = \mathrm{Id} - \varepsilon \cdot \eta \colon C \to C$ and inductively for $m > 1$

$$\widehat{\Delta}^m = \big\langle C \xrightarrow{\widehat{\Delta}^{m-1}} C^{\odot m} \xrightarrow{\widehat{\Delta}^1 \odot 1_{C^{\odot(m-1)}}} C^{\odot(m+1)} \big\rangle.$$

Following [13, Section 5.8.6] we define the coradical filtration

$$\mathbb{1} = F_0C \subset F_1C \subset \cdots \subset F_mC \subset \cdots \subset C \tag{37}$$

by subobjects $F_mC = \operatorname{Ker}(\widehat{\Delta}^m \colon C \to C^{\odot(m+1)})$. In the case where $C(0) = 0$ and the canonical morphism $\operatorname{colim}_m F_mC \to C$ is invertible, Loday and Vallette call such an augmented cooperad *conilpotent*.

Proposition 11. *If a non-counital cooperad $\bar{C}$ in C with $\bar{C}(0) = 0$ is conilpotent in the sense of Definition 11 then the augmented cooperad $C = \mathbb{1} \oplus \bar{C}$ is conilpotent in the sense of Loday and Vallette.*

Proof. Assume that $\bar{C}$ is conilpotent in the sense of Definition 11 and $\bar{C}(0) = 0$. Having coradical filtration (37) define

$$\overline{F_mC} = \operatorname{Ker}(F_mC \hookrightarrow C \xrightarrow{\varepsilon} \mathbb{1}).$$

Then $F_mC = \mathbb{1} \oplus \overline{F_mC}$ and we have to show that

$$\overline{F_1C} \subset \overline{F_2C} \subset \cdots \subset \overline{F_mC} \subset \cdots \subset \bar{C}$$

is an exhaustive filtration. Let us compare it with the exhaustive filtration $\bar{C}_1 \subset \bar{C}_2 \subset \cdots \subset \bar{C}_m \subset \cdots \subset \bar{C}$ given by

$$\bar{C}_m = \bigcap_{|\mathrm{v}(t)|>m}^{t\in\mathbf{tr}} \operatorname{Ker} \bar{\Delta}_t.$$

We have

$$\Delta\big|_{\bar{C}(n)} = \sum_{n_1+\cdots+n_k=n}^{n_i>0} \Delta_{t(n_1,\dots,n_k)} \colon \bar{C}(n) \to (C \odot C)(n),$$

$$\widehat{\Delta}^1\big|_{\bar{C}(n)} = \sum_{\substack{N\subset\{i\in\mathbf{k}|n_i=1\}\\ N\neq\mathbf{k}}}^{\substack{n_i>0\\ n_1+\cdots+n_k=n}} \bar{\Delta}_{t(n_1,\dots,n_k)^N} \cdot \mu_{t(n_1,\dots,n_k)}((\zeta(i))_{i=1}^k, \mathrm{id}) \colon \bar{C}(n) \to (C \odot \bar{C})(n),$$

where

$$\zeta(i) = \begin{cases} \eta(1), & \text{for } i \in N, \\ \mathrm{id}, & \text{for } i \notin N. \end{cases} \tag{38}$$

The formula for $\widehat{\Delta}^1$ allows to compute

$$\widehat{\Delta}^m\big|_{\bar{C}} = \big\langle \bar{C} \xrightarrow{\widehat{\Delta}^1} C\odot\bar{C} \xrightarrow{\widehat{\Delta}^1\odot 1} C\odot\bar{C}\odot\bar{C} \xrightarrow{\widehat{\Delta}^1\odot 1\odot 1} \dots \xrightarrow{\widehat{\Delta}^1\odot 1\odot\cdots\odot 1} \odot^{\mathbf{1+m}}(C,{}^m\bar{C})\big\rangle,$$

$$\widehat{\Delta}^m\big|_{\bar{C}(n)} = \sum_{\substack{N\subset\{i\in t(1)\,|\,|t_1^{-1}i|=1\}\\ \forall p\in t(2)\ N\not\supset t_2^{-1}p}}^{t\in\mathbf{sstr}(n,m+1)} \bar{\Delta}_{t^N}\cdot\mu_t\big((\zeta(i))_{i\in t(1)},(\mathrm{id})_{\mathrm{v}(t)\backslash t(1)}\big),$$

where $\zeta(i)$ is given by (38). Clearly

$$\overline{F_mC}(n) = \operatorname{Ker}\widehat{\Delta}^m\big|_{\bar{C}(n)} = \bigcap_{\substack{N\subset\{i\in t(1)\,|\,|t_1^{-1}i|=1\}\\ \forall p\in t(2)\ N\not\supset t_2^{-1}p}}^{t\in\mathbf{sstr}(n,m+1)} \operatorname{Ker}\bar{\Delta}_{t^N}\big|_{\bar{C}(n)}.$$

This is bigger than $\bar{C}_m = \bigcap_{|\mathrm{v}(\tau)|\geqslant m+1}^{\tau\in\mathbf{tr}}\operatorname{Ker}\bar{\Delta}_\tau$ since $|\mathrm{v}(t^N)|\geqslant m+1$. □

Remark 7. For the cooperad $C = \perp\!\!\!\perp X$ the subcollection $\overline{F_2C}$ is strictly bigger than $\bar{C}_2$. I guess the two conilpotency conditions are not equivalent.

5. Homotopy cooperads.

Remark 8. If $\mathsf{B} = \mathbf{1}$ is the terminal $\mathcal{C}at$-operad ($\mathbf{1}(n) = \mathit{1}$ is the one-morphism category) then the strict $\mathcal{C}at$-operad $Q_c\mathbf{1}$ is $\mathbf{TR}^{\mathrm{op}}$, where the strict $\mathcal{C}at$-operad $\mathbf{TR}$ is introduced by T. Leinster in [12, Section 7.3]. The bijection on morphisms assigns to $(\xi^{t_p})_{p\in\mathrm{v}(t)}\in Q_c\mathbf{1}$ the morphism $[t;(t_p)]\equiv(t;(t_p))^{\mathrm{op}}\colon t\to I(t;(t_p))\in\mathbf{TR}^{\mathrm{op}}$. As noticed in Corollary 3 a pair of composable morphisms f, g of $\mathbf{Tr}$ identifies with a 3-cluster tree

$$(t;(t_p);(t^q_p)) = \Big[I(t;(t_p);(t^q_p)) \xrightarrow[f]{(I(t;(t_p));(t^q_p)^{q\in\mathrm{v}(t_p)}_{p\in\mathrm{v}(t)})} I(t;(t_p)) \xrightarrow[g]{(t;(t_p))} t\Big],$$

whose composition is $(t;(I(t_p;(t^q_p)_{q\in\mathrm{v}(t_p)}))_{p\in\mathrm{v}(t)})\in\mathbf{Tr}$. On the other hand,

$$(\xi^{t_p})_{p\in\mathrm{v}(t)}\cdot(\xi^{t^q_p})_{(p,q)\in\mathrm{v}(I(t;(t_p)))} = (\xi^{I(t_p;(t^q_p)_{q\in\mathrm{v}(t_p)})})_{p\in\mathrm{v}(t)}$$

since for all $p\in\mathrm{v}(t)$ we have $\xi^{t_p}\cdot(\xi^{t^q_p})_{q\in\mathrm{v}(t_p)} = \xi^{I(t_p;(t^q_p)_{q\in\mathrm{v}(t_p)})}$ due to (26). Thus the compositions in $Q_c\mathbf{1}$ and $\mathbf{TR}^{\mathrm{op}}$ agree. Obviously the operadic multiplications μ_t agree as well.

Let C be a strong $\mathcal{C}at$-operad. According to Proposition 8, a cooperad $C\colon\mathbf{1}\to\mathsf{C}$ is isomorphic as a colax $\mathcal{C}at$-multifunctor to $\mathbf{1}\xrightarrow{e}\mathbf{TR}^{\mathrm{op}}\xrightarrow{D}\mathsf{C}$ for some strong $\mathcal{C}at$-multifunctor D. Moreover, the category of cooperads in C is equivalent to the category of strong $\mathcal{C}at$-multifunctors $\mathbf{TR}^{\mathrm{op}}\to\mathsf{C}$.

Definition 12. A *homotopy cooperad* in C is a lax $\mathcal{C}at$-multifunctor $\mathbf{TR}^{\mathrm{op}}\to\mathsf{C}$. The category of homotopy cooperads in C is the category ${}_p\mathbb{T}\text{-}\mathrm{Alg}_l(\mathbf{TR}^{\mathrm{op}},\mathsf{C})$.

D. Borisov and Yu. I. Manin ([3, Section 4.6]) studied generalized operads, i.e. lax $\mathcal{C}at$-multifunctors $\mathsf{C}\to\mathcal{E}$, concentrating on the case in which C is a collection of groupoids and $\mathcal{E}$ is the endomorphism $\mathcal{C}at$-operad of a category $\mathcal{C}$, see Example 7.

As noticed above, the category of ordinary cooperads in C admits a full and faithful functor to the category of homotopy cooperads in C.

Description 5. A homotopy cooperad (C,χ) in C consists of

— an object $C(t)\in\mathsf{C}(n)$ for each $t\in\mathbf{tr}(n)$;
— morphism $C[t;(t_p)]\colon C(t)\to C(I(t;(t_p)))$ for each 2-cluster tree $(t;(t_p))$;
— morphism $\chi(t;(t_p))\colon \mu_t(C(t_p)\mid p\in \mathrm{v}(t))\to C(I(t;(t_p)))$ for each 2-cluster tree $(t;(t_p))$;

such that

- C is a functor $\mathbf{Tr}^{\mathrm{op}}\to\mathsf{C}$;
- naturality of χ holds: for each 3-cluster tree $(t;(t_p);(t_p^q))$

$$\begin{array}{ccc}
\mu_t(C(t_p)\mid p\in\mathrm{v}(t)) & \xrightarrow{\chi(t;(t_p))} & C(I(t;(t_p))) \\
{\scriptstyle \mu_t(C[t_p;(t_p^q)_q]\mid p\in\mathrm{v}(t))}\big\downarrow & = & \big\downarrow{\scriptstyle C(I(t;[t_p;(t_p^q)]))} \\
\mu_t(C(I(t_p;(t_p^q)_q))\mid p\in\mathrm{v}(t)) & \xrightarrow{\chi(t;(I(t_p;(t_p^q)_q))_p)} & C(I(t;(I(t_p;(t_p^q)_q))_p))
\end{array}$$

- normalization: for each $t\in\mathbf{tr}(n)$

$$\bigl\langle C(t)\xrightarrow[\cong]{\iota}\mu_{\tau[n]}C(t)\xrightarrow{\chi(\tau[n];t)}C(t)\bigr\rangle=\mathrm{id};$$

- multiplicativity holds: for each 3-cluster tree $(t;(t_p);(t_p^q))$

$$\begin{array}{ccc}
\mu_t(\mu_{t_p}(C(t_p^q)\mid q\in\mathrm{v}(t_p))\mid p\in\mathrm{v}(t)) & \xrightarrow{\mu_t(\chi(t_p;(t_p^q)_q)\mid p\in\mathrm{v}(t))} & \mu_t(C(I(t_p;(t_p^q)_q))\mid p\in\mathrm{v}(t)) \\
{\scriptstyle \alpha_{t;(t_p)}}\big\downarrow & = & \big\downarrow{\scriptstyle \chi(t;(I(t_p;(t_p^q)_q))_p)} \\
\mu_{I(t;(t_p))}(C(t_p^q)\mid q\in\mathrm{v}(t_p),\,p\in\mathrm{v}(t)) & \xrightarrow{\chi(I(t;(t_p));(t_p^q)_p^q)} & C(I(t;(t_p);(t_p^q)))
\end{array} \tag{39}$$

Description 6. A *morphism of homotopy cooperads*

$$f\colon (C,\chi)\to(G,\gamma)$$

in C consists of a family of maps $f(t)\colon C(t)\to G(t)\in\mathsf{C}(n)$ for $t\in\mathbf{tr}(n)$, $n\in\mathbb{N}$, such that

- naturality holds: for each 2-cluster tree $(t;(t_p))$

$$\begin{array}{ccc}
C(t) & \xrightarrow{f(t)} & G(t) \\
{\scriptstyle C[t;(t_p)]}\big\downarrow & = & \big\downarrow{\scriptstyle G[t;(t_p)]} \\
C(I(t;(t_p))) & \xrightarrow{f(I(t;(t_p)))} & G(I(t;(t_p)))
\end{array}$$

- multiplicativity holds: for each 2-cluster tree $(t;(t_p))$

$$\begin{array}{ccc}
\mu_t(C(t_p)\mid p\in\mathrm{v}(t)) & \xrightarrow{\mu_t(f(t_p)\mid p\in\mathrm{v}(t))} & \mu_t(G(t_p)\mid p\in\mathrm{v}(t)) \\
{\scriptstyle \chi(t;(t_p))}\big\downarrow & = & \big\downarrow{\scriptstyle \gamma(t;(t_p))} \\
C(I(t;(t_p))) & \xrightarrow{f(I(t;(t_p)))} & G(I(t;(t_p)))
\end{array}$$

As was shown in the proof of Proposition 8, the functor S from (32) in

$$\mathrm{Coop}\equiv{}_p\mathbb{T}\text{-}\mathrm{Alg}_c(1,\mathsf{C})\xrightarrow{S}{}_p\mathbb{T}\text{-}\mathrm{Alg}_p(Q_c1,\mathsf{C})\hookrightarrow{}_p\mathbb{T}\text{-}\mathrm{Alg}_l(Q_c1,\mathsf{C})\equiv\mathrm{hCoop}$$

is a split embedding and an equivalence. The second functor is a full embedding. The functor S takes a cooperad (C, Δ_t) to the following homotopy cooperad (SC, χ):

$$
\begin{gathered}
SC(t) = \mu_t(C|p| \mid p \in \mathrm{v}(t)), \\
SC[t;(t_p)] = \big\langle \mu_t(C|p| \mid p \in \mathrm{v}(t)) \xrightarrow{\mu_t(\Delta_{t_p}|p\in \mathrm{v}(t))} \\
\mu_t(\mu_{t_p}(C|q| \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t)) \xrightarrow{\alpha_{t;(t_p)}} \mu_{I(t;(t_p))}(C|q| \mid q \in \mathrm{v}(t_p),\, p \in \mathrm{v}(t))\big\rangle, \\
\chi(t;(t_p)) = \alpha_{t;(t_p)} \colon \mu_t(\mu_{t_p}(C|q| \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t)) \to \mu_{I(t;(t_p))}(C|q| \mid q \in \mathrm{v}(t_p),\, p \in \mathrm{v}(t)).
\end{gathered}
$$

Remark 9. Let C be a strong $\mathcal{C}at$-operad. A homotopy cooperad (G, γ) is isomorphic to a one of the form (SC, χ) for some cooperad C in C if and only if all γ are invertible. In fact, S is an equivalence of the category of cooperads in C and the category of strong $\mathcal{C}at$-multifunctors $\mathbf{TR}^{\mathrm{op}} \to \mathsf{C}$.

Example 11. Let C be a strong $\mathcal{C}at$-operad. Let $C = \amalg_\circ X$ be the cooperad of Example 10. Then $G = (SC, \chi)$ is specified by

$$
\begin{aligned}
G(t) &= \mu_t\Bigg(\coprod_{\tau_p \in \mathbf{tr}\,|p|} \mu_{\tau_p}(X|r| \mid r \in \mathrm{v}(\tau_p)) \mid p \in \mathrm{v}(t)\Bigg) \\
&\cong \coprod_{(\tau_p) \in \prod_{p\in \mathrm{v}(t)} \mathbf{tr}\,|p|} \mu_t\big(\mu_{\tau_p}(X|r| \mid r \in \mathrm{v}(\tau_p)) \mid p \in \mathrm{v}(t)\big),
\end{aligned}
$$

$$
\begin{gathered}
G[t;(t_p)] = \Big\langle G(t) = \mu_t\Bigg(\coprod_{\tau_p \in \mathbf{tr}\,|p|} \mu_{\tau_p}(X|r| \mid r \in \mathrm{v}(\tau_p)) \mid p \in \mathrm{v}(t)\Bigg) \xrightarrow{\mu_t(A_p^{-1}|p\in \mathrm{v}(t))} \\
\mu_t\Bigg(\coprod_{(t_p^q) \in \prod_{q\in \mathrm{v}(t_p)} \mathbf{tr}\,|q|} \mu_{t_p}\big(\mu_{t_p^q}(X|r| \mid r \in \mathrm{v}(t_p^q)) \mid q \in \mathrm{v}(t_p)\big) \mid p \in \mathrm{v}(t)\Bigg) \cong \\
\mu_t\big(\mu_{t_p}((\amalg_\circ X)|q| \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t)\big) \xrightarrow{\alpha_{t;(t_p)}} \\
\mu_{I(t;(t_p))}\big((\amalg_\circ X)|v| \mid v \in \mathrm{v}(I(t;(t_p)))\big) = G(I(t;(t_p)))\Big\rangle,
\end{gathered}
$$

where $\bar{A}_p$ is a matrix, whose entry is $\alpha^{-1}_{t_p;(t_p^q)_q}$ if $\tau_p = I(t_p;(t_p^q)_q)$ and vanishes otherwise. Furthermore,

$$
\begin{aligned}
\chi(t;(t_p)) = \alpha_{t;(t_p)} \colon \mu_t(\mu_{t_p}((\amalg_\circ X)|q| \mid q \in \mathrm{v}(t_p)) \mid p \in \mathrm{v}(t)) \\
\to \mu_{I(t;(t_p))}((\amalg_\circ X)|q| \mid q \in \mathrm{v}(t_p),\, p \in \mathrm{v}(t)).
\end{aligned}
$$

Example 12. Let C be a strong $\mathcal{C}at$-operad. We shall provide with a non-trivial example of a homotopy cooperad. For a collection $X(n) \in \mathsf{C}(n)$ define

$$
\begin{aligned}
(\tilde{\amalg}_\circ X)(t) = &\coprod_{(\tau_p) \in \prod_{p\in \mathrm{v}(t)} \mathbf{tr}\,|p|} \mu_t\big(\mu_{\tau_p}(X|r| \mid r \in \mathrm{v}(\tau_p)) \mid p \in \mathrm{v}(t)\big) \\
&\xrightarrow[\cong]{\prod \alpha_{t;(\tau_p)}} \coprod_{f\colon \tau \to t \in \mathbf{Tr}} \mu_\tau(X|v| \mid v \in \mathrm{v}(\tau)),
\end{aligned}
$$

$$(\tilde{\perp\!\!\!\perp}_\circ X)[t;(t_p)] = \Big\langle (\tilde{\perp\!\!\!\perp}_\circ X)(t) = \prod_{(\tau_p)\in\prod_{p\in \mathrm{v}(t)} \mathbf{tr}\,|p|} \mu_t\big(\mu_{\tau_p}(X|r| \mid r\in \mathrm{v}(\tau_p)) \mid p\in \mathrm{v}(t)\big)$$
$$\xrightarrow{\prod \mu_t(\bar{A}_p|p\in \mathrm{v}(t))} \prod_{(t^q_p)\in\prod_{p\in \mathrm{v}(t)}\prod_{q\in \mathrm{v}(t_p)} \mathbf{tr}\,|q|} \mu_t\big(\mu_{t_p}(\mu_{t^q_p}(X|r| \mid r\in \mathrm{v}(t^q_p)) \mid q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t)\big) \xrightarrow{\prod \alpha_{t;(t_p)}}$$
$$\prod_{(t^q_p)\in\prod_{p\in \mathrm{v}(t)}\prod_{q\in \mathrm{v}(t_p)} \mathbf{tr}\,|q|} \mu_{I(t;(t_p))}\big(\mu_{t^q_p}(X|r| \mid r\in \mathrm{v}(t^q_p)) \mid q\in \mathrm{v}(t_p), p\in \mathrm{v}(t)\big) = (\tilde{\perp\!\!\!\perp}_\circ X)(I(t;(t_p)))\Big\rangle.$$

Here the first map postcomposed with $\mathrm{pr}_{(t^q_p)^q_p}$ equals $\mathrm{pr}_{(I(t_p;(t^q_p)_q))_p} \cdot \mu_t(\alpha^{-1}_{t_p;(t^q_p)_q} \mid p\in \mathrm{v}(t))$

$$(\tilde{\perp\!\!\!\perp}_\circ X)[t;(t_p)]\cdot \mathrm{pr}_{(t^q_p)^q_p} = \Big\langle \prod_{(\tau_p)\in\prod_{p\in \mathrm{v}(t)} \mathbf{tr}\,|p|} \mu_t\big(\mu_{\tau_p}(X|r| \mid r\in \mathrm{v}(\tau_p)) \mid p\in \mathrm{v}(t)\big)$$
$$\xrightarrow{\mathrm{pr}_{(I(t_p;(t^q_p)_q))_p}} \mu_t\big(\mu_{I(t_p;(t^q_p)_q)}(X|r| \mid r\in \mathrm{v}(I(t_p;(t^q_p)_q))) \mid p\in \mathrm{v}(t)\big)$$
$$\xrightarrow{\mu_t(\alpha^{-1}_{t_p;(t^q_p)_q}|p\in \mathrm{v}(t))} \mu_t\big(\mu_{t_p}(\mu_{t^q_p}(X|v| \mid v\in \mathrm{v}(t^q_p)) \mid q\in \mathrm{v}(t_p)) \mid p\in \mathrm{v}(t)\big)$$
$$\xrightarrow{\alpha_{t;(t_p)}} \mu_{I(t;(t_p))}\big(\mu_{t^q_p}(X|v| \mid v\in \mathrm{v}(t^q_p)) \mid q\in \mathrm{v}(t_p), p\in \mathrm{v}(t)\big)\Big\rangle.$$

Furthermore, for any 2-cluster tree $(t;(t_p))$ the morphism

$$\chi(t;(t_p))\colon \mu_t((\tilde{\perp\!\!\!\perp}_\circ X)(t_p) \mid p\in \mathrm{v}(t)) \to (\tilde{\perp\!\!\!\perp}_\circ X)(I(t;(t_p)))$$

is determined by its composition with the following projection

$$\chi(t;(t_p))\cdot \mathrm{pr}_{(t^q_p)^q_p} = \Big\langle \mu_t\Big(\prod_{(t^q_p)\in\prod_{q\in \mathrm{v}(t_p)} \mathbf{tr}\,|q|} \mu_{t_p}\big(\mu_{t^q_p}(X|r| \mid r\in \mathrm{v}(t^q_p)) \mid q\in \mathrm{v}(t_p)\big) \mid p\in \mathrm{v}(t)\Big)$$
$$\xrightarrow{\mu_t(\mathrm{pr}_{(t^q_p)_q}|p\in \mathrm{v}(t))} \mu_t\big(\mu_{t_p}\big(\mu_{t^q_p}(X|r| \mid r\in \mathrm{v}(t^q_p)) \mid q\in \mathrm{v}(t_p)\big) \mid p\in \mathrm{v}(t)\big)$$
$$\xrightarrow{\alpha_{t;(t_p)}} \mu_{I(t;(t_p))}\big(\mu_{t^q_p}(X|r| \mid r\in \mathrm{v}(t^q_p)) \mid q\in \mathrm{v}(t_p), p\in \mathrm{v}(t)\big)\Big\rangle.$$

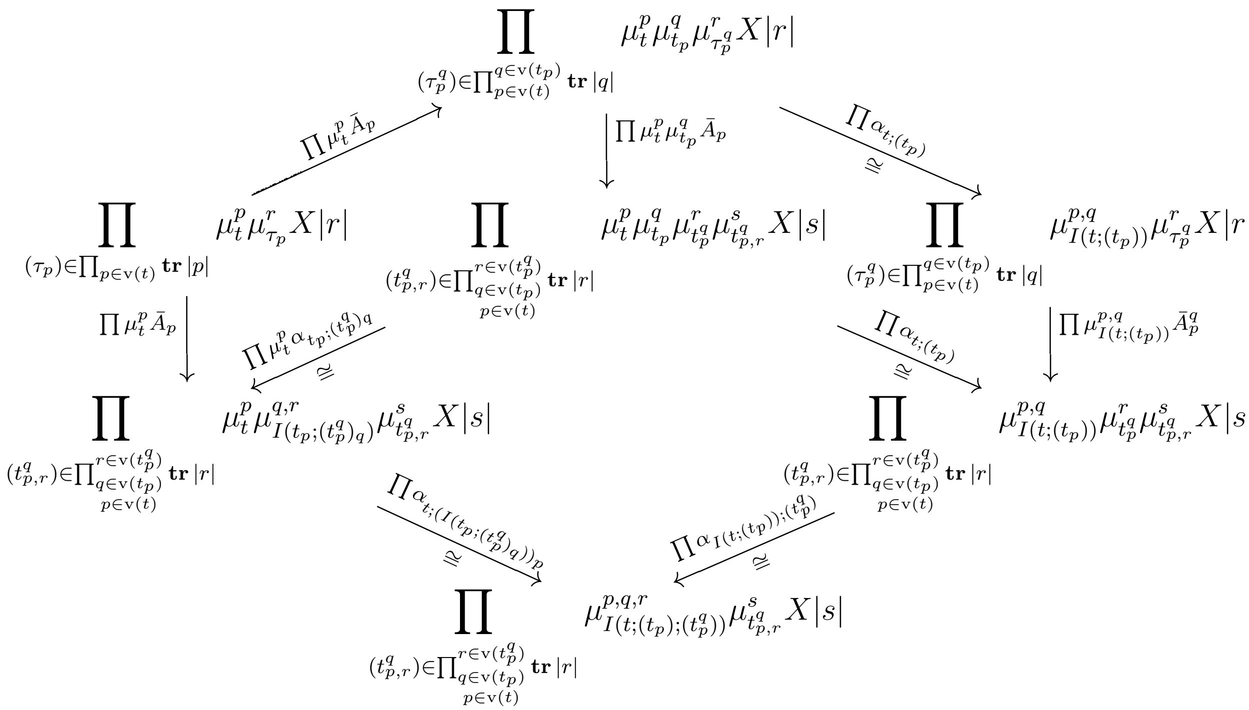
Let us introduce the endofunctor $\hat{\perp\!\!\!\perp}_\circ\colon \prod_{n\geqslant 0}\mathsf{C}(n) \to \prod_{n\geqslant 0}\mathsf{C}(n)$:

$$(\hat{\perp\!\!\!\perp}_\circ X)(n) = (\tilde{\perp\!\!\!\perp}_\circ X)(\tau[n]) = \prod_{t\in \mathbf{tr}(n)} \mu_t\big(X|p| \mid p\in \mathrm{v}(t)\big).$$

Let us verify the conditions of Description 5. In order to be a functor $\tilde{\perp\!\!\!\perp}_\circ X\colon \mathbf{Tr}^{\mathrm{op}} \to \mathsf{C}$ has to satisfy

$$(\tilde{\perp\!\!\!\perp}_\circ X)[fg] = \big((\tilde{\perp\!\!\!\perp}_\circ X)(t) \xrightarrow{(\tilde{\perp\!\!\!\perp}_\circ X)[g]} (\tilde{\perp\!\!\!\perp}_\circ X)(I(t;(t_p))) \xrightarrow{(\tilde{\perp\!\!\!\perp}_\circ X)[f]} (\tilde{\perp\!\!\!\perp}_\circ X)(I(t;(t_p);(t^q_p)))\big)$$
$$\text{for } fg = \big(I(t;(t_p);(t^q_p)) \xrightarrow{f} I(t;(t_p)) \xrightarrow{g} t\big),$$

which is expanded to the exterior of the commutative diagram



Here $\mu^p_t{}_{-} = \mu_t^{p\in \mathrm{v}(t)}{}_{-} = \mu_t({}_{-} \mid p \in \mathrm{v}(t))$. Squares commute due to (19).

Let us prove the naturality of χ: for each 3-cluster tree $(t;(t_p);(t_p^q))$ the square below commutes

$$\begin{array}{ccc}
\mu^p_t \prod\limits_{(\tau^q_p)\in\prod_{q\in\mathrm{v}(t_p)}\mathbf{tr}\,|q|} \mu^q_{t_p}\mu^r_{\tau^q_p}X|r| & \xrightarrow{\chi(t;(t_p))} & \prod\limits_{(\tau^q_p)\in\prod_{p\in\mathrm{v}(t)}^{q\in\mathrm{v}(t_p)}\mathbf{tr}\,|q|} \mu^{p,q}_{I(t;(t_p))}\mu^r_{\tau^q_p}X|r| \\
\Big\downarrow{\scriptstyle \mu^p_t(\tilde{\perp\!\!\!\perp}_\circ X)[t_p;(t^q_p)_q]} & & \Big\downarrow{\scriptstyle (\tilde{\perp\!\!\!\perp}_\circ X)(t\colon [t_p;(t^q_p)_q]_p)} \\
\mu^p_t \prod\limits_{(t^q_{p,r})\in\prod_{q\in\mathrm{v}(t_p)}^{r\in\mathrm{v}(t^q_p)}\mathbf{tr}\,|r|} \mu^{q,r}_{I(t_p;(t^q_p)_q)}\mu^s_{t^q_{p,r}}X|s| & \xrightarrow{\chi(t;(I(t_p;(t^q_p)_q))_p)} & \prod\limits_{(t^q_{p,r})\in\prod_{q\in\mathrm{v}(t_p),\ p\in\mathrm{v}(t)}^{r\in\mathrm{v}(t^q_p)}\mathbf{tr}\,|r|} \mu^{p,q,r}_{I(t;(t_p);(t^q_p))}\mu^s_{t^q_{p,r}}X|s|
\end{array}$$

Whiskering the equation with the projection $\mathrm{pr}_{(t^q_p)}$ for arbitrary (t^q_p) we rewrite it as

$$\Big\langle \mu^p_t \prod_{(\tau^q_p)\in\prod_{q\in\mathrm{v}(t_p)}\mathbf{tr}\,|q|} \mu^q_{t_p}\mu^s_{\tau^q_p}X|s| \xrightarrow{\mu^p_t\,\mathrm{pr}_{(I(t^q_p;(t^q_{p,r})_r))_q}} \mu^p_t\mu^q_{t_p}\mu^s_{I(t^q_p;(t^q_{p,r})_r)}X|s| \xrightarrow{\alpha_{t;(t_p)}}$$
$$\mu^{p,q}_{I(t;(t_p))}\mu^s_{I(t^q_p;(t^q_{p,r})_r)}X|s| \xrightarrow{\mu^{p,q}_{I(t;(t_p))}\alpha^{-1}_{t^q_p;(t^q_{p,r})_r}} \mu^{p,q}_{I(t;(t_p))}\mu^r_{\tau^q_p}\mu^s_{t^q_{p,r}}X|s| \xrightarrow{\alpha_{I(t;(t_p));(t^q_p)^q_p}} \mu^{p,q,r}_{I(t;(t_p);(t^q_p))}\mu^s_{t^q_{p,r}}X|s|\Big\rangle$$
$$= \Big\langle \mu^p_t \prod_{(\tau^q_p)\in\prod_{q\in\mathrm{v}(t_p)}\mathbf{tr}\,|q|} \mu^q_{t_p}\mu^s_{\tau^q_p}X|s| \xrightarrow{\mu^p_t\,\mathrm{pr}_{(I(t^q_p;(t^q_{p,r})_r))_q}} \mu^p_t\mu^q_{t_p}\mu^s_{I(t^q_p;(t^q_{p,r})_r)}X|s| \xrightarrow{\mu^p_t\mu^q_{t_p}\alpha^{-1}_{t^q_p;(t^q_{p,r})_r}}$$
$$\mu^p_t\mu^q_{t_p}\mu^r_{t^q_p}\mu^s_{t^q_{p,r}}X|s| \xrightarrow{\mu^p_t\alpha_{t_p;(t^q_p)_q}} \mu^p_t\mu^{q,r}_{I(t_p;(t^q_p)_q)}\mu^s_{t^q_{p,r}}X|s| \xrightarrow{\alpha_{t;(I(t_p;(t^q_p)_q))_p}} \mu^{p,q,r}_{I(t;(t_p);(t^q_p))}\mu^s_{t^q_{p,r}}X|s|\Big\rangle.$$

Commuting the second and the third arrows in the left hand side, we reduce the equation to (19).

The normalisation property holds due to (18).

The multiplicativity holds: for each 3-cluster tree $(t;(t_p);(t^q_p))$

$$\begin{array}{ccc}
\mu^p_t\mu^q_{t_p}\prod\limits_{(t^q_{p,r})\in\prod_{r\in\mathrm{v}(t^q_p)}\mathbf{tr}|r|}\mu^r_{t^q_p}\mu^s_{t^q_{p,r}}X|s| & \xrightarrow{\mu^p_t\chi(t_p;(t^q_p)_q)} & \mu^p_t\prod\limits_{(t^q_{p,r})\in\prod^{r\in\mathrm{v}(t^q_p)}_{q\in\mathrm{v}(t_p)}\mathbf{tr}|r|}\mu^{q,r}_{I(t_p;(t^q_p)_q)}\mu^s_{t^q_{p,r}}X|s| \\
\Big\downarrow{\scriptstyle \alpha_{t;(t_p)}} & & \Big\downarrow{\scriptstyle \chi(t;(I(t_p;(t^q_p)_q))_p)} \\
\mu^{p,q}_{I(t;(t_p))}\prod\limits_{(t^q_{p,r})\in\prod_{r\in\mathrm{v}(t^q_p)}\mathbf{tr}|r|}\mu^r_{t^q_p}\mu^s_{t^q_{p,r}}X|s| & \xrightarrow{\chi(I(t;(t_p));(t^q_p)^q_p)} & \prod\limits_{(t^q_{p,r})\in\prod^{r\in\mathrm{v}(t^q_p)}_{q\in\mathrm{v}(t_p),\,p\in\mathrm{v}(t)}\mathbf{tr}|r|}\mu^{p,q,r}_{I(t;(t_p);(t^q_p))}\mu^s_{t^q_{p,r}}X|s|
\end{array}$$

Whiskering this equation with the projection $\mathrm{pr}_{(t^q_{p,r})}$ for arbitrary $(t^q_{p,r})$ we get an equivalent

$$\begin{aligned}
&\Big\langle\mu^p_t\mu^q_{t_p}\prod_{(t^q_{p,r})\in\prod_{r\in\mathrm{v}(t^q_p)}\mathbf{tr}|r|}\mu^r_{t^q_p}\mu^s_{t^q_{p,r}}X|s| \xrightarrow{\alpha_{t;(t_p)}} \mu^{p,q}_{I(t;(t_p))}\prod_{(t^q_{p,r})\in\prod_{r\in\mathrm{v}(t^q_p)}\mathbf{tr}|r|}\mu^r_{t^q_p}\mu^s_{t^q_{p,r}}X|s| \\
&\qquad\xrightarrow{\mu^{p,q}_{I(t;(t_p))}\mathrm{pr}_{(t^q_{p,r})}} \mu^{p,q}_{I(t;(t_p))}\mu^r_{t^q_p}\mu^s_{t^q_{p,r}}X|s| \xrightarrow{\alpha_{I(t;(t_p));(t^q_p)^q_p}} \mu^{p,q,r}_{I(t;(t_p);(t^q_p))}\mu^s_{t^q_{p,r}}X|s|\Big\rangle \\
&= \Big\langle\mu^p_t\mu^q_{t_p}\prod_{(t^q_{p,r})\in\prod_{r\in\mathrm{v}(t^q_p)}\mathbf{tr}|r|}\mu^r_{t^q_p}\mu^s_{t^q_{p,r}}X|s| \xrightarrow{\mu^p_t\mu^q_{t_p}\mathrm{pr}_{(t^q_{p,r})}} \mu^p_t\mu^q_{t_p}\mu^r_{t^q_p}\mu^s_{t^q_{p,r}}X|s| \xrightarrow{\mu^p_t\alpha_{t_p;(t^q_p)_q}} \\
&\qquad\mu^p_t\mu^{q,r}_{I(t_p;(t^q_p)_q)}\mu^s_{t^q_{p,r}}X|s| \xrightarrow{\alpha_{t;(I(t_p;(t^q_p)_q))_p}} \mu^{p,q,r}_{I(t;(t_p);(t^q_p))}\mu^s_{t^q_{p,r}}X|s|\Big\rangle.
\end{aligned}$$

Commuting the first and the second arrows at the left hand side, we reduce the equation to (19). Thus $\tilde{\amalg}_\circ X$ is indeed a homotopy cooperad.

Remark 10. Passing to the opposite $\mathcal{C}at$-operad one defines the notion of homotopy operad and constructs an example of such without the assumption that μ_t preserves colimits.

Proposition 12. *Let C be a strong $\mathcal{C}at$-operad, C be a cooperad in C, and $X=(X(n))_{n\geqslant0}$, $X(n)\in\mathrm{Ob}\,\mathsf{C}(n)$. Then the map*

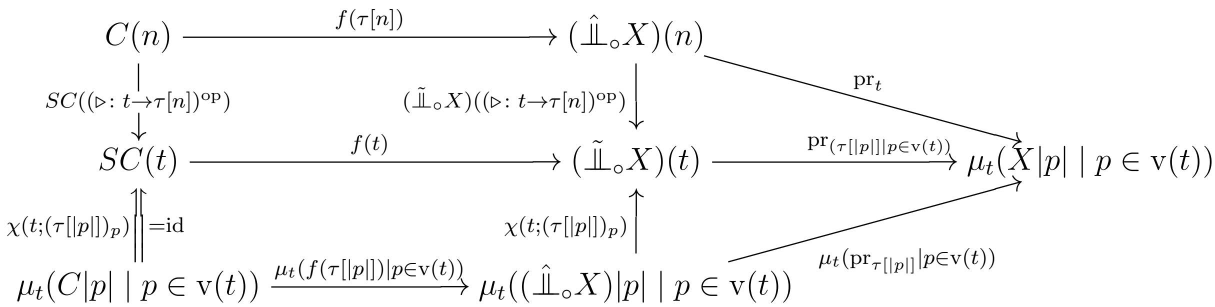
$$\mathrm{hCoop}_{\mathsf{C}}(SC,\tilde{\amalg}_\circ X)\to\prod_{n\in\mathbb{N}}\mathsf{C}(n)(C(n),X(n)),\quad f\mapsto\check{f},\quad \check{f}(n)=$$
$$=\big(C(n)\xrightarrow[\cong]{\iota}\mu_{\tau[n]}C(n)\xrightarrow{f(\tau[n])}(\tilde{\amalg}_\circ X)(\tau[n])\xrightarrow{\mathrm{pr}_{\tau[n]}}\mu_{\tau[n]}\mu_{\tau[n]}X(n)\xrightarrow[\cong]{\iota^{-1}}\mu_{\tau[n]}X(n)\xrightarrow[\cong]{\iota^{-1}}X(n)\big)$$

is bijective.

Proof. Without loss of generality we may assume that $\mu_{\tau[n]}=\mathrm{Id}_{\mathsf{C}(n)}$ and $\iota=\mathrm{id}\colon\mathrm{Id}\to\mu_{\tau[n]}$ for $n\geqslant0$. First we prove that a morphism $f\colon SC\to\tilde{\amalg}_\circ X\in\mathrm{hCoop}_{\mathsf{C}}$ is determined by

$$\check{f}=\big(C(n)\xrightarrow{f(\tau[n])}(\hat{\amalg}_\circ X)(n)\xrightarrow{\mathrm{pr}_{\tau[n]}}X(n)\big)_{n\geqslant0}.$$

For an arbitrary $t \in \mathbf{tr}(n)$ consider the commutative diagram



It follows that

$$\big\langle C(n) \xrightarrow{f(\tau[n])} (\hat{\mathbb{1}}_\circ X)(n) \xrightarrow{\mathrm{pr}_t} \mu_t(X|p| \mid p \in \mathrm{v}(t))\big\rangle$$
$$= \big\langle C(n) \xrightarrow{SC((\triangleright\colon t \to \tau[n])^{\mathrm{op}})} SC(t) = \mu_t(C|p| \mid p \in \mathrm{v}(t)) \xrightarrow{\mu_t(\check{f}|p||p\in \mathrm{v}(t))} \mu_t(X|p| \mid p \in \mathrm{v}(t))\big\rangle.$$

Denote by $\hat{\Delta}(n)\colon C(n) \to (\hat{\mathbb{1}}_\circ C)(n)$, $n \geqslant 0$, the unique collection of maps such that $\hat{\Delta}(n) \cdot \mathrm{pr}_t = \Delta_t \equiv SC((\triangleright\colon t \to \tau[n])^{\mathrm{op}})\colon C(n) \to SC(t)$ for all $t \in \mathbf{tr}(n)$. Thus,

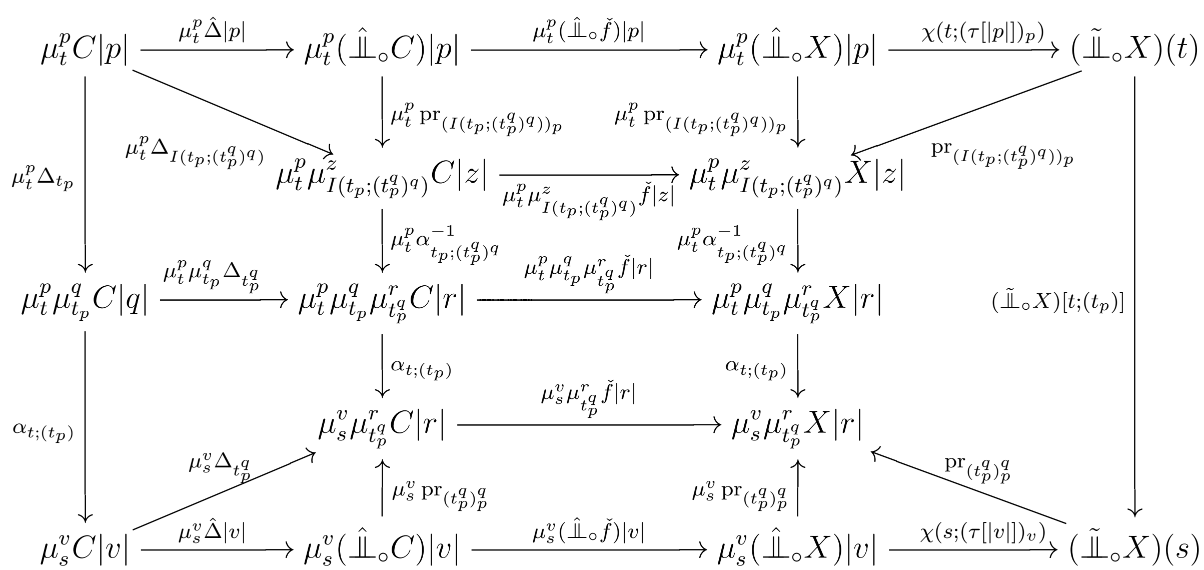
$$f(\tau[n]) = \big\langle C(n) \xrightarrow{\hat{\Delta}(n)} (\hat{\mathbb{1}}_\circ C)(n) \xrightarrow{(\hat{\mathbb{1}}_\circ \check{f})(n)} (\hat{\mathbb{1}}_\circ X)(n)\big\rangle.$$

At last, the left bottom square of the diagram gives

$$f(t) = \big\langle SC(t) = \mu_t(C|p| \mid p \in \mathrm{v}(t)) \xrightarrow{\mu_t(f(\tau[|p|])|p\in \mathrm{v}(t))} \mu_t((\hat{\mathbb{1}}_\circ X)|p| \mid p \in \mathrm{v}(t))$$
$$\xrightarrow{\chi(t;(\tau[|p|])_p)} (\tilde{\mathbb{1}}_\circ X)(t)\big\rangle$$
$$= \big\langle SC(t) = \mu_t^p C|p| \xrightarrow{\mu_t^p \hat{\Delta}|p|} \mu_t^p(\hat{\mathbb{1}}_\circ C)|p| \xrightarrow{\mu_t^p(\hat{\mathbb{1}}_\circ \check{f})|p|} \mu_t^p(\hat{\mathbb{1}}_\circ X)|p| \xrightarrow{\chi(t;(\tau[|p|])_p)} (\tilde{\mathbb{1}}_\circ X)(t)\big\rangle. \quad (40)$$

This proves the injectivity of the map $f \mapsto \check{f}$. The bijectivity will be proven once we show that for an arbitrary $\check{f}\colon C(n) \to X(n) \in \mathsf{C}(n)$, $n \geqslant 0$, formulae (40) define a morphism of homotopy cooperads $f\colon SC \to \tilde{\mathbb{1}}_\circ X$.

The naturality of this f means the commutativity of the exterior of the following diagram for an arbitrary map $(t;(t_p))\colon s = I(t;(t_p)) \to t \in \mathbf{Tr}$



Notice that $v \in \mathrm{v}(s)$ is identified with some pair (p,q), $p \in \mathrm{v}(t)$, $q \in \mathrm{v}(t_p)$. In order to prove the commutativity of the exterior, whisker it with $\mathrm{pr}_{(t_p^q)_p^q}$ (directed inward as in the diagram). Then this diagram is partitioned into commutative cells and the naturality is proven.

The multiplicativity of f is expressed by the exterior of the following diagram commutative for each 2-cluster tree $(t;(t_p))$ with $s = I(t;(t_p))$

$$\begin{array}{ccccccc}
\mu_t^p\mu_{t_p}^q C|q| & \xrightarrow{\mu_t^p\mu_{t_p}^q\hat{\Delta}|q|} & \mu_t^p\mu_{t_p}^q(\hat{\perp\!\!\!\perp}_\circ C)|q| & \xrightarrow{\mu_t^p\mu_{t_p}^q(\hat{\perp\!\!\!\perp}_\circ\check{f})|q|} & \mu_t^p\mu_{t_p}^q(\hat{\perp\!\!\!\perp}_\circ X)|q| & \xrightarrow{\mu_t^p\chi(t_p;(\tau[|q|])_q)} & \mu_t^p(\tilde{\perp\!\!\!\perp}_\circ X)(t_p) \\
{\scriptstyle\alpha_{t;(t_p)}}\downarrow & & {\scriptstyle\alpha_{t;(t_p)}}\downarrow & & {\scriptstyle\alpha_{t;(t_p)}}\downarrow & & \downarrow{\scriptstyle\chi(t;(t_p))} \\
\mu_s^v C|v| & \xrightarrow{\mu_s^v\hat{\Delta}|v|} & \mu_s^v(\hat{\perp\!\!\!\perp}_\circ C)|p| & \xrightarrow{\mu_s^v(\hat{\perp\!\!\!\perp}_\circ\check{f})|v|} & \mu_s^v(\hat{\perp\!\!\!\perp}_\circ X)|v| & \xrightarrow{\chi(s;(\tau[|v|])_v)} & (\tilde{\perp\!\!\!\perp}_\circ X)(s)
\end{array}$$

The rightmost square is a particular case of multiplicativity (39) for the 3-cluster tree $(t;(t_p);(\tau[|q|])_{p\in\mathrm{v}(t)}^{q\in\mathrm{v}(t_p)})$. Therefore, f is a morphism of homotopy cooperads. $\square$

Introduce the filtration $\mathbf{tr}_K(n) = \{t \in \mathbf{tr}(n) \mid |\mathrm{v}(t)| \leqslant K\}$ of the sets $\mathbf{tr}(n)$. We say that a morphism $f\colon \hat{\perp\!\!\!\perp}_\circ X \to Y$ has a *finite support* if for all $n \in \mathbb{N}$ there is $K \in \mathbb{N}$ such that

$$f(n) = \bigl\langle (\hat{\perp\!\!\!\perp}_\circ X)(n) = \prod_{t\in\mathbf{tr}(n)} \mu_t^p X|p| \longrightarrow\!\!\triangleright \prod_{t\in\mathbf{tr}_K(n)} \mu_t^p X|p| \xrightarrow{\cong} \coprod_{t\in\mathbf{tr}_K(n)} \mu_t^p X|p| \xrightarrow{(f_t)} Y(n)\bigr\rangle.$$

Proposition 13. *Let a morphism $f\colon \hat{\perp\!\!\!\perp}_\circ X \to Y$ have finite support. For each $t \in \mathbf{tr}$ define a morphism $\hat{f}(t)\colon (\tilde{\perp\!\!\!\perp}_\circ X)(t) \to (\tilde{\perp\!\!\!\perp}_\circ Y)(t) \in \mathsf{C}(\operatorname{Inp} t)$ by the equations for a 2-cluster tree $(t;(t_p))$*

$$\hat{f}(t)\cdot\mathrm{pr}_{(t_p|p\in\mathrm{v}(t))} = \sum_{(t_p^q)\in\prod_{p\in\mathrm{v}(t)}\prod_{q\in\mathrm{v}(t_p)}\mathbf{tr}_K|q|} \Bigl\langle \prod_{(\tau_p)\in\prod_{p\in\mathrm{v}(t)}\mathbf{tr}|p|} \mu_t^p\mu_{\tau_p}^r X|r| \xrightarrow{\mathrm{pr}_{(I(t_p;(t_p^q)_q))_p}}$$

$$\mu_t^p\mu_{I(t_p;(t_p^q)_q)}^r X|r| \xrightarrow{\mu_t^p\alpha^{-1}_{t_p;(t_p^q)_q}} \mu_t^p\mu_{t_p}^q\mu_{t_p^q}^v X|v| \xrightarrow{\mu_t^p\mu_{t_p}^q f_{t_p^q}} \mu_t^p\mu_{t_p}^q Y|q|\Bigr\rangle.$$

Then the family $\hat{f}\colon \tilde{\perp\!\!\!\perp}_\circ X \to \tilde{\perp\!\!\!\perp}_\circ Y$ is a morphism of homotopy cooperads.

Proof. We have to prove the naturality of $\hat{f}$, expressed by the following square for each 2-cluster tree $(t;(t_p))$

$$\begin{array}{ccccc}
(\tilde{\perp\!\!\!\perp}_\circ X)(t) & \xrightarrow{\qquad f(t)\qquad} & (\tilde{\perp\!\!\!\perp}_\circ Y)(t) & & \\
{\scriptstyle(\tilde{\perp\!\!\!\perp}_\circ X)[t;(t_p)]}\downarrow & = & \downarrow{\scriptstyle(\tilde{\perp\!\!\!\perp}_\circ Y)[t;(t_p)]} & & \\
(\tilde{\perp\!\!\!\perp}_\circ X)(I(t;(t_p))) & \xrightarrow{f(I(t;(t_p)))} & (\tilde{\perp\!\!\!\perp}_\circ Y)(I(t;(t_p))) & \xrightarrow{\mathrm{pr}_{(t_p^q)_p^q}} & \mu_{I(t;(t_p))}^{p,q}\mu_{t_p^q}^v X|v|.
\end{array}$$

The commutativity of the square is equivalent to the equality of two compositions in this diagram for all $(t_p^q) \in \prod_{p\in\mathrm{v}(t)}\prod_{q\in\mathrm{v}(t_p)}\mathbf{tr}\,|q|$. Both compositions here are sums over $(t_p^{q,r}) \in \prod_{p\in\mathrm{v}(t)}\prod_{q\in\mathrm{v}(t_p)}\prod_{r\in\mathrm{v}(t_p^q)}\mathbf{tr}_K\,|r|$ of expressions that begin with

$$\mathrm{pr}_{(I(t_p;(t_p^q)_q;(t_p^{q,r})_q^r))_p}\colon \prod_{(\tau_p)\in\prod_{p\in\mathrm{v}(t)}\mathbf{tr}\,|p|} \mu_t^p\mu_{\tau_p}^r X|r| \to \mu_t^p\mu_{I(t_p;(t_p^q)_q;(t_p^{q,r})_q^r)}^z X|z|.$$

The remaining two compositions are presented as the exterior of the following diagram

$$
\begin{array}{ccccc}
\mu^p_t\mu^z_{I(t_p;(t^q_p)_q;(t^{q,r}_p)^r_q)}X|z| & \xrightarrow{\mu^p_t\alpha^{-1}_{I(t_p;(t^q_p)_q);(t^{q,r}_p)^r_q}} & \mu^p_t\mu^{q,r}_{I(t_p;(t^q_p)_q)}\mu^v_{t^{q,r}_p}X|v| & \xrightarrow{\mu^p_t\mu^{q,r}_{I(t_p;(t^q_p)_q)}f_{t^{q,r}_p}} & \mu^p_t\mu^z_{I(t_p;(t^q_p)_q)}Y|z| \\
{\scriptstyle \mu^p_t\alpha^{-1}_{t_p;(I(t^q_p;(t^{q,r}_p)_r))_q}}\downarrow & & \downarrow{\scriptstyle \mu^p_t\alpha^{-1}_{t_p;(t^q_p)_q}} & & \downarrow{\scriptstyle \mu^p_t\alpha^{-1}_{t_p;(t^q_p)_q}} \\
\mu^p_t\mu^q_{t_p}\mu^v_{I(t^q_p;(t^{q,r}_p)_r)}X|v| & \xrightarrow{\mu^p_t\mu^q_{t_p}\alpha^{-1}_{t^q_p;(t^{q,r}_p)_r}} & \mu^p_t\mu^q_{t_p}\mu^r_{t^q_p}\mu^v_{t^{q,r}_p}X|v| & \xrightarrow{\mu^p_t\mu^q_{t_p}\mu^r_{t^q_p}f_{t^{q,r}_p}} & \mu^p_t\mu^q_{t_p}\mu^r_{t^q_p}Y|r| \\
{\scriptstyle \alpha_{t;(t_p)}}\downarrow & & \downarrow{\scriptstyle \alpha_{t;(t_p)}} & & \downarrow{\scriptstyle \alpha_{t;(t_p)}} \\
\mu^{p,q}_{I(t;(t_p))}\mu^v_{I(t^q_p;(t^{q,r}_p)_r)}X|v| & \xrightarrow{\mu^{p,q}_{I(t;(t_p))}\alpha^{-1}_{t^q_p;(t^{q,r}_p)_r}} & \mu^{p,q}_{I(t;(t_p))}\mu^r_{t^q_p}\mu^v_{t^{q,r}_p}X|v| & \xrightarrow{\mu^{p,q}_{I(t;(t_p))}\mu^r_{t^q_p}f_{t^{q,r}_p}} & \mu^{p,q}_{I(t;(t_p))}\mu^r_{t^q_p}Y|r|
\end{array}
$$

Let us verify the multiplicativity of $\hat{f}$, which is the commutativity of the following square for each 2-cluster tree $(t;(t_p))$

$$
\begin{array}{ccccc}
\mu^p_t(\tilde{\perp\!\!\!\perp}_\circ X)(t_p) & \xrightarrow{\mu^p_t f(t_p)} & \mu^p_t(\tilde{\perp\!\!\!\perp}_\circ Y)(t_p) & & \\
{\scriptstyle \chi(t;(t_p))}\downarrow & = & \downarrow{\scriptstyle \chi(t;(t_p))} & & \\
(\tilde{\perp\!\!\!\perp}_\circ X)(I(t;(t_p))) & \xrightarrow{f(I(t;(t_p)))} & (\tilde{\perp\!\!\!\perp}_\circ Y)(I(t;(t_p))) & \xrightarrow{\mathrm{pr}_{(t^q_p)^q_p}} & \mu^{p,q}_{I(t;(t_p))}\mu^v_{t^q_p}X|v|
\end{array}
$$

Equivalently, two compositions in this diagram are equal for all $(t^q_p) \in \prod_{p\in\mathrm{v}(t)}\prod_{q\in\mathrm{v}(t_p)} \mathbf{tr}\,|q|$. Both compositions here are sums over $(t^{q,r}_p) \in \prod_{p\in\mathrm{v}(t)}\prod_{q\in\mathrm{v}(t_p)}\prod_{r\in\mathrm{v}(t^q_p)} \mathbf{tr}_K\,|r|$ of expressions that begin with

$$
\mu^p_t\,\mathrm{pr}_{(I(t^q_p;(t^{q,r}_p)_r))_q} : \mu^p_t \prod_{(\tau^q_p)\in\prod_{q\in\mathrm{v}(t_p)}\mathbf{tr}\,|q|} \mu^q_{t_p}\mu^v_{t^q_p}X|v| \to \mu^p_t\mu^q_{t_p}\mu^v_{I(t^q_p;(t^{q,r}_p)_r)}X|v|.
$$

The remaining two compositions are presented as the exterior of the following diagram

$$
\begin{array}{ccccc}
\mu^p_t\mu^q_{t_p}\mu^v_{I(t^q_p;(t^{q,r}_p)_r)}X|v| & \xrightarrow{\mu^p_t\mu^q_{t_p}\alpha^{-1}_{t^q_p;(t^{q,r}_p)_r}} & \mu^p_t\mu^q_{t_p}\mu^r_{t^q_p}\mu^v_{t^{q,r}_p}X|v| & \xrightarrow{\mu^p_t\mu^q_{t_p}\mu^r_{t^q_p}f_{t^{q,r}_p}} & \mu^p_t\mu^q_{t_p}\mu^r_{t^q_p}Y|r| \\
{\scriptstyle \alpha_{t;(t_p)}}\downarrow & = & {\scriptstyle \alpha_{t;(t_p)}}\downarrow & = & \downarrow{\scriptstyle \alpha_{t;(t_p)}} \\
\mu^{p,q}_{I(t;(t_p))}\mu^v_{I(t^q_p;(t^{q,r}_p)_r)}X|v| & \xrightarrow{\mu^{p,q}_{I(t;(t_p))}\alpha^{-1}_{t^q_p;(t^{q,r}_p)_r}} & \mu^{p,q}_{I(t;(t_p))}\mu^r_{t^q_p}\mu^v_{t^{q,r}_p}X|v| & \xrightarrow{\mu^{p,q}_{I(t;(t_p))}\mu^r_{t^q_p}f_{t^{q,r}_p}} & \mu^{p,q}_{I(t;(t_p))}\mu^r_{t^q_p}Y|r|
\end{array}
$$

Thus $\hat{f}$ is a morphism of homotopy cooperads. □

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